Fredholm Operators and Atkinson's Theorem

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Introduction. Let X and Y be vector spaces and $T : X \to Y$ a linear operator. We know that T is an isomorphism if and only if $\ker(T) = \{0\}$ and $\operatorname{im}(T) = Y$, that is, $\operatorname{coker}(T) \coloneqq Y/\operatorname{im}(T) = \{0\}$. Equivalently, T is an isomorphism if and only if $\dim\ker(T) = 0$ and $\dim\operatorname{coker}(T) = 0$. We see that the "closeness" of T to becoming an isomorphism is related to the subspaces $\ker(T)$ and $\operatorname{coker}(T)$. Fredholm operators can be viewed as operators "close" to being isomorphisms.

Definition. Let $T : X \to Y$ be a bounded linear operator between Hilbert spaces. T is *Fredholm* if $\dim\ker(T) < \infty$ and $\dim\operatorname{coker}(T) < \infty$. The *index* of T is the integer

$$\mathit{index}(T) = \dim\ker(T) - \dim\operatorname{coker}(T).$$

Remark. Any linear isomorphism is Fredholm with index 0 and any linear operator between finite dimensional spaces is Fredholm.

Proposition 1. *Let $T : X \to Y$ be Fredholm. Then the image of T is closed in Y and*

$$\mathit{index}(T) = \dim\ker(T) - \dim\ker(T^*).$$

Proof. Since $\dim\ker(T) < \infty$, then $\ker(T)$ is a finite dimensional subspace of X, and hence closed. Then we may assume without loss of generality that $\ker(T) = \{0\}$ as we can restrict T to $\ker(T)^\perp$. Since $\dim\operatorname{coker}(T) < \infty$, there exists a finite dimensional subspace $V \subseteq Y$ such that $Y = \operatorname{im}(T) \oplus V$. As V is finite dimensional, V is closed so $Y = V \oplus V^\perp$. Let $\pi : Y \to V^\perp$ be the orthogonal projection operator.

Define $G = \pi T : X \to V^\perp$. Note that G is continuous as π and T are continuous. As $Y = \operatorname{im}(T) \oplus V$, then $\operatorname{im}(T) \cap V = \{0\}$. Then for $x \in \ker(G)$, $\pi T(x) = 0$ so $T(x) \in \ker(\pi) = V$. Then $Tx \in \operatorname{im}(T) \cap V = \{0\}$. Thus G is injective. Let $y \in V^\perp \subseteq Y$. Then there exists unique vectors $y_1 \in \operatorname{im}(T), y_2 \in V$ such that $y = y_1 + y_2$. As $y_1 \in \operatorname{im}(T)$, then there exists $x \in X$ such that $T(x) = y_1$. Note that as $y_1 = y - y_2$ where $y \in V^\perp, y_2 \in V$, then $\pi(y_1) = y$. Thus $G(x) = \pi T(x) = \pi(y_1) = y$. Hence G is an isomorphism. Then by the open mapping theorem, G is an open map so $G^{-1} : V^\perp \to X$ is continuous.

Let $\{x_n\}_{n=1}^\infty$ be a sequence in X such that $\{T(x_n)\}_{n=1}^\infty$ converges in Y, say to $y \in Y$. Then $\lim_{n\to\infty} G(x_n) = \pi(y)$ exists in $V^\perp$. Then $x \coloneqq \lim_{n\to\infty} x_n = G^{-1}\pi(y)$ exists in X. Thus $T(x_n) \to T(x) \in \operatorname{im}(T)$ so the image of T is closed.

