Spectral Invariants in Noncommutative Geometry

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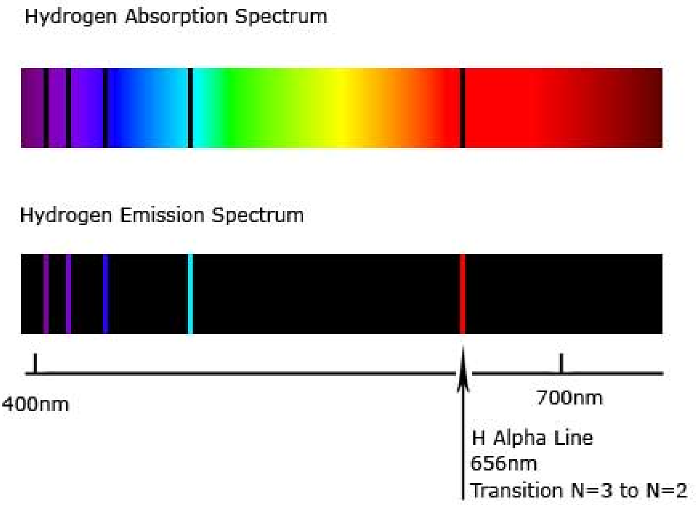
Spectra in Physics and Geometry

- The (discrete) emission or absorption spectra of atoms and molecules as well as the (continuous) spectrum of the termal radiation in solids has palyed an important role in the development of quantum mechanics and later on in Alain Connes' Noncommutative Geometry.


Hydrogen Absorption Spectrum
Hydrogen Emission Spectrum
400nm
700nm
H Alpha Line
656nm
Transition N=3 to N=2

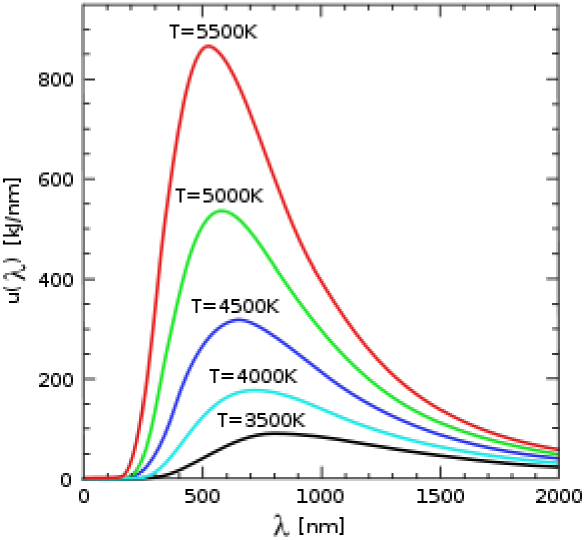


Figure: Black body spectrum

Spectral Geometry and Classical-Quantum Correspondence

- Classical Spectral Geometry teaches us how to express topological and geometric invariants of manifolds in terms of the spectrum of operators like Laplacian or Dirac. A celebrated example is Weyl's Law which relates the asymptotic distribution of eigenvalues of the Laplacian to the volume of a manifold. It has its roots in accoustics and in Planck's Radiation Law.

Planck's Radiation Law

- From 1859 (Kirchhoff) till 1900 (Planck) a great effort went into finding the right formula for spectral energy density function of a radiating black body (T = temperature, ν = frequency, h= Planck's constant, k= Boltzmann's constant, c = speed of light):

$$\rho(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

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- Kirchhoff predicted: ρ will be independent of the shape of the cavity and should only depend on its volume.

- The conjecture of Lorentz (1910; proved by Weyl in 1911): ' *It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between* ν *and* $\nu + d\nu$ *is independent of the shape of the enclosure and is simply proportional to its volume. ....There is no doubt that it holds in general even for multiply connected spaces'.*

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- Weyl:

  One can hear the volume of a cavity.

  But the ultimate question is

  What else can one hear about the shape of a cavity?

