Fredholm Determinant

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1 DEFINITION OF FREDHOLM DETERMINANT

Suppose if $f(x) \in C[0,1]$, $K(x,y) \in C([0,1]\times[0,1])$, we hope to find out the solution of the following equation:

$$u(x) + \int_0^1 K(x,y)u(y)dy = f(x), \tag{1.1}$$

where $u(x) \in C[0,1]$.
Actually, we can write (1.1) as

$$(\mathbf{I}+\mathbf{K})u(x) = f(x). \tag{1.2}$$

Fredholm solved it by replacing the integral in (1.1) by a Riemann sum over n intervals of length h. This yields a system of n linear equations for the values u_j of u at the n nodes j/n of the subdivision. Fredholm expressed the solution of these equations as $n \to \infty$.The discretized form of (1.1) is

$$u_i + h\sum K_{ij}u_j = f_i, \quad i = 1,\dots,n, \tag{1.3}$$

where $f_i = f(ih)$, $h = 1/n$ and $K_{ij} = K(ih, jh)$.Denote by $D(h)$ the determinant of the matrix acting on the vector u in (1.3):

$$D(h) = det(I + hK_{ij}) \tag{1.4}$$

We can write $D(h)$ as a polynomial in h:

$$D(h) = \sum_{m=0}^{n} a_m h^m. \tag{1.5}$$

a_m can be written as Taylor coefficients:

$$a_m = \frac{1}{m!}\left(\frac{d}{dh}\right)^m D(h)|_{h=0} \tag{1.6}$$

In general, we have a rule:

$$\frac{d}{dh}det(C_1,...,C_n) = \sum_l det\left(C_1,...,\frac{d}{dh}C_l,...,C_n\right). \tag{1.7}$$

And notice that at $h=0$, $C_j(0) = E_j$, thus using (1.4) in (1.6), we have

$$D(h) = 1 + h\sum_i K_{ij} + \frac{h^2}{2}\sum_{i,j} det\begin{pmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{pmatrix} + \cdots \tag{1.8}$$

Fredholm denoted

$$K\begin{pmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{pmatrix} = detK(x_i, y_j), \quad 1 \leqslant i,j \leqslant k. \tag{1.9}$$

Now we set $h = 1/n$, and let n tend to ∞, then we change (1.8) into

$$D = \sum_0^\infty \frac{1}{k!}\int\cdots\int K\begin{pmatrix} x_1 & \cdots & x_k \\ x_1 & \cdots & x_k \end{pmatrix} dx_1 \cdots dx_k. \tag{1.10}$$

Definition. D is called the Fredholm determinant of the operator $(\mathbf{I}+\mathbf{K})$acting on the left of (1.2).

Theorem 1. The series (1.10) is convergent.
Proof.
Recall the Hadamard's inequality says that

$$|det(C_1,\cdots,C_k)| \leqslant \prod_{j=0}^k ||C_j||,$$

where $||C||$ denotes the Euclidean length of the vector C. Since the kernel K is continuous, it is bounded, let's say,

$$|K(x,y)| \leqslant M,$$

for all x, y, so the length of each column vector of the $k \times k$ matrix (1.9) is less than $M\sqrt{k}$. Thus, according to Hadamard's inequality,

$$|K\begin{pmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{pmatrix}| \leqslant M^k k^{k/2}.$$

So the kth term in series (1.10)is $\leqslant M^k k^{k/2}/k!$, by Stirling's formula,

$$M^k k^{k/2}/k! \leqslant (Me)^k k^{-k/2}.$$

Therefore,

$$D \leqslant \sum_0^\infty (Me)^k k^{-k/2} < +\infty.$$

That is, the Fredholm Determinant is well-defined.