As $\operatorname{im}(T)$ is closed, we have $Y = \operatorname{im}(T) \oplus \operatorname{im}(T)^\perp$. As $\operatorname{im}(T)^\perp = \ker(T^*)$, we have $\operatorname{coker}(T) = Y/\operatorname{im}(T) \cong \ker(T^*)$. Thus $\dim\operatorname{coker}(T) = \dim\ker(T^*)$. □

Example. Let $S : \ell^2 \to \ell^2$ be the left shift operator, that is, $S(x_1, x_2, ...) = (x_2, x_3, ...)$. Then we see that $\dim\ker(S) = 1$ and it is easy to see that S is surjective so $\dim\operatorname{coker}(S) = 0$. Thus $\mathit{index}(S) = 1$. Also, $S^* : \ell^2 \to \ell^2$ is the right shift operator which is injective and its image has codimension 1 so $\mathit{index}(S^*) = -1$. Furthermore, $\mathit{index}(S^k) = k$ and $\mathit{index}((S^*)^k) = -k$ for all $k \geq 0$.

Lemma 2. *If $K : X \to Y$ is a finite rank operator, then there exists $\{\phi_n\}_{n=1}^k \subseteq X$ and $\{\psi_n\}_{n=1}^k \subseteq Y$ such that*

a) $Kx = \sum_{n=1}^k \langle x, \phi_n \rangle \psi_n$ *for all* $x \in X$.

b) $K^*y = \sum_{n=1}^k \langle y, \psi_n \rangle \phi_n$ *for all* $y \in Y$. *In particular,* K^* *is finite rank.*

For the c) and d), suppose further that $X = Y$.

c) $\dim \ker(I + K) < \infty$.

d) $\dim \mathit{coker}(I + K) < \infty$.

Proof. a) As K is finite rank, $\dim \mathrm{im}(K) < \infty$ so let $\{\psi_n\}_{n=1}^k$ be an orthonormal basis for $\mathrm{im}(K)$. Then for $x \in X$,

$$Kx = \sum_{n=1}^k \langle Kx, \psi_n \rangle \psi_n = \sum_{n=1}^k \langle x, K^*\psi_n \rangle \psi_n = \sum_{n=1}^k \langle x, \phi_n \rangle \psi_n$$

where $\phi_n = K^*\psi_n$.

b) Extend $\{\psi_n\}_{n=1}^k$ to an orthonormal Hilbert basis $\{\psi_n\}_{n=1}^k \cup \{\varphi_i\}_{i \in I}$. Then for $y \in Y$, $y = \sum_{n=1}^k \langle y, \psi_n \rangle \psi_n + \sum_{i \in I} \langle y, \varphi_i \rangle \varphi_i$. Thus for $x \in X$

$$\left\langle K^*y - \sum_{n=1}^k \langle y, \psi_n \rangle \phi_n, x \right\rangle = \left\langle K^* \left(y - \sum_{n=1}^k \langle y, \psi_n \rangle \psi_n \right), x \right\rangle$$

$$= \left\langle \sum_{n=1}^k \langle y, \psi_n \rangle \psi_n + \sum_{i \in I} \langle y, \varphi_i \rangle \varphi_i - \sum_{n=1}^k \langle y, \psi_n \rangle \psi_n, Kx \right\rangle$$

$$= \left\langle \sum_{i \in I} \langle y, \varphi_i \rangle \varphi_i, \sum_{n=1}^k \langle x, \phi_n \rangle \psi_n \right\rangle = 0.$$

Thus $K^*y = \sum_{n=1}^k \langle y, \psi_n \rangle \phi_n$.

c) We have $\ker(I + K) = \{x \in X | Kx = -x\} \subseteq \mathrm{im}(K)$, which is finite dimensional. Thus $\dim \ker(I + K) < \infty$.

d) For $x \in \ker(K)$, $x = (I + K)x$ so $\ker(K) \subseteq \mathrm{im}(I + K)$. For $x, y \in X$, with $\langle x, \phi_i \rangle = 0$ for all i, we have $\langle Kx, y \rangle = \langle x, K^*y \rangle = \langle x, \sum_{i=1}^n \langle y, \psi_i \rangle \phi_i \rangle = 0$ so $Kx = 0$. Thus $\{\phi_1, ..., \phi_k\}^\perp \subseteq \ker(K)$. Then $X = \ker(K) + \mathrm{span}(\{\phi_i\}_{i=1}^k)$ as $\mathrm{span}(\{\phi_i\}_{i=1}^k) = \mathrm{im}(K^*)$ and $X = \mathrm{im}(K^*) \oplus \mathrm{im}(K^*)^\perp$. Thus $\dim(X/\ker(K)) < \infty$. Since $\ker(K) \subseteq \mathrm{im}(I + K)$, then $\dim \mathrm{coker}(I + K) = \dim(X/\mathrm{im}(I + K)) < \infty$. □