Weyl's Law

- (M, g) = compact Riemannian manifold

$$\Delta = d^* d : L^2(M) \to L^2(M), \qquad \text{Laplacian}$$

- Is a s. a. positive operator. In local coordinates:

$$\Delta = -g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B$$

- Spectrum of Δ (counting multiplicities):

$$0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- Eigenvalue counting function:

$$N(\lambda) := \#\{\lambda_i \leq \lambda\}$$

- Weyl's Law:

$$N(\lambda) = \frac{\text{Vol }(M)}{(4\pi)^{m/2}\Gamma(1+m/2)}\lambda^{m/2} + \text{ o}(\lambda^{m/2})$$

One can hear the Volume and Dimension of a Riemannian manifold.

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One can hear the Volume and Dimension of a Riemannian manifold.

- Asymptotic expansion of the trace of the heat kernel:

$$\mathrm{Trace}\,(e^{-t\Delta}) \sim \sum_0^\infty a_n t^{\frac{n-m}{2}} \qquad (t \to 0)$$

$$a_n = \int_M a_n(x, \Delta) d\mathrm{Vol}_x \qquad \text{local invariants}$$

- Seeley-DeWitt coefficients $a_n(x, \Delta)$, $n \geq 0$

$$a_0(x, \Delta) = (4\pi)^{-m/2}$$

$$a_0 = \int_M a_0(x, \Delta) dVol = (4\pi)^{-m/2} \mathrm{Vol}(M)$$

Tauberian theorems $\Rightarrow$ Weyl's law.

Spectral Triples

- $(\mathcal{A}, \mathcal{H}, D)$, $\mathcal{A}=$ involutive unital algebra, acting by bounded operators on a Hilbert space $\mathcal{H}$, D = a s.a. operator on $\mathcal{H}$ with compact resolvent such that all commutators $[D, a]$ are bounded.

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- Assume: an asymptotic expansion of the form

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holds.
- Let $\Delta = D^2$. Spectral zeta function

$$\zeta_D(s) = \operatorname{Tr}(|D|^{-s}) = \operatorname{Tr}(\Delta^{-s/2}), \quad \operatorname{Re}(s) \gg 0.$$

Spectral Triples

- Using the Mellin transform and the asymptotic expansion, easy to show that: ζ_D has a meromorphic extension to all of $\mathbb{C}$ and non-zero terms a_α, $\alpha < 0$, give a pole of ζ_D at -2α with

$$\mathrm{Res}_{s=-2\alpha}\zeta_D(s) = \frac{2a_\alpha}{\Gamma(-\alpha)}.$$

Also, $\zeta_D(s)$ is holomorphic at $s = 0$ and

$$\zeta_D(0) + \dim \ker D = a_0$$

Gauss-Bonnet for Noncommutative Torus

- Fix $\theta \in \mathbb{R}$. $A_\theta = C^*$-algebra generated by unitaries U and V satisfying

$$VU = e^{2\pi i\theta} UV.$$

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- Fix $\theta \in \mathbb{R}$. A_θ = C^*-algebra generated by unitaries U and V satisfying

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- Dense subalgebra of *'smooth functions'*:

$$A_\theta^\infty \subset A_\theta,$$

$a \in A_\theta^\infty$ iff

$$a = \sum a_{mn} U^m V^n$$

where $(a_{mn}) \in \mathcal{S}(\mathbb{Z}^2)$ is rapidly decreasing:

$$\sup_{m,n} (1 + m^2 + n^2)^k \, |a_{mn}| < \infty$$

for all $k \in \mathbb{N}$.

