The Nature of Space in Noncommutative Geometry

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Abstract

The term noncommutative geometry requires justification. After all noncommutativity, as in noncommutativity of matrix multiplication, is an algebraic property and just how it can be tied up to a new notion of space and geometry is not immediate. Mathematicians, physicists, and philosophers have debated the nature of space for centuries while expanding its meaning and scope. In this talk I shall follow some of these threads, specially the evolving relation between geometry and algebra, with a view towards the modern notion of noncommutative space at the heart of noncommutative geometry.

Selected References

- A. Connes, Noncommutative geometry (1994)
- M. Khalkhali, Basic noncommutative geometry (2009)
- M. K. and M. Marcolli, An invitation to noncommutative geometry (2008)
- M. Artin, et al., Interactions between noncommutative algebra and algebraic geometry (BIRS reports)
- A. Okounkov, Noncommutative geometry of random surfaces (arXiv:0907.2322).

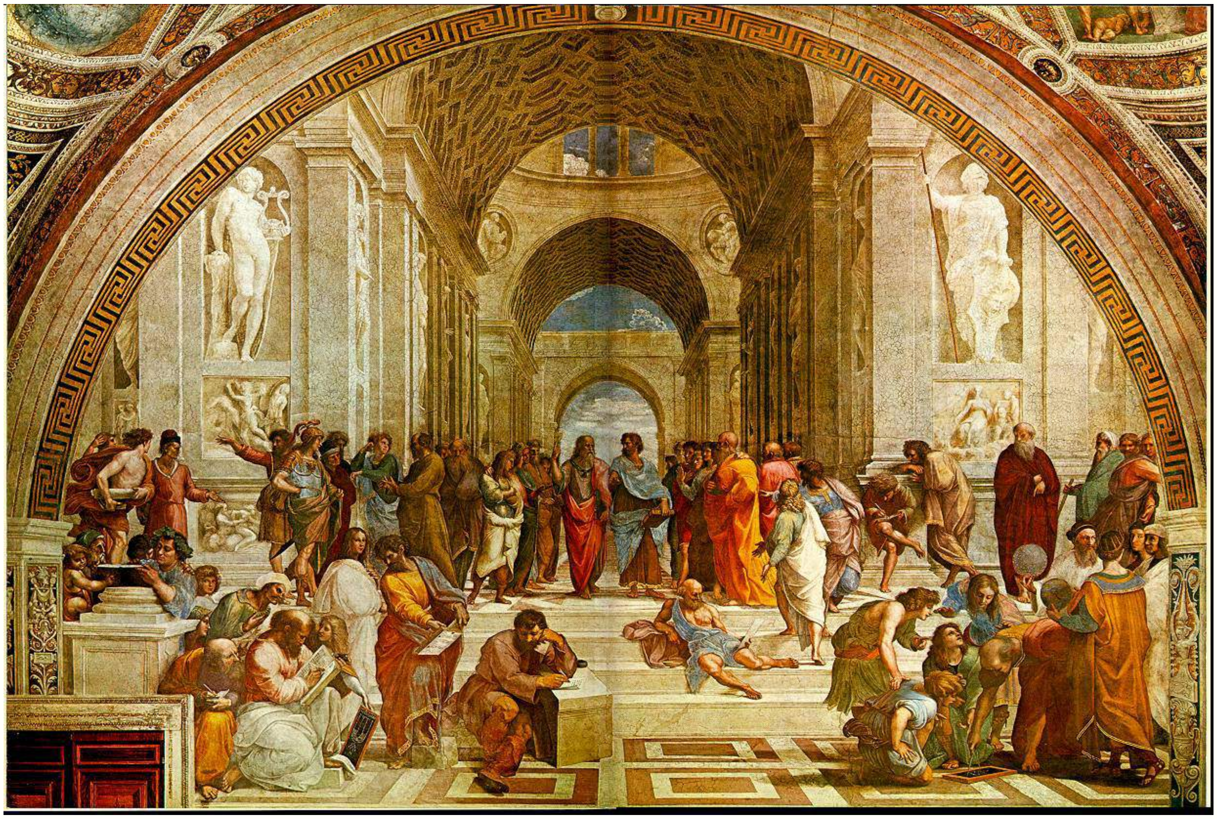
Pythagoras (570 BC-495 BC)

All is Number

This is fine, but let us imagine we are *not* told what *All* means, what *is* means and what *Number* means!

While pythagoreans had a clear view of their meanings, these terms acquired different interpretations throughout history, and my goal is to highlight their evolution during this lecture. This idea is very relevant for the evolution of the notion of space is mathematics.

Raphael's The School of Athens



For Pythagoreans:

Number: positive integers and rationals.

All: everything! including shapes and figures, music, physics and astronomy.

Is: e.g. every length *is* rational, etc.

The correspondence between the two worlds is achieved through measurements, introduction of coordinates.

Remark: Introduction of rationals is a very sophisticated idea; the first example of localization in commutative algebra. Noncommutative localizations, e.g. Ore extensions, was understood only in the 20th century!