Theorem 3. *(Atkinson) A bounded operator $T : X \to Y$ between Hilbert spaces is Fredholm if and only if there exists a bounded operator $A : Y \to X$ such that $AT - I$ and $TA - I$ are both compact. Furthermore, we may choose A so that $AF - I$ and $FA - I$ are both finite rank operators.*

Proof. ($\Rightarrow$) Suppose $T : X \to Y$ is Fredholm. Then $T : \ker(T)^\perp \to \mathrm{im}(T)$ is an isomorphism between Hilbert spaces. Let $\tilde{T}$ be the inverse of this map. Note that $\tilde{T}$ is continuous by the open mapping theorem. As $\mathrm{im}(T)$ is closed in Y, let $P : Y \to \mathrm{im}(T)$ be the orthogonal projection map. Note that $PT = T$. Let $A = \tilde{T}P$. Let $x = x_1 + x_2 \in X$ where $x_1 \in \ker(T)$ and $x_2 \in \ker(T)^\perp$. Then

$$(AT - I)(x) = \tilde{T}PT(x) - x = \tilde{T}T(x) - x = \tilde{T}T(x_2) - x = x_2 - x = -x_1 = -Q(x)$$

where $Q : X \to \ker(T)$ is the orthogonal projection onto $\ker(T)$. Let $y = y_1 + y_2 \in Y$ where $y_1 \in \mathrm{im}(T)$ and $y_2 \in \mathrm{im}(T)^\perp$. Then

$$(TA - I)(y) = T\tilde{T}P(y) - y = T\tilde{T}(y_1) - y = y_1 - y = -(I - P)(y).$$

Note that $I - P$ is the orthogonal projection onto $\mathrm{im}(T)^\perp = \mathrm{coker}(T)$. Since T is Fredholm, $\dim\ker(T) < \infty$ and $\dim\mathrm{coker}(T) < \infty$ so Q and $I - P$ are finite rank projections and hence compact. Thus $AT - I$ and $TA - I$ are compact.

($\Leftarrow$) Proof 1: Let $A : Y \to X$ be such that $AT - I = K_1$ and $TA - I = K_2$ are both compact. Then $AT = I + K_1$ and $TA = I + K_2$. Then by Question 4 on Assignment 3, $\dim\ker(AT) < \infty$ and $\dim\mathrm{coker}(TA) < \infty$. As $\ker(T) \subseteq \ker(AT)$ and $\mathrm{im}(T) \supseteq \mathrm{im}(TA)$, then $\dim\ker(T) < \infty$ and $\dim\mathrm{coker}(T) < \infty$ so T is Fredholm.

Proof 2: For the converse, we will assume the approximation property that Hilbert spaces have: a bounded operator $K : X \to Y$ is compact if and only if there exists finite rank operators $K_n : X \to Y$ such that $\|K - K_n\| \to 0$ as $n \to \infty$ (see Proposition 16.7 in [D]).