- A_θ has a normalized, faithful, and positive trace (unique if θ is irrational):

$$\tau_0 : A_\theta \to \mathbb{C}$$

$$\tau_0\left(\sum a_{mn} U^m V^n\right) = a_{00}.$$

- Derivations $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$; uniquely defined by:

$$\delta_1(U) = U, \qquad \delta_1(V) = 0$$

$$\delta_2(U) = 0, \qquad \delta_2(V) = V.$$

We have

$$\delta_1 \delta_2 = \delta_2 \delta_1, \qquad \delta_i(a^*) = -\delta_i(a)^*,$$

- Invariance property:

$$\tau_0(\delta_i(a)) = 0.$$

- The Hilbert space

$$\mathcal{H}_0 = L^2(A_\theta, \tau_0),$$

completion of A_θ w.r.t. inner product

$$\langle a, b \rangle = \tau_0(b^* a).$$

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$$\langle a, b \rangle = \tau_0(b^* a).$$

- The derivations

$$\delta_1, \delta_2 : \mathcal{H}_0 \to \mathcal{H}_0$$

are formally selfadjoint unbounded operators (analogues of $\frac{1}{i}\frac{d}{dx}, \frac{1}{i}\frac{d}{dy}$).

Complex structure

- Metrics on A_θ will be defined through their conformal class. Fix

$$\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0,$$

and define

$$\partial = \delta_1 + \tau\delta_2, \qquad \partial^* = \delta_1 + \bar{\tau}\delta_2.$$

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- Define the Hilbert space (analogue of $(1,0)$-forms)

$$\mathcal{H}^{(1,0)} \subset \mathcal{H}_0$$

as the completion of the subspace spanned by finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$. Connes and Tretkoff consider $\tau = i$.

Laplacian

- View

$$\partial = \delta_1 + \tau\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(1,0)}$$

as an unbounded operator with the adjoint given by

$$\partial^* = \delta_1 + \bar{\tau}\delta_2.$$

- Define the Laplacian

$$\triangle := \partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2.$$

Conformal perturbation of the metric

- To investigate the Gauss-Bonnet theorem for general metrics, vary the metric by a Weyl factor e^h, $h = h^* \in A_\theta^\infty$: Define a positive linear functional $\varphi : A_\theta \to \mathbb{C}$ by

$$\varphi(a) = \tau_0(ae^{-h}), \qquad a \in A_\theta.$$

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- It is a twisted trace

$$\varphi(ba) = \varphi(a\sigma_i(b))$$

which is the KMS condition at $\beta = 1$ for the automorphisms $\sigma_t : A_\theta \to A_\theta,\ t \in \mathbb{R}$,

$$\sigma_t(x) = e^{ith} x e^{-ith}.$$

In fact

$$\sigma_t = \Delta^{-it}$$

with the modular operator

$$\Delta(x) = e^{-h} x e^h.$$

The perturbed Laplacian

- Let $\mathcal{H}_\varphi$ = completion of A_θ w.r.t. $\langle , \rangle_\varphi$, where

$$\langle a, b\rangle_\varphi = \varphi(b^* a), \qquad a, b \in A_\theta.$$

Let

$$\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

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- It has a formal adjoint ∂_φ^* given by

$$\partial_\varphi^* = R(e^h)\partial^*$$

  where $R(e^h)$ is the right multiplication operator by e^h $(R(e^h)(x) = e^h x)$.

- Define the new Laplacian:

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Lemma (Connes-Tretkoff; continues to hold for general τ)

$\triangle'$ is anti-unitarily equivalent to the positive unbounded operator $k\Delta k$ acting on $\mathcal{H}_0$, where $k = e^{h/2}$.

Spectral Zeta Function

$$\zeta(s) = \sum \lambda_i^{-s} = \text{Trace}\,(\triangle'^{-s}), \qquad \text{Re}(s) > 1.$$

Mellin transform

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} dt$$

gives us

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Trace}^+(e^{-t\triangle'}) t^{s-1} dt,$$

where

$$\text{Trace}^+(e^{-t\Delta'}) = \text{Trace}\,(e^{-t\Delta'}) - \text{Dim Ker}(\Delta').$$

ζ has a moromorphic extension to $\mathbb{C} \setminus 1$ with a simple pole at $s = 1$.