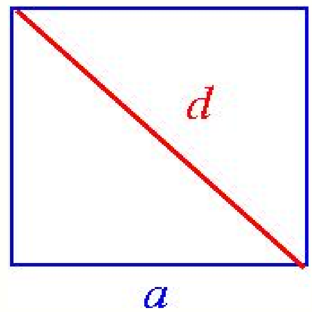
Crisis, Creation, Salvation

The slogan *All Is Number* must be upheld at any cost; it's too valuable to be abandoned! This is usually done by rethinking the meanings of Number, All, and Is. Aspects of the history of maths can be understood as unveiling of this struggle.

Galileo: 'Measure everything that is measurable, and make measurable everything that is not yet so'.

An early example:

Crisis: for $a = 1$, d is *not* a number!



Creation: by expanding the number concept, $\sqrt{2}$ eventually became a number, even a *real* one! Starting with Eudoxus and Euclid, this dream took almost 2400 years to be fully realized (Dedekind 1872).

What is Algebra?

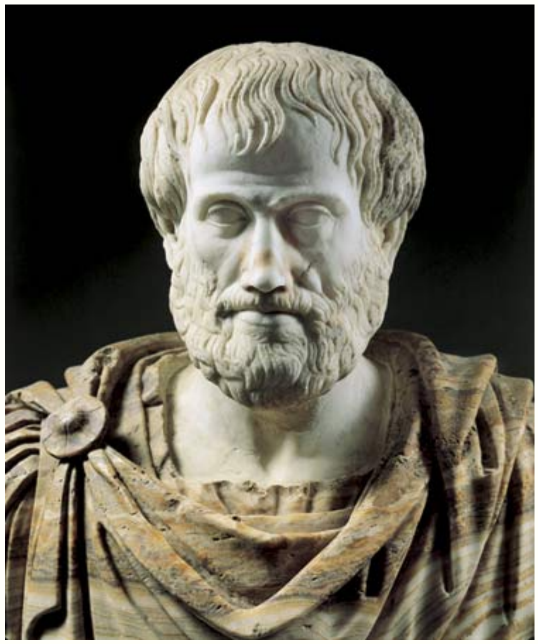
"Thesis. Anything which is the object of mathematical study (curves and surfaces, maps, symmetries, crystals, quantum mechanical quantities and so on) can be 'coordinatised' or 'measured'. However, for such a coordinatisation the 'ordinary' numbers are by no means adequate. Conversely, when we meet a new type of object, we are forced to construct (or to discover) new types of 'quantities' to coordinatise them. The construction and the study of the quantities arising in this way is what characterizes the place of algebra in mathematics (of course, very approximately)"-I. R. Shafarevich, Basic Notions of Algebra.

Example: The Dictionary of Quantum Mechanics.

QM	**Math**
State of system	Line in a Hilbert space
Observable	S.a. operator
Simultaneous measurement	Commuting operators
Observable with value λ	S.a. operator w. eigenvalue λ

Aristotle (384 BC-322 BC)

The importance of Physics



Absolute rest makes sense (Earth). There is absolute space and absolute time. Aristotelian space-time:

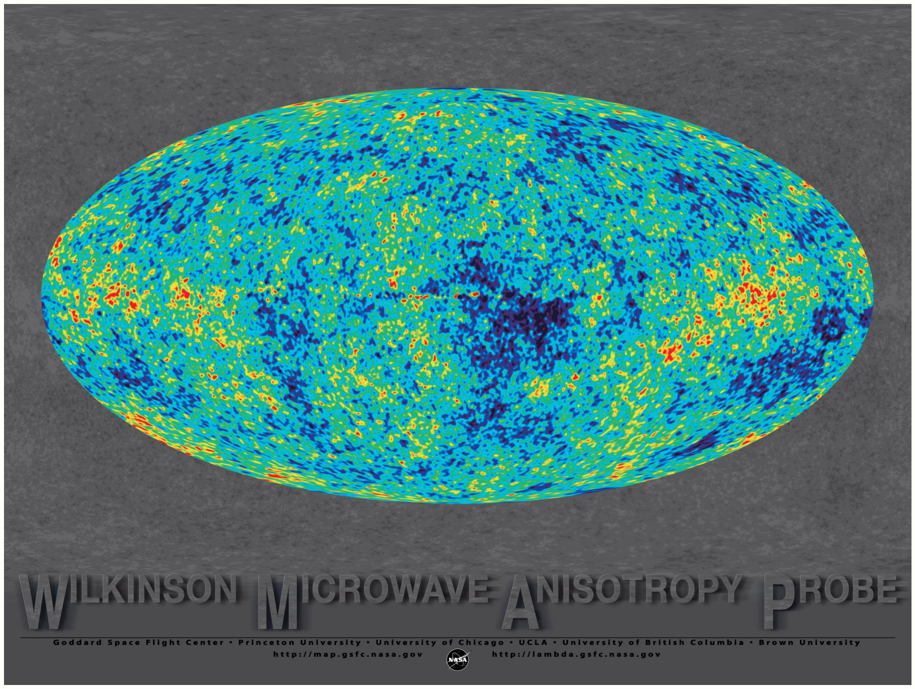
$$\mathcal{M} = E^1 \times E^3$$

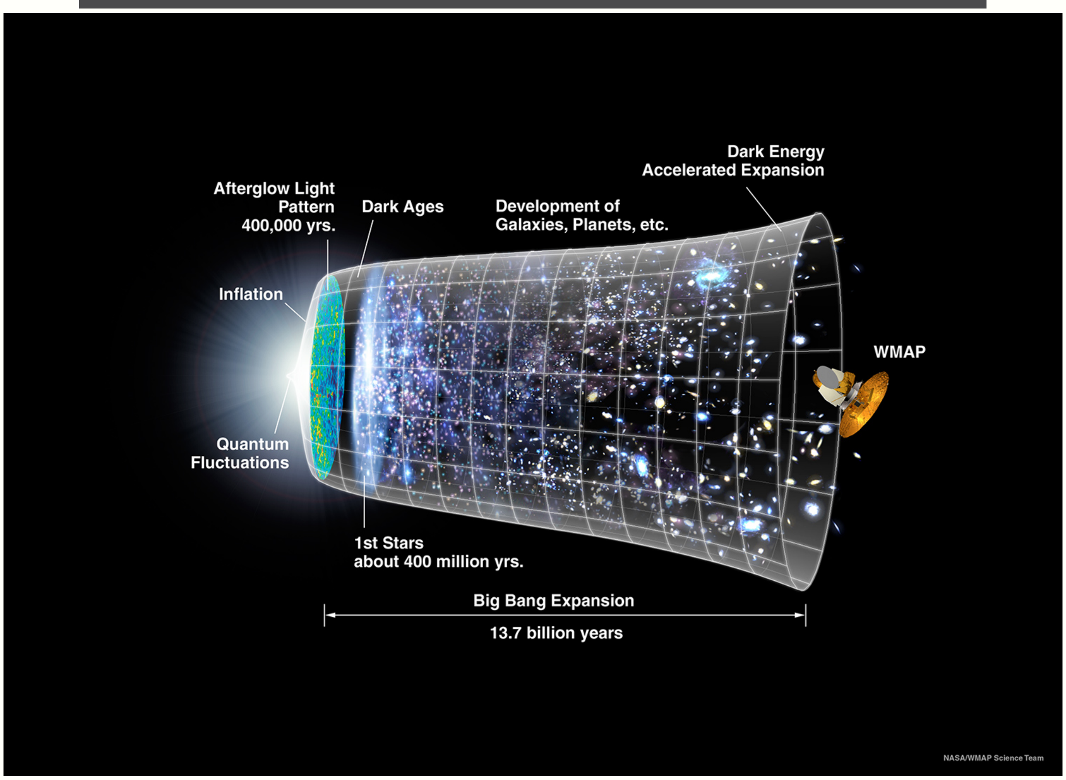
A cartesian product of affine Euclidean spaces.

- Simultaneity of events make sense.
- It makes sense to say two events took place at different times at the same place.
- Though utterly wrong (see the next Section on Galileo), this view of spacetime is the one that is commonly held by the majority of people even today!

Warning: $E^1 \neq \mathbb{R}$. A choice of the origin and length scale gives an isomorphism $E^1 \simeq \mathbb{R}$.

What is the curvature of the universe? The most recent WMAP satellite image (2006) of the CMB (cosmic microwave background radiation) shows that the the universe is flat (with %2 margin of error).


WILKINSON MICROWAVE ANISOTROPY PROBE
Goddard Space Flight Center • Princeton University • University of Chicago • UCLA • University of British Columbia • Brown University
http://map.gsfc.nasa.gov
NASA
http://lambda.gsfc.nasa.gov


Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
400,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years
NASA/WMAP Science Team

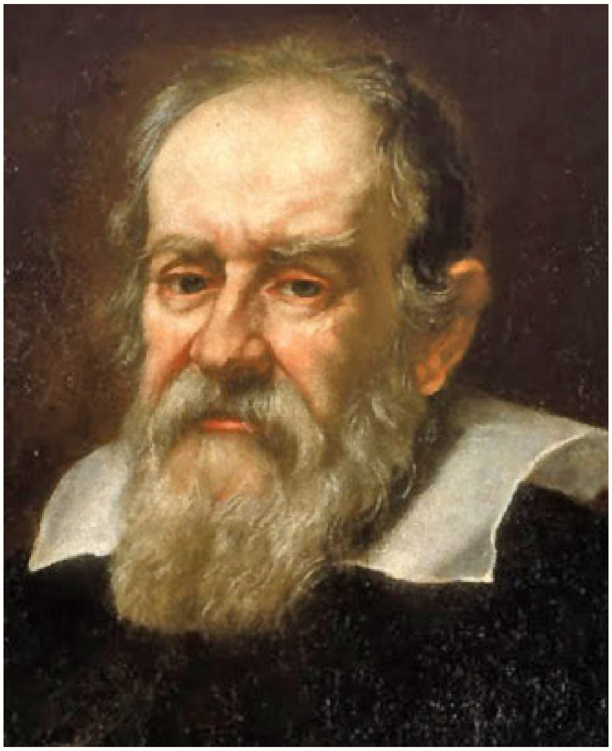
So the spacetime is curved, but the universe is flat!

