Mathematics 9020B/4120B, Field Theory
Winter 2016, Western University
Instructor: Masoud Khalkhali

A very brief outline of Galois theory from past to present: Galois theory, as originally conceived by Évariste Galois around 1830, was developed to address the question of solvability by radicals of polynomial equations. The theory has however a much broader reach and can be vastly extended. This was even clear to Galois himself as he called his theory the "theory of ambiguities" and was planing to apply his ideas to questions of "transcendental analysis". Over the years the original approach of Galois was reformulated and streamlined using modern algebraic language and tools. This culminated in Emil Artin's formulation of Galois theory of field extensions around 1940. This modern approach then paved the way for Grothendieck and his school to obtain a vast generalization of Galois theory merging ideas of Galois groups and field extensions with fundamental groups and covering spaces in very general contexts like schemes in algebraic geometry and Tannakian categories. A very readable survey is the recent article by Yves André, cited below.

Course outline: This course is mostly about fields and the Galois theory of field extensions and its applications. The following topics will be covered.

1) Field extensions, automorphism group of a field and Galois group of an extension, Galois extensions, fundamental theorem of Galois theory.

2) Applications, finite fields, Abelian extensions, Kummer extensions, algebraic number fields, Hilbert theorem 90, algebraic closure, function field-number field analogies.

3) Solvability by radicals, Abel's theorem on insolvability of general equation of degree bigger or equal to five, Galois criterion for solvability of algebraic equations, cyclotomic fields, geometric constructions, Gauss' theorem on constructibility of regular 17 gons, how to compute the Galois group of an equation?

4) (Time permitting) Galois groups and fundamental groups, a very brief sketch of modern Galois theory according to Grothendieck.

Textbook for self study : Galois Theory, by S. H. Weintraub, Springer Paperbacks (free online version available through Western library).

A refreshingly different look at Galois theory: Galois theory, coverings, and Riemann surfaces, by A. G. Khovanskii (free online copy available through Western library).

Grading: %60 assignments (4 sets of assignments), %40 final exam.

Suggested extra reading: The following sources contain a lot of valuable information on Galois theory and its many connections to other fields.

Supplementary textbooks:

1. Galois Theory, by Emil Artin.

2. Galois Theory, by D. A. Cox.

3. Field and Galois Theory, by P. Morandi.

4. Galois Groups and Fundamental Groups, by T. Szamuely.
Publisher's blurb: *'Ever since the concepts of Galois groups in algebra and fundamental groups in topology emerged during the nineteenth century, mathematicians have known of the strong analogies between the two concepts. This book presents the connection starting at an elementary level, showing how the judicious use of algebraic geometry gives access to the powerful interplay between algebra and topology that underpins much modern research in geometry and number theory. Assuming as little technical background as possible, the book starts with basic algebraic and topological concepts, but already presented from the modern viewpoint advocated by Grothendieck. This enables a systematic yet accessible development of the theories of fundamental groups of algebraic curves, fundamental groups of schemes, and Tannakian fundamental groups. The connection between fundamental groups and linear differential equations is also developed at increasing levels of generality. Key applications and recent results, for example on the inverse Galois problem, are given throughout'.*

Texts of historical interest (but still quite technical):

1. The mathematical writings of Évariste Galois, by Peter M. Neumann.
Publisher's blurb: *'the present work contains English translations of almost all the Galois material. They are presented alongside a new transcription of the original French, and are enhanced by three levels of commentary. An introduction explains the context of Galois' work, the various publications in which it appears, and the vagaries of his manuscripts. Then there is a chapter in which the five mathematical articles published in his lifetime are reprinted. After that come the Testamentary Letter and the First Memoir (in which Galois expounded the ideas now called Galois Theory), which are the most famous of the manuscripts. There follow the less well known manuscripts, namely the Second Memoir and the many fragments. A short epilogue devoted to myths and mysteries concludes the text.*

2. Galois Theory, by H. M. Edwards.
Publisher's blurb: *'This is an introduction to Galois Theory along the lines of Galois Memoir on the Conditions for Solvability of Equations by Radicals. It puts Galois ideas into historical perspective by tracing their antecedents in the works of Gauss, Lagrange, Newton, and even the ancient Babylonians. It also explains the modern formulation of the theory. It includes many exercises, with their answers, and an English translation of Galoiss memoir.'*

3. Galois' Theory of Algebraic Equations, by Jean-Pierre Tignol.
Publisher's blurb: *'Galois' Theory of Algebraic Equations gives a detailed account of the development of the theory of algebraic equations, from its origins in ancient times to its completion by Galois in the nineteenth century. The main emphasis is placed on equations of at least the third degree, i.e. on the developments during the period from the sixteenth to the nineteenth century. The appropriate parts of works by Cardano, Lagrange, Vandermonde, Gauss, Abel and Galois are reviewed and placed in their historical perspective, with the aim of conveying to the reader a sense of the way in which the theory of algebraic equations has evolved and has led to such basic mathematical notions as "group" and "field." A brief discussion on the fundamental theorems of modern Galois theory is included.'*

4. Ambiguity theory, old and new, by Yves Andrè.
http://arxiv.org/pdf/0805.2568v1.pdf.

Authors' abstract: *This is a introductory survey of some recent developments of Galois ideas in Arithmetic, Complex Analysis, Transcendental Number Theory and Quantum Field Theory, and of some of their interrelations.*

Online notes:

1. Fields and Galois Theory, by James Milne; available at www.jmilne.org/math/CourseNotes/FT.pdf.

On Youtube:

1. A quick proof of the insolvability of quintics by radicals: https://www.youtube.com/watch?v=RhpVSV6iCko.

2. The Memoirs and Legacy of Évariste Galois, by Peter Neumann: http://www.youtube.com/watch?v=xfJ4vwQ3vpo.

3. Évariste Galois et la théorie de l'ambiguité, by Alain Connes: http://www.youtube.com/watch?v=rMb9UE5msH8.