Friedrichs Extension

Di Wang

April 29, 2015

Recall that in the class we always consider the self-adjoint operators defined on the whole space H. But actually there are many operators(for example unbounded operators) cannot defined on the whole space. So today we mainly concentrate on the symmetry operator B defined on a dense subset $D(B)$.

Definition 1 *If B is a linear map $: D(B) \to H$ and satisfy:*

$$< Bu, v >=< u, Bv >$$

for all u, v in $D(B)$ where $D(B)$ is a dense set of H Then we call B is a symmetric operator.

Notice that when $D(B) = H$, then we know B is bounded by Hellinger-Toeplitz theorem.

Definition 2 *$D(B)$ is a dense subset of H and B is a linear operator defined on $D(B)$, then the adjoint operator B^* is the operator whose domain is $D(B^*)$, consists of all v in H, satisfy $\exists z$ such that $< Bu, v >=< u, z >$ for $\forall u \in D(B)$,we denote $z = B^*v$. Since $D(B)$ is dense so $\forall v$ there can be only one z, and B is called self-adjoint if $D(B^*) = D(B)$ and $B^* = B$.*

Note that it's obviously that $B^* \supseteq B$.

Remark 1 *Note that not all the symmetry operator B is self-adjoint since we may have $D(B^*) \neq D(B)$. And even there are some symmetry operators which doesn't have self-adjoint extension. But we will prove that there's a large number of symmetry operators have self-adjoint extension. And the method due to Friedrichs.*

Remark 2 *The closure $\overline{B}$ of B is defined by setting $\overline{B}u = w$ for all u, w satisfying*

$$(w, v) = (u, Bv)$$

for all v in $D(B)$.
Let $H = L^2(R)$,define $B = i\frac{d}{dx}$ on $D(B) = C_0^1$=all once differentiable functions whose support is a compact subset of R. Then B is symmetric and $\overline{B}$ is self-adjoint.

Remark 3 *Let* $H = L^2(0,\infty)$*, define* $B = i\frac{d}{dx}$ *on* $D(B) = C_0^1$*=all once differentiable functions whose support is a compact subset of* $(0.\infty)$*.*
Then B *is symmetric, but* B *has no self-adjoint extensions.*

Remark 4 *Let* $H = L^2(0,1)$*,define* $B = i\frac{d}{dx}$ *on* $D(B) = C_0^1$*=continuously differential functions on* $[0,1]$ *that vanish at* $x = 0$ *and* $x = 1$*.*
Then B *is symmetric,* $\overline{B}$ *is not self-adjoint operator, but* B *has self-adjoint extensions.*
Let α *be any complex number* $\neq 1$ *but of absolute value 1:*$\mid \alpha \mid= 1$*.Define* A_α *to be the operator* $i\frac{d}{dx}$ *acting on all* C^1 *functions that satisfy the boundary condition*

$$u(1) = \alpha u(0)$$

Then A_α *is an extension of* B *and we can show that it is self-adjoint.*

The next theorem determines when a symmetry operator is a self-adjoint operator.

Theorem 1 *A symmetry operator* T *is self-adjoint if and only if all non-real complex numbers belong to its resolvent set.*

Proof 1 *If* T *is self-adjoint,we first show that the image of* $T - zI$ *is a closed subspace of* H*,then for* $u \in Im(T - zI)$*,which is:*

$$Tv - zv = u$$

so:

$$< Tv, v > -z < v, v >=< u, v >$$

Since T *is symmetric operator,*$< Tv, v >$ *is real,so the imaginary part on the left side is* $-Im(z)\|v\|^2$*,*$\mid Im(z) \mid \|v\|^2 \leq \|u\|\|v\|$ *so:*

$$\|v\| \leq \frac{1}{\mid Imz \mid}\|u\|$$

let $\{u_n\}_{n=1}^{\infty} \to u, u_n \in Im(T - zI)$ *with*

$$Tv_n - zv_n = u_n$$

So we have $\|v_n - v_m\| \leq \frac{1}{|Imz|}\|u_n - u_m\|$*,so* $\{v_n\} \to v$ *we will show that* $v \in D$*. To see this we take the limit of the above relations as* $n \to \infty$*. The right side tends to* u*,and the second term on the left tend to* $-zv$*. Therefore the first term* Tv_n *tends to a limit,call it* r*:*

$$r - zv = u$$

$\forall w \in D$*,we have:*

$$< Tv_n, w >=< v_n, Tw >$$

So:

$$< r, w >=< v, Tw >$$

By the definition of the self-adjointness,this shows that $v \in D$ *and* $Tv = r$*,so* $u \in Im(T - zI)$*,so* $Im(T - zI)$ *is closed.*
If the range of $T - zI$ *were not* H*,there* $\exists k$ *where:*

$$< Tv - zv, k >=< Tv, k > - < v, \bar{z}k >= 0$$

for all $v \in D$*. By the definition of self-adjointness, we have* $k \in D$*, and* $Tk = \bar{z}k$*. But then* $< k, Tk >= z < k, k >$ *is not real, contradict to* T *is symmetric operator.*
Thus $Im(T-zI) = H$*, and it's obviously that it is injective: since if* $Tv-zv = 0$*, so by*

$$\|v\| \leq \frac{1}{\mid Imz \mid}\|u\| = 0$$

contradict.
The converse,if z *is non real, then* z *and* $\bar{z} \in C - \sigma(T)$
We first show that $((T - z)^{-1})^* = (T - \bar{z})^{-1}$
$\forall f, g \in H$

$$< (T - z)^{-1}f, g >=< f, (T - \bar{z})^{-1}g >$$

let $(T - z)^{-1}f = x \in DT, (T - \bar{z})^{-1}g = y \in D(T)$*,then we have*

$$< x, (T - \bar{z})y >=< (T - z)x, y >$$

And this is true since T *is symmetric.*
Next we will prove T *is self-adjoint.*
To prove it is self adjoint,we will show $D(T^*) \subseteq D(T)$ *and* $T^* = T$*.*
let $v \in D(T^*)$*,* $T^*v = w$*,and by the definition we have,*

$$\forall x \in D(T), < Tx, v >=< x, w >\Longrightarrow < (T - z)x, v >=< x, w - \bar{z}v >$$

