A noncommutative view of zeta regularized determinants and analytic torsion

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Heat kernel

- Heat equation: $\partial_t + \triangle = 0$.
- $k(t, x, y) =$ kernel of $e^{-t\triangle}$. Asymptotic expansion near $t = 0$:

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}}\left(a_0(x, \triangle) + a_1(x, \triangle)t + a_2(x, \Delta)t^2 + \cdots\right)$$

- $a_i(x, \triangle)$, Seeley-De Witt-Gilkey coefficients.

- Theorem: $a_i(x, \triangle)$ are universal polynomials in curvature tensor R and its covariant derivatives:

$$
\begin{aligned}
a_0(x, \triangle) &= 1 \\
a_1(x, \triangle) &= \frac{1}{6} S(x) \qquad \text{scalar curvature} \\
a_2(x, \triangle) &= \frac{1}{360}\left(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2\right) \\
a_3(x, \triangle) &= \cdots\cdots
\end{aligned}
$$

Spectral invariants

Compute $\text{Trace}(e^{-t\triangle})$ in two ways:

$$\text{Spectral Sum} = \text{Geometric Sum}.$$

$$\sum e^{-t\lambda_i} = \int_M k(t,x,x)d\text{vol}_x.$$

Hence

$$a_j = \int_M a_j(x,\triangle)d\text{vol}_x,$$

are manifestly spectral invariants:

$$\begin{aligned} a_0 &= \int_M d\text{vol}_x = \text{Vol}(M), && \Longrightarrow \text{Weyl's law} \\ a_1 &= \frac{1}{6}\int_M S(x)d\text{vol}_x, && \text{total scalar curvature} \end{aligned}$$

Meromorphic extension of spectral zeta functions

$$\zeta_\triangle(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \qquad \mathrm{Re}(s) > \frac{m}{2}$$

Mellin transform + asymptotic expansion:

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1}\, dt \qquad \mathrm{Re}(s) > 0$$

$$\begin{aligned} \zeta_\triangle(s) &= \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Trace}(e^{-t\triangle}) - \mathrm{Dim\,Ker}\,\triangle) t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \{ \int_0^c \cdots + \int_c^\infty \cdots \} \end{aligned}$$

The second term defines an entire function, while the first term has a meromorphic extension to $\mathbb{C}$ with simple poles within the set

$$\frac{m}{2} - j, \qquad j = 0, 1, \cdots$$

Also: 0 is always a regular point.

Simplest example: For $M = S^1$ with round metric, we have

$$\zeta_{\triangle}(s) = 2\zeta(2s) \qquad \text{Riemann zeta function}$$

Scalar curvature

The spectral invariants a_i in the heat asymptotic expansion

$$\mathrm{Trace}(e^{-t\triangle}) \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)$$

are related to residues of spectral zeta function by

$$\mathrm{Res}_{s=\alpha}\zeta_{\triangle}(s) = (4\pi)^{-\frac{m}{2}} \frac{a_{\frac{m}{2}-\alpha}}{\Gamma(\alpha)}, \qquad \alpha = \frac{m}{2} - j > 0$$

Focusing on subleading pole $s = \frac{m}{2} - 1$ and using $a_1 = \frac{1}{6}\int_M S(x) dvol_x$, we obtain a formula for scalar curvature density as follows:

Let $\zeta_f(s) := \mathrm{Tr}\,(f\triangle^{-s}),\ f \in C^\infty(M)$.

$$\mathrm{Res}\,\zeta_f(s)|_{s=\frac{m}{2}-1} = \frac{(4\pi)^{-m/2}}{\Gamma(m/2-1)}\int_M fS(x)dvol_x, \quad m \geq 3$$

$$\zeta_f(s)|_{s=0} = \frac{1}{4\pi}\int_M fS(x)dvol_x - \mathrm{Tr}(fP) \qquad m = 2$$

$$\log\det(\triangle) = -\zeta'(0), \qquad \text{Ray-Singer regularized determinant}$$

Curved noncommutative tori A_θ

$A_\theta = C(\mathbb{T}^2_\theta) =$ universal C^*-algebra generated by unitaries U and V

$$VU = e^{2\pi i\theta} UV.$$

$$A_\theta^\infty = C^\infty(\mathbb{T}^2_\theta) = \Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n \,:\, (a_{m,n})\,\text{Schwartz class} \Big\}.$$

- Differential operators $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$

$$\delta_1(U) = U, \qquad \delta_1(V) = 0$$

$$\delta_2(U) = 0, \qquad \delta_2(V) = V$$

- Integration $\mathfrak{t} : A_\theta \to \mathbb{C}$ on smooth elements:

$$\mathfrak{t}\Big(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n\Big) = a_{0,0}.$$

Complex structure on A_θ

- Fix $\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0$. Dolbeault operators

$$\partial := \delta_1 + \tau\delta_2, \qquad \partial^* := \delta_1 + \bar{\tau}\delta_2.$$

- Flat Dolbeault Laplacian: $\partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$.

