Functional Analysis
Problem set 1

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1. Show that any two o.n. basis in a Hilbert space have the same cardinality.

2. Show that the Hilbert dimension of a Hilbert space is less than or equal to its vector space dimension

$$\text{Hilbert dimension}\, H \leq \dim_{\mathbb{C}} H$$

with equality only if H is finite dimensional. For example, compare the Hilbert dimension of ℓ^2 with its vector space dimension.

3. Apply the Gram-Schmidt orthonormalization process to the sequence

$$1, x, x^2, \cdots$$

in $L^2[-1,1]$ and compute the first 4 terms. The resulting sequence of polynomials

$$p_0(x), p_1(x), p_2(x), ...$$

are *Legendre polynomials.*

4. Consider the space of all measurable functions f on $\mathbb{R}$ such that

$$\int_{-\infty}^{+\infty} |f(x)|^2 w(x) dx < \infty$$

where

$$w(x) = e^{-\frac{1}{2}x^2}.$$

Define an inner product on the space of all such functions and turn it into a Hilbert space. Define the *Hermite polynomials* by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad n = 0, 1, 2, \cdots$$

Compute the first 3 Hermite polynomials. Show that they form an orthogonal set in $L^2_w(\mathbb{R})$:

$$\langle H_n, H_m \rangle = n! \sqrt{2\pi}\, \delta_{n,m}.$$

Show that H_n satisfies the *eigenvalue problem*:

$$u'' - xu' = \lambda u, \qquad \lambda = n.$$

5. Parseval's equality and Fourier series provide a very powerful method to sum some remarkable series. Here are a few examples:
a) Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re} s > 1$$

denote the *Riemann zeta function*. Let $f(x) = x$. Apply Parseval's equality

$$||f||^2 = \sum_{i=-\infty}^{\infty} |\langle f, e_i \rangle|^2$$

in $L^2[-\pi, \pi]$ to deduce

$$\zeta(2) = \frac{\pi^2}{6} \qquad \text{Euler (1739).}$$

Show that

$$\zeta(4) = \frac{\pi^4}{90} \qquad \text{Euler (1739)}$$

by a similar method applied to $f(x) = x^2$. Can you show that

$$\zeta(2n) = (-1)^n \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

as well? Here B_n denotes the n th Bernoulli number.
b) Let $f(x) = e^{sx}$ and show that

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2 + s^2} = \frac{\pi}{s} \coth(\pi s).$$

6. Apart from the *Pythagorean theorem* and the parallelogram law, there is yet a third identity in an inner product space, the *polarization identity*:

$$\langle x, y \rangle = \frac{1}{4}\{||x + y||^2 - ||x - y||^2 - i(||x + iy||^2 - ||x - iy||^2)\}$$

7. (Uncertainty Principle). Let $A : H \to H$ be a selfadjoint linear operator and $x \in H$ a *unit* vector in H. The *expectation* value (or *mean value*) of A in the state x is defined as

$$\langle A \rangle_x := \langle Ax, x \rangle,$$

and the *standard deviation* or *dispersion* of A in the state x is defined as

$$\Delta_x A := \sqrt{\langle (A - \langle A \rangle_x)^2 \rangle_x} = \sqrt{\langle A^2 \rangle_x - \langle A \rangle_x^2}.$$

It gives the 'error' involved in measuring a quantum mechanical observable represented by the selfadjoint operator A when the system is in the state x. Let A and B be selfadjoint operators and let $[A, B] = AB - BA$ denote their commutator. Use the Cauchy-Schwartz inequality to prove *Heisenberg's uncertainty principle*:

$$\Delta_x A \, \Delta_x B \geq \frac{1}{2} |\langle [A, B] \rangle_x|.$$

8. We saw that for each cardinal number there is a unique, up to isomorphism, Hilbert space whose Hilbert dimension is the given cardinal number. In sharp contrast to Hilbert spaces, show that there are uncountably many different, i.e., non-isometric, Banach norms on $\mathbb{R}^2$.

9. Let $S^1 = \{z \in \mathbb{C}; |z| = 1\}$ denote the unit circle. Show that the set of trigonometric polynomials

$$\mathcal{A} = \{\sum_{k=-n}^{n} a_k e^{ikx};\ n \geq 0\},$$

considered as functions on the circle, satisfies conditions of the Stone-Weirestrass approximation theorem and hence is dense in $C(S^1)$. Use this to show that $e_n = e^{2\pi inx}, n \in \mathbb{Z}$, is an o.n. *basis* for $L^2[0,1]$.

10. Prove that $C[0,1]$ is *not* complete under the L^2-norm,

$$||f||_2^2 = \int_0^1 |f(x)|^2 dx,$$

but *is* complete under the sup norm

$$||f||_\infty = \sup\{|f(x)|;\ x \in [0,1]\}.$$