(Dali, Searching for the 4th dimension, 1979)

Galileo Galilei (1564-1642)

The importance of Physics for the structure of spacetime

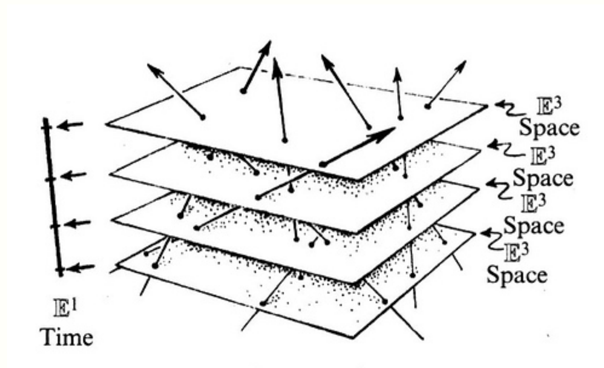
Principle of Relativity and Equivalence Principle



- Absolute rest is an illusion!
- There is absolute time but *no* absolute space!
- Galilean relativity in geometric terms: spacetime

is a fiber bundle, not a cartesian product, with base

E^1 and fiber E^3: $\pi : \mathcal{M} \longrightarrow E^1$.

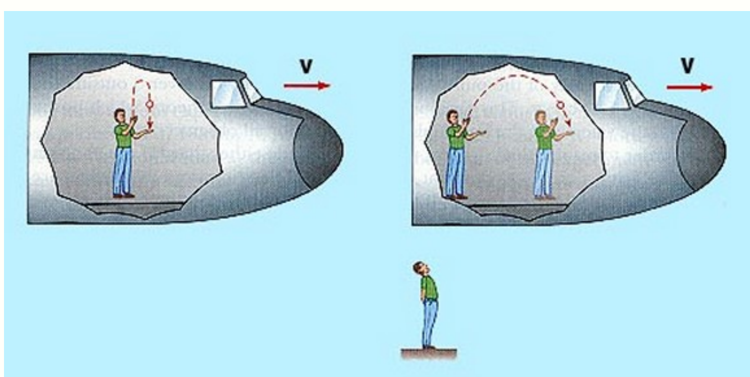


A more down to earth view of Galilean relativity: laws of classical mechanics must be invariant under the Galilean group of transformations of $\mathbb{R}^4$:

$$\therefore \quad \text{(8a)} \quad \left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \text{Galilean Transformation Equations}$$

Two fundamental discoveries of Galileo

• Principle of Relativity: the (local) geometric structure of spacetime is defined by a group; laws of physics must be invariant under the action of this group. (Only experiment can fix the structure of this group). This was later absorbed in Felix Klein's Erlnagen Program.



- The Equivalence Principle

gravitational mass = inertial mass

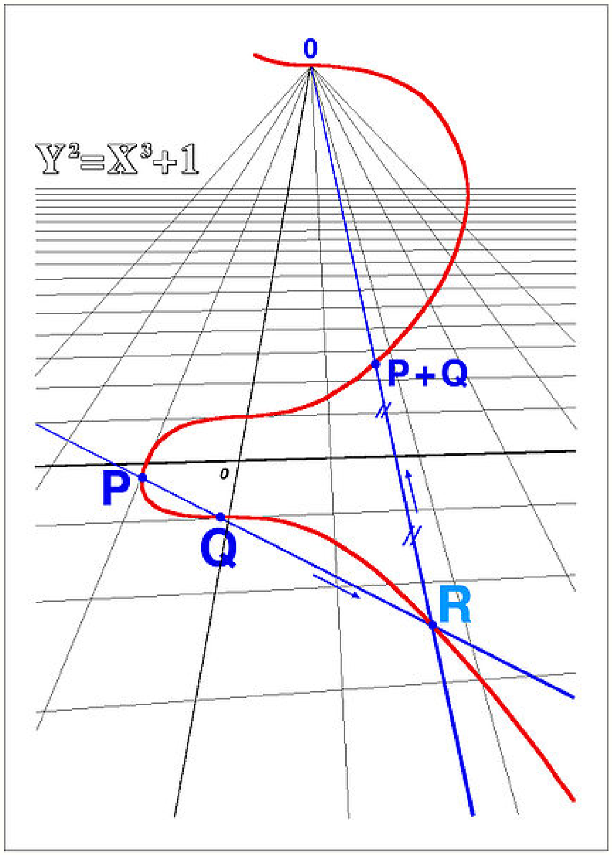
In the hands of Einstein, these two laws eventually led to the special and general theory of relativity, respectively.

René Descartes (1596-1650)

A happy marriage of geometry and algebra!

This is the same old idea of Pythagoras, only stated more precisely.

Main idea: break the symmetry of your geometric (or algebraic, physical,....) object and introduce a coordinate system!



Three poignant remarks by Hermann Weyl:

- *"The introduction of numbers as coordinates...is an act of violence..."*

In fact by introducing a coordinate, or gauge, one breaks the symmetry of the object, and only by a careful act of book keeping via group theory one can maintain the original symmetry and still work with coordinates in a meaningful way.