2 THE INVERSE OF $\mathbf{I}+\mathbf{K}$

If we denote

$$R(x,y) = K(x,y) + \int K\begin{pmatrix} x & x_1 \\ y & x_1 \end{pmatrix} dx_1 + \cdots = \sum_0^{\infty} \frac{1}{k!} \int \cdots \int K\begin{pmatrix} x & x_1 & \cdots & x_k \\ y & x_1 & \cdots & x_k \end{pmatrix} dx_1 \cdots dx_k, \tag{2.1}$$

and notice that

$$K\begin{pmatrix} x & x_1 & \cdots & x_k \\ y & x_1 & \cdots & x_k \end{pmatrix} = K(x,y)K\begin{pmatrix} x_1 & \cdots & x_k \\ x_1 & \cdots & x_k \end{pmatrix} - K(x,x_1)K\begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ y & x_2 & \cdots & x_k \end{pmatrix} + \cdots, \tag{2.2}$$

It is obvious that the integrals of the last k terms on the right are all equal. In fact, by interchanging in the jth integral the names of the variables x_1 and x_j, and then performing one row permutation and $j-2$ column permutations, we have

$$\begin{aligned}
&(-1)^j \int \cdots \int K(x,x_j) K\begin{pmatrix} x_1 & x_2 & & x_j & \cdots & & x_k \\ y & x_1 & \cdots & x_{j-1} & x_{j+1} & \cdots & x_k \end{pmatrix} dx_1 \cdots dx_k \\
&= (-1)^j \int \cdots \int K(x,x_1) K\begin{pmatrix} x_j & x_2 & & x_1 & \cdots & & x_k \\ y & x_j & \cdots & x_{j-1} & x_{j+1} & \cdots & x_k \end{pmatrix} dx_1 \cdots dx_k \\
&= (-1)^{j+1} \int \cdots \int K(x,x_1) K\begin{pmatrix} x_1 & x_2 & & x_j & \cdots & & x_k \\ y & x_j & \cdots & x_{j-1} & x_{j+1} & \cdots & x_k \end{pmatrix} dx_1 \cdots dx_k \\
&= (-1)^{j+1}(-1)^{j-2} \int \cdots \int K(x,x_1) K\begin{pmatrix} x_1 & x_2 & & x_j & \cdots & & x_k \\ y & x_2 & \cdots & x_j & x_{j+1} & \cdots & x_k \end{pmatrix} dx_1 \cdots dx_k \\
&= (-1) \int \cdots \int K(x,x_1) K\begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ y & x_2 & \cdots & x_k \end{pmatrix} dx_1 \cdots dx_k
\end{aligned} \tag{2.3}$$

therefore,

$$\begin{aligned}
&\int \cdots \int K\begin{pmatrix} x & x_1 & \cdots & x_k \\ y & x_1 & \cdots & x_k \end{pmatrix} dx_1 \cdots dx_k \\
&= K(x,y) \int \cdots \int K\begin{pmatrix} x_1 & \cdots & x_k \\ x_1 & \cdots & x_k \end{pmatrix} dx_1 \cdots dx_k \\
&- k \int \cdots \int K(x,x_1) K\begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ y & x_2 & \cdots & x_k \end{pmatrix} dx_1 \cdots dx_k
\end{aligned} \tag{2.4}$$

Divide it by $k!$ and sum. According to (2.1) of $R(x,y)$ and (1.10) of D, we have

$$R(x,y) = K(x,y)D - \int K(x,x_1)R(x_1,y)dx_1,$$

or, we can write it as

$$R(x,y) + \int K(x,z)R(z,y)dz - DK(x,y) = 0 \tag{2.5}$$

If we expand the determinants in (2.1) according to the first column instead of the first row, the we have an analogous identity as

$$R(x,y)+\int K(t,y)R(x,t)dt-DK(x,y)=0 \tag{2.6}$$

Now we can compute the inverse of $(\mathbf{I}+\mathbf{K})$ now.