We first show we can choose A so that $AT - I$ and $TA - I$ are finite rank operators. Let $G = AT - I$, a compact operator. Choose a finite rank approximation G_1 to G such that $G = G_1 + \mathcal{E}$ where $\|\mathcal{E}\| < 1$. Then $I - \mathcal{E}$ is invertible. Let $A_L : Y \to X$ be the operator $A_L = (I + \mathcal{E})^{-1}A$. As $AT = I + G = I + \mathcal{E} + G_1$, then

$$\begin{aligned} A_L T &= (I + \mathcal{E})^{-1}AT = (I + \mathcal{E})^{-1}(I + \mathcal{E} + G_1) \\ &= I + (I + \mathcal{E})^{-1}G_1 = I + K_L \end{aligned}$$

where K_L is a finite rank operator as G_1 is a finite rank operator. Similarly, there exists a bounded operator $A_R : Y \to X$ and a finite rank operator K_R such that $TA_R = I + K_R$. We have $A_L T A_R = (I + K_L)A_R = A_R + K_L A_R$ and $A_L T A_R = A_L + A_L K_R$. Thus

$$A_L - A_R = A_L K_R - K_L A_R = S,$$

a finite rank operator as K_L and K_R are finite rank. Then $TA_L = T(A_R + S) = TA_R + TS = I + K_R + TS$ so $TA_L - I = K_R - TS$, a finite rank operator. Thus $A_L T - I$ and $TA_L - I$ are finite rank.

We may now assume that A is chosen such that $AT - I = G_1$ and $TA - I = G_2$ are finite rank. We see that $\ker(T) \subseteq \ker(AT) = \ker(I + G_1)$ and $\mathrm{im}(T) \supseteq \mathrm{im}(TA) = \mathrm{im}(I + G_2)$. By the previous lemma, $\dim\ker(I + G_1) < \infty$ and $\dim\mathrm{coker}(I + G_2) < \infty$. Thus $\dim\ker(T) < \infty$ and $\dim\mathrm{coker}(T) < \infty$ so T is Fredholm. ☐

Corollary 4. *If $T : X \to Y$ is Fredholm, then T^* is Fredholm and index$(T^*) = -$index(T).*

Proof. Choose $A : Y \to X$ such that $AT - I$ and $TA - I$ are compact. Then $(AT - I)^* = T^*A^* - I$ and $(TA - I)^* = A^*T^* - I$ are compact so T^* is Fredholm. As $index(T) = \dim\ker(T) - \dim\ker(T^*)$, then $index(T^*) = \dim\ker(T^*) - \dim\ker(T) = -index(T)$. ☐

Lemma 5. *A bounded operator $T : X \to Y$ is Fredholm if and only if there exists orthogonal decompositions $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ such that*

a) X_1 and Y_1 are closed subspaces.

b) X_2 and Y_2 are finite dimensional subspaces.

c) T has the block diagonal form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : X_1 \oplus X_2 \to Y_1 \oplus Y_2$$

where $T_{ij} : X_j \to Y_i$, and $T_{11} : X_1 \to Y_1$ is a bounded invertible operator.

Furthermore, given this decomposition, index$(T) = \dim(X_2) - \dim(Y_2)$.

Proof. ($\Rightarrow$) Suppose T is Fredholm. Let $X_1 = \ker(T)^\perp, X_2 = \ker(T), Y_1 = \operatorname{im}(T)$, and $Y_2 = \operatorname{im}(T)^\perp$. We note that a) and b) are satisfied. We have $T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$ where $T_{11} = T|_{X_1} : \ker(T)^\perp \to \operatorname{im}(T)$ is invertible. Also, $\mathit{index}(T) = \dim\ker(T) - \dim\operatorname{coker}(T) = \dim(X_2) - \dim(Y_2)$.

($\Leftarrow$) Let $A = \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} : Y \to X$. Then

$$AT = \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} I & T_{11}^{-1}T_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & T_{11}^{-1}T_{12} \\ 0 & -I \end{pmatrix}.$$

As $T_{12} : X_2 \to Y_1$ and X_2 is finite dimensional, T_{12} is finite rank so $T_{11}^{-1}T_{12}$ is finite rank. Similarly, $-I : Y_2 \to Y_2$ is finite rank. Thus $AT - I$ is finite rank. Similarly, $TA - I$ is finite rank, so T is Fredholm.