Conformal perturbation of the metric

- Fix $h = h^* \in A_\theta^\infty$. Replace the volume form $\mathfrak{t}$ by $\varphi : A_\theta \to \mathbb{C}$,

$$\varphi(a) := \mathfrak{t}(ae^{-h}), \quad a \in A_\theta.$$

- It is a twisted trace:

$$\varphi(ab) = \varphi(b\Delta(a)), \quad \forall a, b \in A_\theta.$$

where

$$\Delta(x) = e^{-h} x e^{h},$$

Perturbed Dolbeault operator

- Hilbert space $\mathcal{H}_\varphi$: completion of A_θ w.r.t.

$$\langle a, b\rangle_\varphi := \varphi(b^* a), \quad a, b \in A_\theta$$

- Let $\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$, and

$$\partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi.$$

Scalar curvature for A_θ

- The scalar curvature of the curved nc torus $(\mathbb{T}^2_\theta, \tau, k)$ is the unique element $R \in A_\theta^\infty$ satisfying

$$\mathrm{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \mathrm{Trace}\,(aP) = \mathfrak{t}\,(aR), \qquad \forall a \in A_\theta^\infty,$$

where P is the projection onto the kernel of $\triangle$.

Symbol of the Green operator $(\triangle + 1)^{-1}$

- The equation

$$(b_0 + b_1 + b_2 + \cdots)((a_2 + 1) + a_1 + a_0) \sim 1,$$

has a solution with each b_j a symbol of order $-2 - j$.

Final formula for the scalar curvature (Connes-Moscovici, Fathizadeh-K, Oct. 2011)

Theorem: The scalar curvature of (A_θ, τ, k), up to an overall factor of $\frac{-\pi}{\tau_2}$, is equal to

$$R_1(\log \Delta)\big(\triangle_0(\log k)\big) +$$

$$R_2(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\delta_1(\log k)^2 + |\tau|^2 \delta_2(\log k)^2 + \tau_1 \big\{\delta_1(\log k), \delta_2(\log k)\big\}\Big) +$$

$$iW(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\tau_2 \big[\delta_1(\log k), \delta_2(\log k)\big]\Big)$$

where

$$R_1(x) = -\frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) = (1+\cosh((s+t)/2))\times$$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s+\sinh t-\sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)},$$

$$W(s,t) = -\frac{(-s-t+t\cosh s+s\cosh t+\sinh s+\sinh t-\sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

The limiting case

In the commutative case, $\theta \to 0$, and for $\tau = i$, the above curvature reduces to a constant multiple of the formula of Gauss (Theorema Egregium):

$$R = \partial\bar{\partial} \log k,$$

where $ds^2 = e^{h/2}(dx^2 + dy^2)$ (isothermal coordinates), and $k = e^h$.

First application: Ray-Singer determinant and conformal anomaly (Connes-Moscovici)

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Analogue of Polyakov's formula: The log-det of the perturbed Laplacian $\triangle$ on $\mathbb{T}^2_\theta$ is given by

$$\log \mathrm{Det}(\triangle) - \log \mathrm{Det}(\triangle_0) = \log \varphi(1) - \frac{\pi}{12\tau_2}\varphi_0(h\triangle_0 h) -$$

$$\frac{\pi}{4\tau_2}\varphi_0\left(K_2(\nabla_1)(\square_{\Re}(h))\right),$$

Second application: the Gauss-Bonnet theorem for A_θ

- Heat trace asymptotic expansion relates geometry to topology, thanks to MacKean-Singer formula:

$$\sum_{p=0}^{m}(-1)^p \operatorname{Tr}\left(e^{-t\Delta_p}\right) = \chi(M) \qquad \forall t > 0$$

- This gives the spectral formulation of the Gauss-Bonnet theorem:

$$\zeta(0) + 1 = \frac{1}{12\pi}\int_\Sigma R\, dv = \frac{1}{6}\chi(\Sigma)$$

Theorem (Connes-Tretkoff; Fathizadeh-K.): Let $\theta \in \mathbb{R}$, $\tau \in \mathbb{C} \setminus \mathbb{R}$, $k \in A_\theta^\infty$ be a positive invertible element. Then

$$\text{Trace}(\triangle^{-s})_{|_{s=0}} + 2 = \mathfrak{t}(R) = 0,$$

where $\triangle$ is the Laplacian and R is the scalar curvature of the spectral triple attached to (A_θ, τ, k).

The determinant line bundle (after Quillen)

- Fred $=$ Fred $(H) = \{T : H \to H;\ \ker T \text{ and } \operatorname{coker} T \text{ are f. d.}\}$
- index: Fred $\to \mathbb{Z}$, $\quad T \mapsto \dim \ker T - \dim \operatorname{coker} T$.

