Functional Analysis
Problem set 2

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1. (extension of functions) Let X and Y be metric spaces and $X' \subset X$ a dense subset. Show that a continuous map $f : X' \to Y$ need not have an extension to a continuous map $X \to Y$, even if Y in complete. Show that if f is *uniformly continuous* and Y is complete then there is a unique such extension and the extension is uniformly continuous as well. (This result is useful in functional analysis since continuous linear maps between normed spaces are automatically uniformly continuous).

2. (integral operators) a) Let $k : [0,1] \times [0,1] \to \mathbb{C}$ be an square summable kernel. Show that for the corresponding integral operator T_k we have

$$||T_k||^2 \leq \int_0^1 \int_0^1 |k(s,t)|^2 ds dt = ||k||^2.$$

In particular for the Volterra integral operator V we have

$$||V|| \leq \frac{1}{\sqrt{2}}.$$

b) Let $k^*(s,t) = \overline{k(t,s)}$ for all $0 \leq s,t \leq 1$. Show that

$$T_k^* = T_{k^*}.$$

In particular T_k is selfadjoint iff $k = k^*$, i.e. $k(s,t) = \overline{k(t,s)}$ for all s,t.

3. We saw that the operator

$$\frac{d}{dx} : \mathcal{D} \to L^2[0,1]$$

is unbounded with respect to the L^2-norm on $\mathcal{D}$. Show that it is bounded with respect to the inner product on $\mathcal{D}$ defined by

$$\langle f, g \rangle = \int_0^1 (f\bar{g} + f'\bar{g}').$$

4. Let $k : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic continuous function and define an integral operator $K : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$ by

$$(Kf)(x) = \int_{-\pi}^{\pi} k(x-y)f(y)dy.$$

Compute the matrix of K with respect to the o.n. basis $(\frac{1}{\sqrt{2\pi)}}e^{inx})_{n\in\mathbb{Z}}$.

5. (norms and adjoints) a) Let T be a bounded linear operator on a Hilbert space H. Show that

$$||T^*|| = ||T||.$$

b) Show that if T is selfadjoint then

$$||T|| = \text{Sup}\, |\langle Tx, x\rangle|$$

where the sup is over the set $||x|| = 1$. Show that this formula is not necessarily correct for *non-selfadjoint* operator.
c) Use b) to show that for any T

$$||T^*T|| = ||T||^2.$$

6. (more on spectrum) a) Let T be a *nilpotent operator*, i.e., $T^n = 0$ for some positive integer n. Show that $1 + T$ is invertible.
b) Use a) to show that $\sigma(T) = \{0\}$, if T is nilpotent.

c) Generalize a) by proving the *spectral mapping theorem*: for any polynomial $f(x)$ with complex coefficients we have

$$\sigma(f(T)) = f(\sigma(T)).$$

d) Show that for any two operators a and b we have

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\},$$

that is ab and ba have the same spectrum except for one point 0.

7. Solve Problems 3.3, 3.9, 3.27, 3.28, 3.34 from the book 'Elementary Functional Analysis' by B. MacCluer (available online in our Library's website).