Functional Analysis
Topics discussed per day

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This document is meant to record the topics discussed in my course, Functional Analysis, per each class. It can also be useful to draw your attention to important points discussed in each class.

• First two weeks of December: Students presented various topics related to the course material and functional analysis (the course ended on December 14, 2009):

1. Saugmann: Hilbert space and quantum mechanics (Chapter 6 of Kowalski's ETH notes)

2. Sinnamon: Fredholm operators and index theory (pp 108-124 in R. G. Douglas' book Banach algebra techniques in operator theory)

3. Mamun: Operator algebras and MASA'a from G. Pederson's book (Analysis now)

4. Gao: Sturm Liouville theory (from Young's book 'An introduction to Hilbert space')

5. Acar: Groups of unitary operators (Chapter 5 of Pederson's book 'Analysis now')

6. Wagley: Chapter 5 and Sections 4.6 of Kowalski's ETH notes.

• The last week of November: Functional analysis and quantum mechanics-postulates of quantum mechanics: state space, observables, measurements and Born's probabilistic interpretation, quantum dynamics (Schroedinger equation); Stone's theorem on (strongly continuous) one parameter groups of unitary operators on a Hilbert space; the role of unbounded operators in QM; Heisenberg's commutation relations; A proof of Heisenberg's uncertainty principle.

• Third week of November: Spectral theorem for compact selfadjoint operators. (I intended to give applications to Sturm-Liouville theory, and the Peter-Weyl theorem. No time!).

• Second week of November: Spectral theory basics: the spectrum is compact and non-empty (Gelfand); spectral radius formula; examples of spectrum computations; spectrum of compact operators (discrete with 0 as the only limit point).

• First week of November: Theory of compact operators (mostly in Hilbert space); equivalent formulations, finite rank operators, Hilbert-Schmidt operators, examples: integral operators with L^2-kernels;

• Wednesday, October 28: As an application of the closed graph theorem, proved the Hellinger-Toepliz theorem on automatic continuity of symmetric everywhere defined operators; consequences for quantum mechanics. Spectrum of a bounded linear operator on a Banach space (the word *spectrum* was used by Newton in his spectrum of light; amazingly the two uses of the word are related!) Eigenvalues and why they are not all of the spectrum in general (shift operator); Spectrum of a diagonal infinite matrix; first goal: to show the spectrum is *compact and non-empty*; compactness is done, non-empty will be done next Monday.

• Monday, October 26: Finished proof of open mapping theorem; corollary: continuity of the inverse (when it exists); proof of *closed graph theorem*, counterexample in the nonlinear case. So, done with 'Big Three'.

• Wednesday, October 21: Discussed the idea of open mapping theorem, counter examples in the nonlinear case. started its proof.

- Monday, October 19: No classes-I gave a colloquium in Toronto (York).

- Wednesday, October 14: Finished proof of *Hahn-Banach Theorem* (real case); corollaries: the continuous dual X^* of a normed space X, showed $(l^1)^* = l^\infty$, tried to identify l^∞-no time; the canonical isometric embedding $X \to X^{**}$; *reflexive Banach spaces*, examples: finite dimensional normed spaces, Hilbert spaces, l^p for $1 < p < \infty$ is reflexive (not prove yet); warning: l^1 is NOT reflexive! the Baire category theorem: nowhere dense sets in complete metric spaces; application: proved the uniform boundedness principle; Banach-Steinhaus theorem.

- Monday, October 12: Thanksgiving Monday-no class.

- Wednesday, October 7: talked a bit more about integral operators and the linear algebra-functional analysis analogy (the 'Rosetta Stone'); simplest example of a differential operator

$$\frac{d}{dx} : \mathrm{Dom}(\frac{d}{dx}) \subset L^2[0,1] \to L^2[0,1];$$

why it is not L^2-bounded, problems with its domain; how to think about $L^2(S^1)$, why

$$-i\frac{d}{dx} : L^2(S^1) \to L^2(S^1)$$

is a *symmetric operator*; the 3 pillars of functional analysis; started with the Hahn-Banach theorem: first tried the real case-the main technical lemma (how to extend f to a bit bigger subspace first) not quite finished yet! gave one application: continuous linear functionals separate the points of X; what Hahn-Banach means? there are plenty of cont. linear functionals on any normed space.