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- index: Fred $\to \mathbb{Z}, \quad T \mapsto \dim \ker T - \dim \operatorname{coker} T.$
- $K(X) = [X,\ \text{Fred}]$
- A virtual, non-existent, bundle: Index $\to$ Fred,

$$(\text{Index})_T = \ker T - \operatorname{coker} T.$$

The determinant line

- Given $T : V \to W$, let $\lambda(T) : \lambda(V) \to \lambda(W)$, and get

$$\det T \in \lambda(V)^* \otimes \lambda(W) \qquad \text{determinant line}$$

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- Goal: globalize this and construct a line bundle $\mathrm{Det} \to \mathrm{Fred}$ s.t.

$$\mathrm{Det}_T \simeq \lambda(kerT)^* \otimes \lambda(cokerT)$$

The determinant line bundle

- Open cover $\text{Fred} = \bigcup U_F$, where $dimF < \infty$, and

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- Open cover $\text{Fred} = \bigcup U_F$, where $dimF < \infty$, and
$$U_F = \{T \in \text{Fred};\ Im(T) + F = H\}$$
- Over U_F define
$$\text{Det}_T = \lambda(T^{-1}F)^* \otimes \lambda(F)$$
- Fact: These glue together nicely to define a line bundle over Fred.

Families of Cauchy-Riemann operators

- $E \to M$ smooth vector bundle over a compact Riemann surface.

$$\Omega^{p,q}(E) = \Omega^{p,q}(M) \otimes_{C^\infty(M)} C^\infty(E).$$

- Holomorphic connection $D : \Omega^{0,0}(E) \to \Omega^{0,1}(E)$.

$$D = \bar{\partial} + A, \qquad A = A(z)d\bar{z} \in \Omega^{0,1}(End(E)).$$

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- Let $\mathcal{A}$= space of connections on E. An affine space over the v.s. $\mathcal{B} = \Omega^{0,1}(\mathrm{End}(E))$.

$$\mathcal{A}/\mathcal{G} \simeq \{\text{holomorphic structures on } E\}$$

Quillen's metric

- D is an elliptic 1st order PDE and defines a Fredholm operator

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- This defines a map $f : \mathcal{A} \to Fred(H_0, H_1)$. Pull back DET along f

$$\mathcal{L} := f^*(DET)$$

- $\mathcal{L}$ is a holomorphic line bundle over $\mathcal{A}$. Quillen defined a metric on $\mathcal{L}$, using regularized determinants.

Regularized determinants (Ray-Singer)

- Laplacians: take the adjoint of $D : \Omega^{0,0}(E) \to \Omega^{0,1}(E)$, and let

$$\Delta = D^* D : C^\infty(E) \to C^\infty(E)$$

  Spectrum of Δ: $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$

- Spectral zeta function:

$$\zeta_\Delta(s) = \sum \frac{1}{\lambda_i^s}, \qquad Re(s) > \frac{m}{2}$$

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- What is $\det(\Delta) = \prod \lambda_i$?

$$\log \det(\Delta) = -\zeta'_\Delta(0)$$

e.g. $\infty! = \sqrt{2\pi}$. (what about ∞^∞?)

Quillen's metric on $\mathcal{L}$

- Pick an o. n. basis for $ker(D)$ and $ker(D^*)$. Get a basis v for $\mathcal{L}_D \simeq \lambda(kerD) \otimes \lambda(kerD^*)$. Let

$$||v||^2 = \exp(-\zeta'_\Delta(0)) = \det\Delta.$$

- Prop: This defines a smooth Hermitian inner product on $\mathcal{L}$.

- A Hermitian metric on a holomorphic line bundle has a unique compatible connection. Its curvature can be computed from

$$\bar{\partial}\partial \log ||s||^2,$$

where s is any local holomorphic frame.

The curvature of $\mathcal{L}$

- A Hermitian metric on $\mathcal{A} = \Omega^{0,1}(EndE)$

$$\langle A, B \rangle = \frac{i}{2\pi} \int_M Tr_E(A^* B)$$

- Its Kaehler form

$$\omega(A, B) = \int_M Tr_E(A \wedge B)$$

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- Theorem (Quillen, JFAA, 1985): The curvature of the determinant line bundle is the symplectic form ω.

Conformal anomaly

- Quillen's result should be thought of as an extension of Polyakov's formula for conformal anomaly of log det:

$$\log\det(\Delta_1) - \log\det(\Delta_0) = -\frac{1}{12\pi}\int_M (2K_0 u + (\nabla_0 u)^2)$$

- Polyakov's formula can be extended to noncommutative tori (as will be explained soon). What about Quillen's theorem?

Thanks You!