Since $(T - z)^{-1}$ *is surjective onto* $D(T)$*,so* $\exists f$ *and* $x = (T - z)^{-1}f$
then

$$< f, v >=< (T - z)^{-1}f, w - \bar{z}v >=< f, (T - \bar{z})^{-1}(w - \bar{z}v) >$$

And this is holds $\forall x \in D(T)$ *so its hold for all* $f \in H$*.*
So we have $v = (T - \bar{z})^{-1}(w - \bar{z}v)$
Since $Im(T - \bar{z})^{-1} = D(T)$*,so* $v \in D(T)$ *and*

$$(T - \bar{z})v = w - \bar{z}v \Longrightarrow Tv = w = T^*v$$

Then we will introduce a method of extension by Friedrichs.

Definition 3 *A symmetric operator* L *defined on a dense subspace* D *of a Hilbert space* H *is semibounded if*

$$\exists c, \forall u \in D, c\|u\|^2 \leq < u, Lu >$$

Remark 5 *We can always assume that* $c = 1$ *by replacing* L *by* $L - (c-1)I$.

we can define on D a new inner product:

$$< v, w >_L = < v, Lw >$$

and new norm(we call L-norm):

$$\|u\| = < u, Lu >^{\frac{1}{2}}$$

so we have

$$\|u\| \leq \|u\|_L$$

Under $(D, \|u\|_L)$ we can complete it and we denote its completion by H_L, by the above inequality,we have if we have a Cauchy sequence in L-norm,then it is also a sequence in norm, so it has a limit in H,thus we can define a map from H_L to H.

Lemma 1 *This map is injective.*

Proof 2 *Let* $\{u_n\}_{n=1}^{\infty}, u_n \in D$ *is a Cauchy sequence in L-norm, and* $u_n \to u^L$ *in L-norm and denote* $u_n \to u$ *in norm*
we have $\forall v \in D$

$$< u_n, v >_L = < u_n, Lv >$$

so:

$$< u^L, v >_L = < u, Lv >$$

Thus,u^L *is determined by* u. *Since* D *is a dense subspace, so* u^L *is determined by* u.

Thus we can embedding H_L into H,and we canregard H_L as a subspace of H and $D \subset H_L$. Now we will define the Freidrichs Extension of L, as L^F.
For fix $g \in H$,define $l_g(v) = < v, g >$,l_g is a bounded linear functional and $| l_g(v) | \leq \|v\|\|g\|$ so $| l_g(v) | \leq \|v\|_L\|g\|$,so l_g is bounded linear functional on H_L
By Rieze Representation Theorem,$\exists w \in H_L$

$$l_g(v) = < v, g > = < v, w >_L$$

we define all of the w as D^F, so l_g is determined by w, so g is defined by w, which means that there is a function $L^F : D^F \to H$ and denote $L^F(w) = g$
Thus $< v, w >_L = < v, L^F w >, \forall w \in D^F, v \in H_L$.
We will see L^F is an extension of L.
For $g = Lu, u \in D$ we have

$$l_g(v) = < v, g > = < v, Lu > = < v, u >_L = < v, L^F(u) >, \forall v \in D$$

so $Lu = L^F u$ and we have $D \subset D^F \subset H_L$

Theorem 2 *L^F is a self-adjoint extension of L on D^F.*

Proof 3 *Firstly we will prove L^F is symmetric on D^F.Since:*

$$< v, w >_L =< v, L^F w >=< w, v >_L =< w, L^F v >$$

We know that w is determined by g, so L^F is an invertible operator. And we denote it as $M : H \to D^F$.
Let $L^F w = g \Rightarrow Mg = w$ we know M is also a symmetric operator.
By Hellinger-Toeplitz Theorem,M is a bounded operator,so M is self-adjoint.
Then we have every non-real complex number belong to the resolvant set of M,and

$$z^{-1} - M^{-1} = z^{-1}(M - zI)M^{-1}$$

so z^{-1} belongs to the resolvant set of $M^{-1} = L^F$,so every non-real complex number belongs to the resolvant set of L^F,so by thm1 we have L^F is self-adjoint operator.

Example 1 *Let $H = L^2(0, 1)$ and $T = -\frac{d^2}{dx^2} + q$,where q is some continuous function on $[0, 1]$ and $D(L) = C_0^2(0, 1)$,we have L is a symmetry operator and*

$$\|u\|_L^2 =< u, Lu >= \int (u_x)^2 + qu^2 dx \geq c\|u\|^2$$

where $c = minq$,so we can assume that $c = 1$.
And we also have every function in H_L is continuous on the closed interval $[0, 1]$ and vanish at the endpoints.
Since $\forall u \in C_0^2$,we have every $a, b \in [0, 1]$

$$| u(b) - u(a) |=| \int_a^b u_x dx |\leq \sqrt{b-a}(\int_a^b u_x^2 dx)^{\frac{1}{2}} \leq \sqrt{b-a}\|u\|_L$$

It follows that a Cauchy sequence in the L-norm converges uniformly,and that the limit in H_L is zero at the endpoints and is continuous.