Example: Let S be a set with n elements. An identification

$$S \simeq \{1, 2, \cdots, n\}$$

amounts to a choice of coordinates in S and breaks the symmetry of S from S_n to the trivial group!

Thus ambiguity is the source of symmetry and we have an equation

$$\text{symmetry} = \text{invariance} = \text{ambiguity} = \text{beauty}$$

Evariste Galois was acutely aware of all of this of course, and that is why he called his theory "theorie des ambiguite".

These ideas eventually led to Kelin's Erlangen program; which is nothing but the relativity theory of mathematics (see F. Klein: Vorlesungen uber die Entwicklung der Mathematik im 19. Jahrhundert).

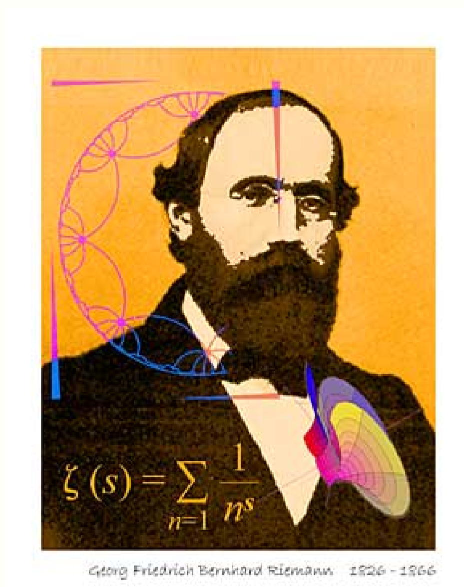
2. *"Galois theory is the theory of relativity of equations"*

3. *"In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics."*

Riemann's blueprint

Über die Hypothesen, welche der Geometrie zu Grunde liegen (1854).

This was directly influenced by Gauss' 1927 memoir on the theory of surface and specially his *Theorema Egregium*.



Riemann's innovations

• Mannigfaltigkeit: intrinsically defined n-dimensional smooth space, without reference to an ambient space. This even inspired Cantor's set theory. He leaves no stones unturned and considers all possibilities for $n = 0, 1, \cdots, \infty$.

• Riemannian metric, and distance

$$\sum g_{\mu\nu} dx^{\mu} dx^{\nu}$$

$$d(p,q) = \inf \int_0^1 \sum g_{\mu\nu} dx^{\mu} dx^{\nu}$$

Compare with a modern noncommutative definition (Connes' distance formula):

$$d(p,q) = \sup\{|f(p) - f(q)|; ||[D, f]|| \leq 1\},$$

where D is the Dirac operator.

Remark: Connes' formula has the advantage that it can be extended to a noncommutative setting (spectral triples).

- Riemann curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y (Z) - \nabla_Y \nabla_X (Z) - \nabla_{[X,Y]}(Z)$$

C is for Classical, Commutative, Cookie,....

Riemann's program, coupled with the progress in algebraic geometry, set theory and Hilbert-Bourbaki formalization of mathematics, inspired a notion of classical space that is still with us: a classical space is a set of points plus an extra structure, say topology, measure, smooth, analytic, algebraic, etc...

But why commutative?

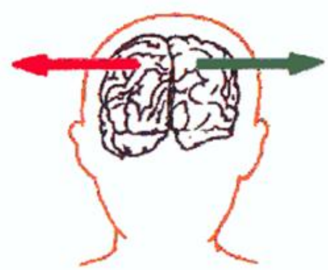
A fundamental duality

(commutative) algebra = (classical) geometry

Each generation of mathematicians find new incarnations of this principle and, *to preserve the principle*, push the boundaries by discovering/introducing new concepts on either side of this equation.

On a physiological level this is perhaps related to a division in human brain: one computes and manipulates symbols with the left hemispheric side of the brain and one visualizes things with the right brain. Computations evolve in time and has a temporal character while visualization is instant and immediate.

"The great pleasure and feeling in my right brain is more than my left brain can find the words to tell you." - Roger Sperry, Nobel Laureate (Medicine 1981; "for his discoveries concerning the functional specialization of the cerebral hemispheres").



Left Brain	***Right Brain***
Verbal	*Non-verbal*
Analytic	*Synthetic, relational*
Parts, detail	*Wholes, big picture*
Logical, rational	*Intuitive, creative*
Sequential, successive	*Random, simultaneous*
Systematic, directed	*Casual, free*
Cautious	*Adventuresome*
Linear	*Holistic*
Factual, words	*Visual, colors*
Abstract, symbolic	*Sensory, concrete*
Digital	*Spatial, analogue*
Rational	*Emotional*
Convergent	*Divergent*
Propositional	*Imaginative*
Objective	*Subjective*
Yang, Masculine	*Yin, Feminine*

“The main theme to emerge... is that there appear to be two modes of thinking, verbal and nonverbal, represented rather separately in left and right hemispheres respectively and that our education system, as well as science in general, tends to neglect the nonverbal form of intellect. What it comes down to is that modern society discriminates against the right hemisphere”.-Roger Sperry (1973).