- Monday, October 5: The adjoint $T^* : H_2 \to H_1$ of a bounded linear operator $T : H_1 \to H_2$ between Hilbert spaces, defined by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

Properties of the adjoint: anti-linear, anti-multiplicative, involutive; better yet: take $H_1 = H_2 = H$, then the adjoint operation $T \to T^*$ turns $\mathcal{L}(H)$ into a C^*-algebra (I shall mention the axioms of a C^*-algebra in the next class).

The matrix of the adjoint in an o.n. basis; important classes of operators: *selfadjoint, isometry, unitary, co-isomtery*; isometry does *not* imply unitary in *infinite* dimensions: the *forward shift* operator is an example of this; the *backward shift* operator; the *Volterra integral operator* $V : L^2[0,1] \to L^2[0,1]$

$$(Vf)(x) = \int_0^x f(y)dy.$$

It is a bounded linear operator. V is an example of an *integral operator*. The class of integral operators provide one of the most important and widely studied classes of linear operators; general definition: given $K : [0,1] \times [0,1] \to \mathbb{C}$, define $T_K : L^2[0,1] \to L^2[0,1]$ by

$$(T_K f)(x) = \int_0^1 K(x,y)f(y)dy$$

K is called the *kernel* of T_K. If K is continuous then T_K is a bounded; this is too restrictive: e.g. the Volterra is an integral operator with a discontinuous kernel. If $T \in L^2([0,1]^2)$, then T_K is bounded. Spent some time to discuss the important analogy between T_K and a linear operator defined by a matrix: this is far reaching and played an important role in Volterra, Fredholm, Hilbert and many others' work up to now!

• Wednesday, September 30: complete metric spaces and how completion works, examples and applications to inner product spaces and normed spaces; bounded linear operators between normed spaces, equivalent definitions of boundedness; why closedness of the kernel is not enough (unlike linear functionals on a Hilbert space), gave a counter example on an inner product space (not a Hilbert space), here is a good question: give an example of a linear operator $T : H \to H$ on a Hilbert space whose kernel is closed, but is not bounded; (orthogonal) projection operators, orthogonal versus non-orthogonal projections.

• Monday, September 28: Gram-Schmidt orthonormalization process; separable Hilbert spaces and equivalent criteria for separability; L^2-spaces (just $L^2[0,1]$ for the moment), how to prove it is separable and how to find a countable o.n. basis for it; Fourier series, how to prove that $e_n = e^{2\pi i n x}, n \in \mathbb{Z}$ form an o.n. basis for $L^2[0,1]$: Weierstrass approximation theorem and its generalization, the Stone-Weierstrass approximation theorem; compare L^2-convergence of functions with other types of convergence, e.g. pointwise or uniform convergence.

• Wednesday, September 23: complete o.n. sets (Hilbert basis) and its various equivalent formulations, existence of Hilbert basis for Hilbert spaces (using Zorn's lemma, what is Zorn's lemma?), comparing the Hilbert dimension with the vector space dimension; isomorphism theorem for Hilbert spaces, moral: there aren't that 'many' Hilbert spaces out there! compare with Banach spaces.

• Monday, September 21: continuous linear functionals, Riesz representation theorem for continuous linear functionals, the dual H^* of a Hilbert space and its relation to H.

• Wednesday, September 16: distance to closed convex sets, orthogonality, orthogonal complement, the orthogonal decomposition theorem.

• Monday, September 14: First lecture: definition of a complex pre-Hilbert space (also called inner product space, Hermitian space); Pythagorean theorem; Cauchy-Schwartz inequality-this was an unusual derivation! norm and metric of a pre-Hilbert space; introduce normed spaces; completeness condition: defined Hilbert space; examples: $\mathbb{C}^n$ and ℓ^2.