Theorem 2. Let K be a continuous kernel, and suppose $D\neq 0$. Then the operator $(\mathbf{I}+\mathbf{K})$ is invertible, and the inverse is $(\mathbf{I}-D^{-1}\mathbf{R})$.
Proof. According to (2.5) and eqref15,

$$\begin{aligned} \mathbf{R}+\mathbf{KR}-D\mathbf{K}&=0.\\ \mathbf{R}+\mathbf{RK}-D\mathbf{K}&=0. \end{aligned} \tag{2.7}$$

Namely,

$$\begin{aligned} (\mathbf{I}+\mathbf{K})(\mathbf{I}-D^{-1}\mathbf{R})&=\mathbf{I}.\\ (\mathbf{I}-D^{-1}\mathbf{R})(\mathbf{I}+\mathbf{K})&=\mathbf{I}. \end{aligned} \tag{2.8}$$

□

Actually we can prove the inverse is also true. Let λ denote a complex parameter. If we replace K by λK in (1.10), we have

$$D(\lambda)=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\int\cdots\int K\begin{pmatrix} x_1 & \cdots & x_k\\ x_1 & \cdots & x_k\end{pmatrix}dx_1\cdots dx_k. \tag{2.9}$$

Similarly, we can define $R(x,y,\lambda)$ as

$$R(x,y,\lambda)=\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{k!}\int\cdots\int K\begin{pmatrix} x & x_1 & \cdots & x_k\\ y & x_1 & \cdots & x_k\end{pmatrix}dx_1\cdots dx_k. \tag{2.10}$$

It is obvious to see that both $D(\lambda)$ and $R(x,y,\lambda)$ are entire analytic functions of λ.

Lemma 3. Suppose that $D(\lambda)$ has a zero of order m at $\lambda=1$. Then there is a value of x such that $R(x,x,\lambda)$ has a zero of order $<m$ at $\lambda=1$.
Proof. According to (2.10),

$$\int R(x,x,\lambda)dx=\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{k!}\int\cdots\int K\begin{pmatrix} x & x_1 & \cdots & x_k\\ x & x_1 & \cdots & x_k\end{pmatrix}dx\,dx_1\cdots dx_k.$$

Notice that the right side is equal to the derivative of (2.9) with respect to λ and times λ, hence,

$$\int R(x,x,\lambda)dx\equiv\lambda\frac{d}{d\lambda}D(\lambda).$$

Since $R(x,x,\lambda)$ is entire analytic, we can write $R(x,x,\lambda)$ as

$$R(x,x,\lambda) = \sum_{k=0}^{\infty} a_k(x)(\lambda-1)^k.$$

If the order of $D(\lambda)$ at $\lambda = 1$ is m, then the order of the right side at $\lambda = 1$ is $m-1$. So the order of the left at $\lambda = 1$ is also equal to $m-1$, which means that

$$\int a_{m-1}(x)dx \neq 0.$$

Hence there must be some x_0 such that $a_{m-1}(x_0) \neq 0$. Then we proved that for some value of x, the order of $R(x,x,\lambda)$ at $\lambda = 1$ is less than m. □

Now we can prove the inverse of ***Theorem 2.***

Theorem 4. Let K be a continuous kernel such that $D = 0$; then the operator $\mathbf{I}+\mathbf{K}$ has a nontrivial null-space and so is not invertible.

Proof. Denote by l the largest number such that $R(x,y,\lambda)$ has a zero of order l at $\lambda = 1$ for every x and y. According to ***Lemma 3***, $l < m$. So we can write

$$R(x,y,\lambda) = g(x,y)(\lambda-1)^l + O(\lambda-1)^{l+1} \tag{2.11}$$

By our definition of l, $g(x,y) \not\equiv 0$. According to (2.6),

$$R(x,y,\lambda) + \int \lambda K(x,t)R(t,y,\lambda)dt = \lambda K(x,y)D(\lambda), \tag{2.12}$$

where $\lambda \neq 1$. Divide both sides by $(\lambda-1)^l$ and let $\lambda \to 1$, then we have

$$g(x,y) + \int K(x,t)g(t,y)dt = 0. \tag{2.13}$$

Since $g \not\equiv 0$, there exists some y_0 such that $g(x,y_0) \not\equiv 0$.
Denote $u(x) = g(x,y_0)$. Hence

$$u(x) + \int K(x,y)u(y)dy = 0. \tag{2.14}$$

That is, $u(x)$ is inside the nullspace of $\mathbf{I}+\mathbf{K}$. □

To sum up, we have that the Fredholm determinant $D \neq 0$ iff the operator $\mathbf{I}+\mathbf{K}$ is invertible.