Gelfand-Naimark Theorem (1943)

A perfect duality between a category of spaces and a category of commutative algebras.

$$\{commutative\ C^* - algebras\} \simeq$$

$$\{compact\ Hausdorff\ spaces\}^{op}$$

Together with the series of papers on 'Rings of Operators I-IV" (later to be named von Neumann algebra by J. Dixmier) by von Neumann and Murray, they form the two pillars of noncommutative geometry. 

The Gelfand-Naimark result suggests: noncommutative C^*-algebras can be regarded as (coordinate rings) of noncommutative topological spaces. Why is this a good idea? Apriori, we have no reason to believe that there is anything deep going on here. Only experience, hard results, and good applications can tell!

Example: The noncommutative torus

Heisenberg's canonical commutation relations

$$pq - qp = \frac{h}{2\pi i} 1$$

No bounded operator realization. Weyl's integrated form of CCR: Let

$$U_t = e^{itp} \quad \text{and} \quad V_s = e^{isq}$$

Then

$$V_s U_t = e^{2\pi i \hbar st} U_t V_s,$$

Let A_θ = universal C^*-algebra generated by unitaries U and V such that:

$$VU = e^{2\pi i\theta} UV$$

Example: The quantum sphere

quantum sphere S_q^2: Generators a, a^* and b. relations

$$aa^* + q^{-4}b^2 = 1,\ a^*a + b^2 = 1,$$

$$ab = q^{-2}ba,\ a^*b = q^2ba^*.$$

S_q^2 is a NC homogeneous space for the quantum group $SU_q(2)$.

• Topological K-theory extends to noncommutative C^*-algebras, even Banach algebras and the Bott periodicity theorem is valid in this generality. Its proof becomes rather simpler even!

• Algebraic topology: fundamental groups of spaces, $G = \pi_1(M)$, are noncommutative in general. The group algebra of G, and its completion $A = C^*(G)$, is in some sense a replacement of the classifying space of G. The Baum-Connes map

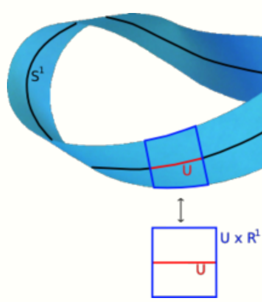
$$\mu : K^*(BG) \to K_*(C^*(G)$$

is an equivariant index map. Surjectivity of this map implies the Novikov conjecture (the homotopy invariance of higher signatures) for the group G. Tools like K-theory, K-homology, and KK-theory play an important role in its proof. Injectivity of μ implies idempotent conjectures for group algebras.

Serre-Swan Theorem

The category of vector bundles on a compact Hausdorff space X is equivalent to the category of finite projective modules over $C(X)$.

$$E \longleftrightarrow \Gamma(E)$$



Serre-Swan Theorem suggest: finite projective modules over A can be regarded as noncommutative vector bundles.

Finite projective A-modules correspond to idempotents

$$e \in M_n(A), \quad e^2 = e.$$

Let $P(A)$ = isomorphism classes of finite projective A-modules. It is a monoid. Let $G(P(A))$= Grotendieck group of $P(A)$. Define

$$K_0(A) = G(P(A))$$

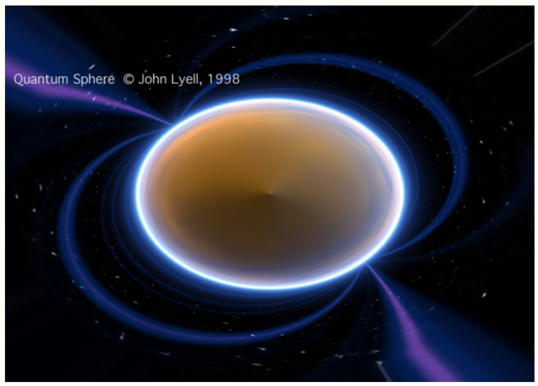
Then

$$K^0(X) = K_0(C(X))$$

A noncommutative example. Hopf line bundle on the quantum sphere S_q^2: Generators a, a^* and b. relations

$$aa^* + q^{-4}b^2 = 1,\ a^*a + b^2 = 1,$$

$$ab = q^{-2}ba,\ a^*b = q^2ba^*.$$



The quantum analogue of the Dirac (or Hopf) monopole line bundle over S^2 is given by the following idempotent in $M_2(S_q^2)$:

$$\mathbf{e}_q = \frac{1}{2}\begin{bmatrix} 1 + q^{-2}b & qa \\ q^{-1}a^* & 1 - b \end{bmatrix}.$$

Example: A finite projective module over the Noncommutative Torus A_θ. Generators U and V. Relation

$$VU = e^{2\pi i\theta}UV$$

Let $E = \mathcal{S}(\mathbb{R})$ = *Schwartz space* of *rapidly decreasing functions* on $\mathbb{R}$. It is an A_θ module via:

$$(Uf)(x) = f(x-\theta), \quad (Vf)(x) = e^{2\pi i x} f(x).$$

E is finite projective.