Note that $(x_1, x_2) \in \ker(T)$, $x_1 \in X_1, x_2 \in X_2$, if and only if $T_{11}x_1 + T_{12}x_2 = 0$ and $T_{21}x_1 + T_{22}x_2 = 0$ if and only if $x_1 = -T_{11}^{-1}T_{12}x_2$ and $(-T_{21}T_{11}^{-1}T_{12} + T_{22})x_2 = 0$. Let $D = (T_{22} - T_{21}T_{11}^{-1}T_{12}) : X_2 \to Y_2$. Then the map $\ker(D) \to \ker(T)$ sending

$$x_2 \mapsto \begin{pmatrix} -T_{11}^{-1}T_{12}x_2 \\ x_2 \end{pmatrix}$$

is a linear isomorphism, as $\ker(D)$ and $\ker(T)$ are finite dimensional and the map is injective. Thus $\ker(T) \cong \ker(D)$. Similarly, as

$$T^* = \begin{pmatrix} T_{11}^* & T_{21}^* \\ T_{12}^* & T_{22}^* \end{pmatrix} : Y_1 \oplus Y_2 \to X_1 \oplus X_2,$$

we have $\ker(T^*) \cong \ker(D^*)$. Thus $\mathit{index}(T) = \mathit{index}(D)$. As D is a linear operator between finite dimensional Hilbert spaces, then by rank nullity, we have $\mathit{index}(D) = \dim(X_2) - \dim(Y_2)$. □

Proposition 6. *Let $T : X \to Y, S : Y \to Z$ be Fredholm and $K : X \to Y$ be compact. Then*

a) The set of Fredholm operators form an open subset of bounded operators. Moreover, if $\mathcal{E} : X \to Y$ is a bounded operator with $\|\mathcal{E}\|$ sufficiently small, then index$(T) =$ index$(T+\mathcal{E})$.

b) $T + K$ is Fredholm and index$(T) =$ index$(T + K)$.

c) ST is Fredholm and index$(ST) =$ index$(S) +$ index(T).

Proof. a) We decompose X, Y, and T as in the previous lemma. Decompose

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix}$$

and choose $\|\mathcal{E}\|$ sufficiently small such that $\|\mathcal{E}_{11}\|$ is sufficiently small to guarantee that $T_{11}+\mathcal{E}_{11}$ is still invertible. We can do this as the set of invertible operators form an open set. Then $T+\mathcal{E}$ is Fredholm by the previous lemma and $index(T+\mathcal{E}) = \dim(X_2) - \dim(Y_2) = index(T)$.

b) Since T is Fredholm, let $A : Y \to X$ be a bounded operator such that $G_1 = AT - I$ and $G_2 = TA - I$ are compact. Then $A(T+K) - I = G_1 + AK$ and $(T+K)A - I = G_2 + KA$ are compact, since the set of compact operators is a two-sided ideal. Thus $T+K$ is Fredholm. By a), the function $f : \mathbb{R} \to \mathbb{Z}$ sending $t \mapsto index(T+tK)$ is a continuous locally constant function, where $\mathbb{Z}$ has the discrete topology, and hence constant. Thus $index(T+K) = f(1) = f(0) = index(T)$.

c) Let $A : Y \to X, B : Z \to Y$ be bounded operators such that $K_1 = AT - I, K_2 = TA - I, L_1 = BS - I$ and $L_2 = SB - I$ are compact. Then

$$STAB - I = S(K_2 + I)B - I = SK_2B + SB - I = SK_2B + L_2$$

is compact, as set of compact operators form a two-sided ideal. Similarly, $ABST - I$ is compact, so ST is Fredholm.