Corollary 5. The complex number k is an eigenvalue of the integral operator $\mathbf{K}$ iff $\lambda = -1/k$ is a zero of $D(\lambda)$.

Proof. Since the determinant of $\mathbf{I}+\mathbf{K}$ is $D(1) = 0$, the determinant of $\mathbf{I}+\lambda\mathbf{K}$ is $D(\lambda)$, according to ***Theorem 2*** and ***Theomrem 4***, $D(1) = 0$ iff there exists some $u(x)$ such that $\mathbf{K}u(x) = -u(x)$, we replace K by λK, then $D(\lambda) = 0 \Leftrightarrow \lambda\mathbf{K}u(x) = -u(x) \Leftrightarrow \mathbf{K}u(x) = -1/\lambda u(x)$. □

3 THE MULTIPLICATIVE PROPERTY OF THE FREDHOLM DETERMINANT

Now we can present Fredholm's extension of the multiplicative property of determinants to operators. Here we denote the determinant of $\mathbf{I}+\mathbf{K}$ by D_K, $\mathbf{I}+\mathbf{H}$ by D_H, and the inverse of $\mathbf{I}+\mathbf{K}$ by $\mathbf{I}-D_K^{-1}\mathbf{R}_K$, the kernel of $\mathbf{R}_K$ by $R_K(x,y)$.

Theorem 6. Let $\mathbf{H}$ and $\mathbf{K}$ be integral operators with continuous kernels, and set $(\mathbf{I}+\mathbf{H})(\mathbf{I}+\mathbf{K})=\mathbf{I}+\mathbf{L}$. Then

$$D_L = D_H D_K. \tag{3.1}$$

Proof. We should notice that

$$(\mathbf{I}+\mathbf{H})(\mathbf{I}+\mathbf{K}) = \mathbf{I}+\mathbf{H}+\mathbf{K}+\mathbf{HK} = \mathbf{I}+\mathbf{L},$$

$$\Rightarrow \mathbf{H}+\mathbf{K}+\mathbf{HK} = \mathbf{L}.$$

therefore the kernel of $\mathbf{L}$ is
Firstly, we prove this when $D_K \neq 0, D_H \neq 0$.

$$L(x,y) = K(x,y) + H(x,y) + \int H(x,t)K(t,y)dt \tag{3.2}$$

We define

$$\delta D_K = \delta \frac{d}{d\epsilon} D_{K+\epsilon\delta K}|_{\epsilon=0} \tag{3.3}$$

First we calculate the variation of the determinant (1.9). According to (1.7)

$$\delta K\begin{pmatrix} x_1 & \cdots & x_k \\ x_1 & \cdots & x_k \end{pmatrix} = \sum_{\ell} detK_{\ell}, \tag{3.4}$$

where the ℓth column of K_ℓ is $\delta K(x_i, y_\ell)$. Expand set K_ℓ with respect to the ℓth column:

$$\delta K\begin{pmatrix} x_1 & \cdots & x_k \\ x_1 & \cdots & x_k \end{pmatrix} = \sum_{m,\ell}(-1)^{m+\ell} K\begin{pmatrix} x_1 & \cdots & (x_m) & \cdots & x_k \\ x_1 & \cdots & (x_\ell) & \cdots & x_k \end{pmatrix}\delta K(x_m, x_\ell), \tag{3.5}$$

where the parentheses indicate that the ℓth column and the mth row are to be omitted.Now we integrate (3.5)over the $k-$dimensional unit cube. We separate the sum into two parts as following:

When $\ell = m$, we notice that all k terms with $\ell = m$ are equal, denoting $x_\ell = x_m = x$, relabelling the remaining variables as $x_1, \cdots x_{k-1}$, hence when $\ell = m$, the sum of the integral is

$$k\int\cdots\int K\begin{pmatrix} x_1 & \cdots & x_{k-1} \\ x_1 & \cdots & x_{k-1} \end{pmatrix} dx_1 \cdots dx_{k-1} \int \delta K(x,x)dx. \tag{3.6}$$

When $\ell \neq m$, relabel $x_m = x, x_\ell = y$, and the remaining variables as $x_1, \cdots, x_{k-2}$. Then each term is equal to

$$(-1)^{\ell+m}(-1)^{\ell-1+m-2}\int\cdots\int K\begin{pmatrix} y & x_1 & \cdots & x_{k-2} \\ x & x_1 & \cdots & x_{k-2}\end{pmatrix}\delta K(x,y)dx_1\cdots dx_{k-2}dxdy,$$

hence all the $k(k-1)$ terms are equal, and the sum is

$$-k(k-1)\int\cdots\int K\begin{pmatrix} y & x_1 & \cdots & x_{k-2} \\ x & x_1 & \cdots & x_{k-2}\end{pmatrix}\delta K(x,y)dx_1\cdots dx_{k-2}dxdy. \tag{3.7}$$

According to (1.10) and (2.1), if we get the sum of (3.6) and (3.7) divided by $k!$, the sum of (3.6) is

$$D_K\int\delta K(x,x)dx,$$

and the sum of (3.7) is

$$-\int\int R_K(y,x)\delta K(x,y)dxdy.$$

Plus them together, we get

$$\delta D_K = D_K\int\delta K(x,x)dx - \int\int R_K(y,x)\delta K(x,y)dxdy.$$

Assume that $D_K \neq 0$,we get

$$\frac{\delta D_K}{D_K} = \delta\log D_K = \int\delta K(x,x)dx - D_K^{-1}\int\int R_K(y,x)\delta K(x,y)dxdy. \tag{3.8}$$

Since

$$\delta K(x,y) - D_K^{-1}\int R(y,x)\delta K(x,y)dx = (\mathbf{I} - D_K^{-1}\mathbf{R}_K)\delta K(x,y),$$

hence,

$$\begin{aligned}\delta\log D_K &= \int(\mathbf{I} - D_K^{-1}\mathbf{R}_K)\delta K(x,y)|_{x=y}dx \\ &= \int(\mathbf{I}+\mathbf{K})^{-\mathbf{1}}\delta K(x,y)|_{x=y}dx.\end{aligned} \tag{3.9}$$

Similarly, if we regard the right part of (3.8) as the applications of the transpose of $\mathbf{I} - D_K^{-1}\mathbf{R}_K$ to the function $\delta K(x,\cdot)$, then we have

$$\delta\log D_K = \int(\mathbf{I}+\mathbf{K}')^{-\mathbf{1}}\delta K(x,y)|_{x=y}dx, \tag{3.10}$$

where $\mathbf{K}'$ represents the transpose of $\mathbf{K}$.

According to (3.2),

$$L(x,y) = K(x,y) + H(x,y) + \int H(x,t)K(t,y)dt,$$

hence

$$
\begin{aligned}
&\delta L(x,y)\\
=&\delta K(x,y)+\delta H(x,y)+\int \delta H(x,t)K(t,y)dt+\int H(x,t)\delta K(t,y)dt\\
=&(\delta K(x,y)+\int H(x,t)\delta K(t,y)dt)+(\delta H(x,y)+\int \delta H(x,t)K(t,y)dt)\\
=&(\mathbf{I}+\mathbf{H})\delta K(x,y)+(\mathbf{I}+\mathbf{K}')\delta H(x,y)
\end{aligned}
\tag{3.11}
$$