From bad quotients to noncommutative spaces

There are more things in

heaven and earth, Horatio,

Than are dreamt of in your

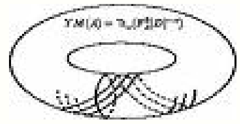
philosophy.

(Hamlet Act 1, scene 5)

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle. A $\mathbb{Z}$-action on $\mathbb{T}$ (rotation by θ):

$$\mathbb{Z} \times \mathbb{T} \to \mathbb{T}, \quad (n, x) \mapsto x + n\theta$$

Note: the orbit space of $\mathbb{T} \longleftrightarrow$ space of leaves of the Kronecker foliation of $\mathbb{T}^2$



For θ =irrational, the action is ergodic, and we have a bad quotient.

Replace $\mathbb{T}/\sim$ by a noncommutative algebra:

$$C(\mathbb{T})^{\mathbb{Z}} \rightsquigarrow C(\mathbb{T}) \rtimes \mathbb{Z}$$

Why is this a reasonable procedure?

Answer: when the action is free and proper the classical quotient and the NC quotient are Morita equivalent:

$$C(X/G) \sim C(X) \rtimes G$$

General idea:

quotient data $\rightsquigarrow$ groupoid $\rightsquigarrow$ groupoid algebra

Example: Identify n points:

Classical quotient: one (fat) point.

NC quotient: $M_n(\mathbb{C})$.

This is the way Heisenberg discovered the matrix mechanics formulation of QM.

Some noncommutative spaces:

- Space of leaves of a foliation $(M, \mathcal{F})$
- The unitary dual of a noncompact (Lie) group

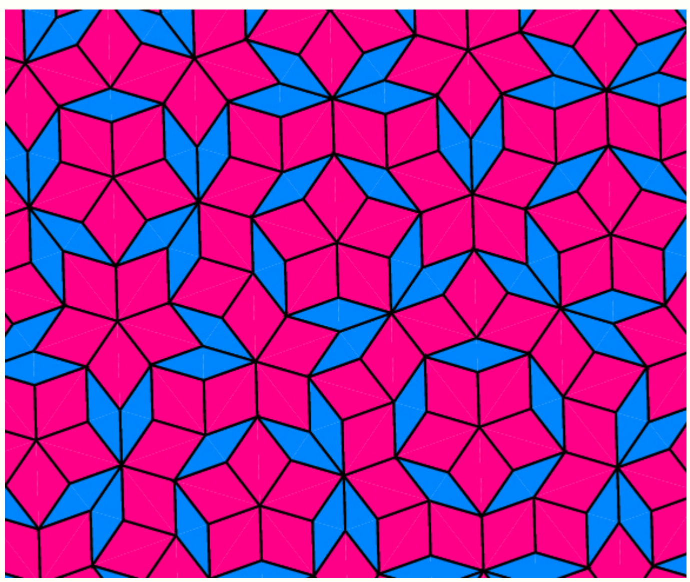
$$G \to C^*(G)$$

It is the NC geometer's version of the 'classifying space' BG of G.

If G is abelian, by Fourier analysis,

$$C^*(G) \simeq C_0(\widehat{G})$$

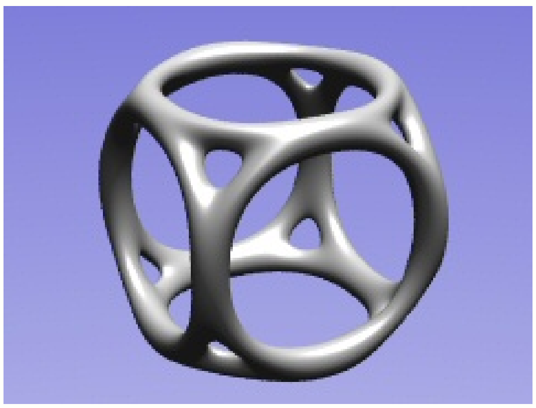
- The space of Penrose tilings.



What is Noncommutative Geometry?

..........In mathematics each object (or subject) can be looked at in two different ways:

$$\mathit{Geometric} \quad \text{or} \quad \mathit{Algebraic}$$

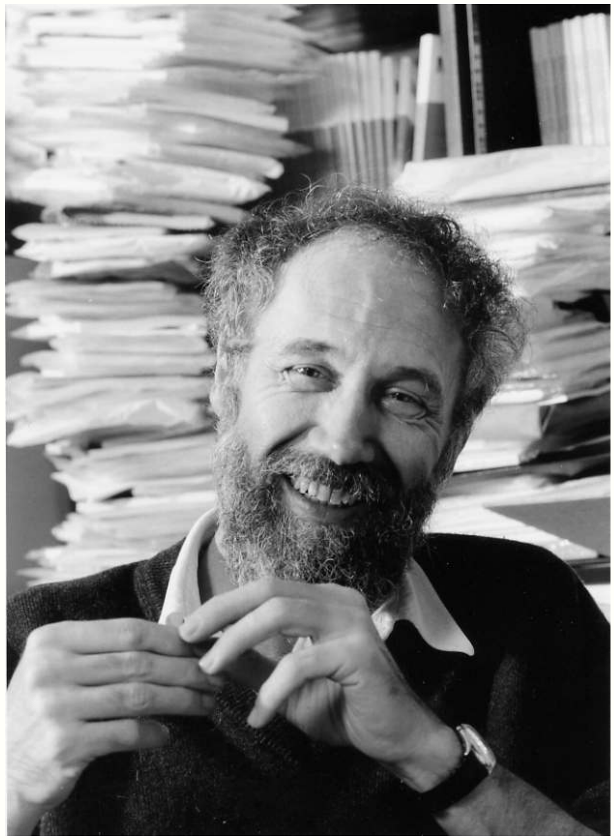


$$= C^\infty(M)$$

Geometry **Algebra**

Classical Geometry $=$ Commutative Algebra

Fundamental duality theorems like Gelfand-Naimark, Serre-Swan, or Hilbert's Nullstellensatz, among others, suggest an equivalence or duality between spaces and commutative algebras.



But Alain Connes, in his Noncommutative Geometry, has taught us that there is a far more fascinating world of noncommutative spaces that have a rich geometry and topology waiting to be explored. Noncommutative Geometry (NCG) builds on this idea of duality between algebra and geometry and vastly extends it by treating special classes of noncommutative algebras as the algebra of coordinates of a noncommutative space.

A paradigm to bear in mind throughout noncommutative geometry is the classical inclusions

$$\text{smooth} \subset \text{continuous} \subset \text{measurable}$$

which reflects in inclusions of algebras

$$C^{\infty}(M) \subset C(M) \subset L^{\infty}(M)$$

This permeates throughout NCG: in the NC world one studies a noncommutative space from a

- measure theoretic point of view (von Neumann algebras)
- continuous topological point of view (C*- algebras)
- differentiable point of view (smooth algebras)
- algebraic geometric point of view (abstract associative algebras, abelian or triangulated categories)

commutative	**noncommutative**
measure space	von Neumann algebra
compact space	C^*-algebra
vector bundle	projective module
complex variable	operator
real variable	s.a. operator
infinitesimal	compact operator
range of a function	spectrum
K-theory	K-theory
vector field	derivation
integral	trace
closed current	cyclic cocycle
de Rham complex	Hochschild homology
de Rham cohomology	cyclic homology
Chern character	NC character
Chern-Weil theory	NC Chern-Weil thoery
elliptic operator	K-cycle
spin^c manifold	spectral triple
index theorem	local index formula
group	quantum group
symmetry	action of QG

From spectral geometry to spectral triples and noncommutative manifolds

Spectral triple (A, H, D)

- $\pi : A \to L(H)$ a representation of A,
- $D : H \to H$, s.a. unbounded operator,

$$[D, \pi(a)] \quad \text{is bounded,}$$

- compact resolvent: $(D + \lambda)^{-1}$ is compact.
- dimension axiom

$$|D|^{-n} \in L^{(1,\infty)}(H)$$

By definition, a positive bounded operator $T \in L^{(1,\infty)}(H)$ if

$$\sum_1^N \lambda_n = O(\mathrm{Log} N)$$

Example: $(C^\infty(S^1), -i\frac{d}{dx}, L^2(S^1))$.

Eigenvalues: $n \in \mathbb{Z}$

$$|D|^{-1} \in L^{(1,\infty)}(H)$$

In general, for any compact spin Riemannian manifold M with Dirac operator D, one gets an spectral triple

$$(C^\infty(M),\, D,\, L^2(S)).$$

But there are many noncommutative examples!

The Dixmier trace

$$Tr_\omega : L^{(1,\infty)}(H) \to \mathbb{C}$$

measures the Log divergence of the satnadard trace (a kind of regularization of the trace).

Noncommutative Volume: the equality

$$⨍ P := \mathrm{Res}_{z=0}\, \mathrm{Trace}(P|D|^{-z})$$

defines a trace on the algebra generated by $\mathcal{A}$, $[D, \mathcal{A}]$, and $|D|^z$, $z \in \mathbb{C}$. In the commutative case, Weyl's Law on the asymptotic distribution of eigenvalues is equivalent to

$$⨍ 1 \sim \int_M \mathrm{dvol}.$$

Before finishing my talk, I would like to recall a famous pronouncement of David Hilbert.

In his only radio address in 1931, a rare event for a mathematician even today, he took to task certain pessimistic ideological undercurrents of his time. At the end of his speech, With characteristic optimism he said:

Wir müssen wissen Wir werden wissen

I guess there is no need to remind ourselves what Gödel proved in that same year! His *Incompleteness Theorem* shattered Hilbert's hopes for fully formalizing mathematics.

Perhaps poets know better, and we should turn to them, for ultimate wisdom.

So I would like to finish my talk by quoting T. S. Eliot (1888 - 1965), that astute observer of human condition. These 4 lines from the 4th quartet capture the essence of our journey, the saga of numbers and figures, their dance and struggle, their cooperation and competition, and the evolution of their meaning in time, from Pythagoras to Connes and beyond.

"We shall never cease from exploration

And the end of all our exploring

Will be to arrive where we started

And know the place for the first time".

(Little Gidding; Four Quartet)