The Gauss-Bonnet theorem in spectral terms

Theorem (Gauss-Bonnet for classical Riemann surfaces)

Let Σ = compact connected oriented Riemann surface with metric g. Then

$$\zeta(0) + 1 = \frac{1}{12\pi}\int_\Sigma R = \frac{1}{6}\chi(\Sigma),$$

where ζ is the zeta function associated to the Laplacian $\triangle_g = d^*d$, and R is the (scalar) curvature. In particular $\zeta(0)$ is a topological invariant; e.g. is invariant under conformal perturbations of the metric $g \mapsto e^f g$.

Theorem (Gauss-Bonnet for NC torus)

Let $k \in A_\theta^\infty$ be an invertible positive element. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is independent of k.

Pseudodifferential calculus

Recall: Connes (1980; C^*-algebras and Noncommutative Differential Geometry)

Differential operators of order n:

$$P : A_\theta^\infty \to A_\theta^\infty, \quad P = \sum_j a_j \delta_1^{j_1} \delta_2^{j_2}$$

with $a_j \in A_\theta^\infty,\ j = (j_1, j_2),\ \ |j| \leq n.$

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Operator valued symbols of order $n \in \mathbb{Z}$: smooth maps

$$\rho : \mathbb{R}^2 \to A_\theta^\infty$$

s.t.

$$||\delta_1^{i_1}\delta_2^{i_2}\,(\partial_1^{j_1}\partial_2^{j_2}\,\rho(\xi))|| \leq c(1+|\xi|)^{n-|j|},$$

where $\partial_i = \frac{\partial}{\partial \xi_i}$, and ρ is homogeneous of order n at infinity:

$$\lim \lambda^{-n}\rho(\lambda\xi_1, \lambda\xi_2), \qquad \lambda \to \infty$$

exists and is smooth.

Given a symbol ρ, define a pseudodifferential operator

$$P_\rho : A_\theta^\infty \to A_\theta^\infty$$

by

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is.\xi} \rho(\xi) \alpha_s(a) ds d\xi,$$

where

$$\alpha_s(U^n V^m) = e^{is.(n,m)} U^n V^m.$$

For pseudodifferential operators P, Q, with symbols $\sigma(P) = \rho, \sigma(Q) = \rho'$:

$$\sigma(PQ) \sim \sum \frac{1}{\ell_1! \ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2} (\rho(\xi)) \delta_1^{\ell_1} \delta_2^{\ell_2} (\rho'(\xi)).$$

Elliptic Symbols: A symbol $\rho(\xi)$ of order n is called elliptic if $\rho(\xi)$ is invertible for $\xi \neq 0$, and, for $|\xi|$ large enough,

$$||\rho(\xi)^{-1}|| \leq c(1+|\xi|)^{-n}.$$

Example:

$$\triangle = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$$

is an elliptic operator with an elliptic symbol

$$\sigma(\Delta) = \xi_1^2 + 2\tau_1\xi_1\xi_2 + |\tau|^2\xi_2^2.$$

Computing $\zeta(0)$

Recall:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Trace}(e^{-t\triangle'})t^{s-1} - 1)dt,$$

$1 = \mathrm{Dim\ Ker}(\triangle')$.

Cauchy integral formula:

$$e^{-t\Delta'} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(\Delta' - \lambda 1)^{-1} d\lambda$$

gives the asymptotic expansion as $t \to 0^+$:

$$\mathrm{Trace}(e^{-t\triangle'}) \sim t^{-1} \sum_0^\infty B_{2n}(\triangle')t^n.$$