By (3.8),

$$
\begin{aligned}
&\delta\log D_L\\
=&\int \delta L(x,x)dx - D_L^{-1}\int\int R_L(y,x)\delta L(x,y)dxdy\\
=&\int ((\mathbf{I}+\mathbf{H})\delta K(x,y)|_{x=y}+(\mathbf{I}+\mathbf{K}')\delta H(x,y)|_{x=y})dx - D_L^{-1}\int\int R_L(y,x)((\mathbf{I}+\mathbf{H})\delta K(x,y)+(\mathbf{I}+\mathbf{K}')\delta H(x,y))dxdy\\
=&(\int ((\mathbf{I}+\mathbf{H})\delta K(x,y)|_{x=y}dx - D_L^{-1}\int\int R_L(y,x)((\mathbf{I}+\mathbf{H})\delta K(x,y)dxdy)\\
&+(\int (\mathbf{I}+\mathbf{K}')\delta H(x,y)|_{x=y})dx - D_L^{-1}\int\int R_L(y,x)(\mathbf{I}+\mathbf{K}')\delta H(x,y))dxdy)\\
=&\int (\mathbf{I}-D_L^{-1}\mathbf{R}_L)(\mathbf{I}+\mathbf{H})\delta K(x,y)|_{x=y}dx+\int (\mathbf{I}-D_L^{-1}\mathbf{R'}_L)(\mathbf{I}+\mathbf{K}')\delta H(x,y)|_{x=y}dx\\
=&\int ((\mathbf{I}+\mathbf{L})^{-1})(\mathbf{I}+\mathbf{H})\delta K(x,y)|_{x=y}dx+\int ((\mathbf{I}+\mathbf{L}')^{-1})(\mathbf{I}+\mathbf{K}')\delta H(x,y)|_{x=y}dx
\end{aligned}
\tag{3.12}
$$

Recalled that $(\mathbf{I}+\mathbf{L})=(\mathbf{I}+\mathbf{H})(\mathbf{I}+\mathbf{K})$, hence force,

$$(\mathbf{I}+\mathbf{L})^{-1}=(\mathbf{I}+\mathbf{K})^{-1}(\mathbf{I}+\mathbf{H})^{-1},$$

$$(\mathbf{I}+\mathbf{L}')^{-1}=(\mathbf{I}+\mathbf{H}')^{-1}(\mathbf{I}+\mathbf{K}')^{-1}.$$

Now we can write down $\delta\log D_L$ as

$$\delta\log D_L=\int (\mathbf{I}+\mathbf{K})^{-1}\delta K(x,y)|_{x=y}dx+\int (\mathbf{I}+\mathbf{H}')^{-1}\delta H(x,y)|_{x=y}dx \tag{3.13}$$

That is ,

$$\delta\log D_L=\delta\log D_K+\delta\log D_H. \tag{3.14}$$

We can deform K and H into 0 so that $D_K\neq 0$ and $D_H\neq 0$ during this deformation. For instance, set $K(t)=\lambda(t)K, H(t)=\lambda(t)H$, where the complex-valued function $\lambda(t)$ avoids all the zero points of $D_K(\lambda)$ and $D_H(\lambda)$. By (3.14),

$$\frac{d}{dt}\left[\log D_{L(t)}-\log(D_{K(t)}D_{H(t)})\right]=0. \tag{3.15}$$

Since $L(0)=K(0)=H(0)=0$, and $D_0=I$, we deduce that

$$\log D_L=\log(D_KD_H).$$

Therefore,

$$D_L = D_K D_H. \tag{3.16}$$

When $D_H = 0$, $\mathbf{I}+\mathbf{H}$ is not surjective, and when $D_K = 0$, $\mathbf{I}+\mathbf{K}$ is not injective. In either case, $(\mathbf{I}+\mathbf{H})(\mathbf{I}+\mathbf{K})$ is not invertible, so $D_L = 0$.

□

REFERENCES

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