Let $X_1 = \ker(T)^\perp, X_2 = \ker(T), Y_1 = \operatorname{im}(T) = T(H_1)$ and $Y_2 = \operatorname{im}(T)^\perp = \ker(T^*)$. Then T decomposes into

$$T = \begin{pmatrix} \tilde{T} & 0 \\ 0 & 0 \end{pmatrix} : X_1 \oplus X_2 \to Y_1 \oplus Y_2$$

where $\tilde{T} = T|_{X_1} : X_1 \to Y_1$ is invertible. Let $Z_1 = S(Y_1)$ and $Z_2 = Z_1^\perp = S(Y_1)^\perp$. Note that $Z_1 = S(Y_1) = SQ(Y_1)$ where $Q : Y \to Y_1$ is orthogonal projection onto Y_1. Since Y_1 is closed and Y_2 is finite dimensional, then Q is Fredholm. Then SQ is Fredholm so $Z_1 = \operatorname{im}(SQ) = SQ(Y)$ is closed in Z and is of finite codimension. Then we can write S in the block form

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} : Y_1 \oplus Y_2 \to Z_1 \oplus Z_2.$$

Since $R = \begin{pmatrix} 0 & S_{12} \\ S_{21} & S_{22} \end{pmatrix} : Y \to Z$ is a finite rank operator, as S_{21} and S_{22} map into Z_2 and S_{12} has domain Y_2, $RT : X \to Z$ is finite rank. Then $index(S-R) = index(S)$ and $index(ST - RT) = index(ST)$ as R and RT are compact. Hence without loss of generality, we may take $S - R$ instead of S, that is, assume S has the form $S = \begin{pmatrix} \tilde{S} & 0 \\ 0 & 0 \end{pmatrix}$. Hence

$$ST = \begin{pmatrix} \tilde{S}\tilde{T} & 0 \\ 0 & 0 \end{pmatrix} : X_1 \oplus X_2 \to Z_1 \oplus Z_2.$$

Note that $\ker(S) = \ker(\tilde{S}) \oplus Y_2$ and $\operatorname{im}(S) = S(Y_1) = \tilde{S}(Y_1) = Z_1$, so $\tilde{S}$ is surjective. Then $\operatorname{coker}(\tilde{S}) = \{0\}$. We have

$$index(S) = \dim\ker(S) - \dim\operatorname{coker}(S)$$

$$= \dim \ker(\tilde{S}) \oplus Y_2 - \dim Z/Z_1$$
$$= index(\tilde{S}) + \dim(Y_2) - \dim(Z_2)$$

Similarly, as $\tilde{T}$ is surjective,

$$index(ST) = index(\tilde{S}\tilde{T}) + \dim(X_2) - \dim(Z_2).$$

Recall that we have

$$index(T) = \dim(X_2) - \dim(Y_2).$$

Thus

$$index(ST) - index(S) - index(T) = index(\tilde{S}\tilde{T}) - index(\tilde{S}).$$

As $\tilde{T}$ is invertible, $\text{im}(\tilde{S}) \cong \text{im}(\tilde{S}\tilde{T})$ and $\ker(\tilde{S}) \cong \ker(\tilde{S}\tilde{T})$. Thus $index(\tilde{S}\tilde{T}) - index(\tilde{S}) = 0$ so $index(ST) = index(S) + index(T)$. □

Fredholm Operators and Spectral Theory. As Fredholm operators can be thought of as being "almost" invertible, it is natural to ask about the relationship between Fredholm operators and the spectrum of an operator.

Definition. The *essential* (or *Fredholm*) *spectrum of a bounded operator* $T : X \to X$ is

$$\sigma_{ess}(T) = \{\lambda \in \mathbb{C} | \lambda - T \text{ is not a Fredholm operator}\}.$$

Much can be said about the essential spectrum, such as the fact that it is nonempty, compact, $\sigma_{ess}(T) = \sigma_{ess}(T^*)$, and if $\sigma_{ess}(T) = \{0\}$, then $\sigma(T)$ is at most countable with 0 as the only possible accumulation point. I invite the reader to see Section 7.5 in [AA] for more details.

References

[AA] Y. A. Abramovich, and C. D. Aliprantis. *An Invitation to Operator Theory.* American Mathematical Society, 2002.

[D] B. K. Driver. *Analysis Tools with Applications.* SPIN Springer's Internal Project Number, 2003.