It follows that:

$$\zeta(0) = B_2(\triangle'),$$

$$B_2(\Delta') = \frac{1}{2\pi i}\int_{\mathbb{R}^2}\int_C e^{-\lambda}\tau_0(b_2(\xi,\lambda))d\lambda d\xi$$

where

$$(b_0(\xi,\lambda) + b_1(\xi,\lambda) + b_2(\xi,\lambda) + \cdots)\sigma(\Delta' - \lambda) \sim 1,$$

$$b_j(\xi,\lambda) \text{ is a symbol of order } -2-j.$$

Can assume $\lambda = -1$:

$$\zeta(0) = -\int \tau_0(b_2(\xi,-1))d\xi.$$

$$\sigma(\Delta' + 1) = \sigma(k\Delta k + 1) = (a_2 + 1) + a_1 + a_0$$

where

$$a_2 = k^2\xi_1^2 + 2\tau_1 k^2 \xi_1\xi_2 + |\tau|^2 k^2 \xi_2^2$$

$$a_1 = (2k\delta_1(k) + 2\tau_1 k\delta_2(k))\xi_1 +$$

$$(2\tau_1 k\delta_1(k) + 2|\tau|^2 k\delta_2(k))\xi_2$$

$$a_0 = k\delta_1^2(k) + 2\tau_1 k\delta_1\delta_2(k) + |\tau|^2 k\delta_2^2(k).$$

Using the calculus for symbols:

$$b_0 = (a_2 + 1)^{-1}$$

$$b_1 = -(b_0 a_1 b_0 + \partial_i(b_0)\delta_i(a_2)b_0)$$

$$b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_i(b_0)\delta_i(a_1)b_0$$

$$+\partial_i(b_1)\delta_i(a_2)b_0 + (1/2)\partial_i\partial_j(b_0)\delta_i\delta_j(a_2)b_0).$$

Integrating $b_2(\xi, -1)$ over the plane

Pass to these coordinates:

$$\xi_1 = r\cos\theta - r\frac{\tau_1}{\tau_2}\sin\theta$$

$$\xi_2 = \frac{r}{\tau_2}\sin\theta$$

where θ ranges from 0 to 2π and r ranges from 0 to ∞.
After integrating $\int_0^{2\pi} \bullet \, d\theta$ we have terms such as

$$4\tau_1 r^3 b_0^3 k^2 \delta_2(k)\delta_1(k),$$

$$2r^3 b_0^2 k^2 \delta_1(k) b_0 \delta_1(k),$$

$$-2r^5 b_0^2 k^2 \delta_1(k) b_0^2 k^2 \delta_1(k),$$

where

$$b_0 = (1 + r^2 k^2)^{-1}.$$

Lemma

Let k be an invertible positive element of A_θ^∞. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is given by

$$\zeta(0)+1=\frac{2\pi}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_1(k))+\frac{2\pi|\tau|^2}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_2(k))+$$

$$\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_2(k))+\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_1(k)),$$

where $\varphi(x)=\tau_0(xk^{-2})$, τ_0 is the unique trace on A_θ, Δ is the modular automorphism, and

$$f(u)=\frac{1}{6}u^{-1/2}-\frac{1}{3}+\mathcal{L}_1(u)-2(1+u^{1/2})\mathcal{L}_2(u)+(1+u^{1/2})^2\mathcal{L}_3(u).$$

($\mathcal{L}_m$ is the modified logarithm.)

The following theorem was proved by Alain Connes and Paula Tretkoff for conformal parameter $\tau = i$, and then for all conformal parameters by Farzad Fathizadeh and M.K.

Theorem (Gauss-Bonnet for NC torus)

Let $k \in A_\theta^\infty$ be an invertible positive element. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is independent of k.

Proof.

$$\varphi(f(\Delta)(\delta_j(k))\delta_j(k)) = 0 \;\text{ for } j = 1, 2,$$
$$\varphi(f(\Delta)(\delta_1(k))\delta_2(k)) = -\varphi(f(\Delta)(\delta_2(k))\delta_1(k)).$$

Therefore

$$\zeta(0) + 1 = \frac{2\pi}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_1(k)) + \frac{2\pi|\tau|^2}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_2(k)) +$$
$$\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_2(k)) + \frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_1(k))$$
$$= 0$$

. $\square$