Lectures on Noncommutative Geometry

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Contents

Abstract

This text is an introduction to a few selected areas of Alain Connes' noncommutative geometry written for the volume of the school/conference "Noncommutative Geometry 2005" held at IPM Tehran. It is an expanded version of my lectures which was directed at graduate students and novices in the subject.

1 Introduction

David Hilbert once said "No one will drive us from the paradise which Cantor created for us" [70]. He was of course referring to set theory and the vast new areas of mathematics which were made possible through it. This was particularly relevant for geometry where our ultimate geometric intuition of *space* so far has been a set endowed with some extra structure (e.g. a topology, a measure, a smooth structure, a sheaf, etc.). With Alain Connes' noncommutative geometry [18, 22, 24], however, we are now gradually moving into a new 'paradise', a 'paradise' which contains the 'Hilbertian paradise' as one of its old small neighborhoods. Interestingly enough though, and I hasten to say this, methods of functional analysis, operator algebras, and spectral theory, pioneered by Hilbert and his disciples, play a big role in Connes' noncommutative geometry.

The following text is a greatly expanded version of talks I gave during the conference on noncommutative geometry at IPM, Tehran. The talks were directed at graduate students, mathematicians, and physicists with no background in noncommutative geometry. It inevitably covers only certain selected parts of the subject and many important topics such as: metric and spectral aspects of noncommutative geometry, the local index formula, connections with number theory, and interactions with physics, are left out. For an insightful and comprehensive introduction to the current state of the art I refer to Connes and Marcolli's article 'A Walk in the Noncommutative Garden' [34] in this volume. For a deeper plunge into the subject one can't do better than directly going to Connes' book [24] and original articles.

I would like to thank Professors Alain Connes and Matilde Marcolli for their support and encouragement over a long period of time. Without their kind support and advice this project would have taken much longer to come to its conclusion, if ever. My sincere thanks go also to Arthur Greenspoon who took a keen interest in the text and kindly and carefully edited the entire manuscript. Arthur's superb skills resulted in substantial improvments in the original text.

2 From C^*-algebras to noncommutative spaces

Our working definition of a *noncommutative space* is a noncommutative algebra, possibly endowed with some extra structure. Operator algebras, i.e. algebras of bounded linear operators on a Hilbert space, provided the first really deep insights into this noncommutative realm. It is generally agreed that the classic series of papers of Murray and von Neumann starting with [103], and Gelfand and Naimark [61] are the foundations upon which the theory of operator algebras is built. The first is the birthplace of von Neumann algebras as the noncommutative counterpart of measure theory, while in the second C^*-algebras were shown to be the noncommutative analogues of locally compact spaces. For lack of space we shall say nothing about von Neumann algebras and their place in noncommutative geometry (cf. [24] for the general theory as well as links with noncommutative geometry). We start with the definition of C^*-algebras and results of Gelfand and Naimark. References include [7, 24, 49, 57].

2.1 Gelfand-Naimark theorems

By an *algebra* in these notes we shall mean an *associative algebra* over the field of complex numbers $\mathbb{C}$. Algebras are not assumed to be commutative or unital, unless explicitly specified so. An *involution* on an algebra A is a conjugate linear map $a \mapsto a^*$ satisfying

$$(ab)^* = b^*a^* \quad \text{and} \quad (a^*)^* = a$$

for all a and b in A. By a *normed algebra* we mean an algebra A such that A is a normed vector space and

$$\|ab\| \leq \|a\|\|b\|,$$

for all a, b in A. If A is unital, we assume that $\|1\| = 1$. A *Banach algebra* is a normed algebra which is *complete* as a metric space. One of the main consequences of completeness is that norm convergent series are convergent; in particular if $\|a\| < 1$ then the geometric series $\sum_{n=1}^{\infty} a^n$ is convergent. From this it easily follows that the group of invertible elements of a unital Banach algebra is open in the norm topology.

Definition 2.1. *A C^*-algebra is an involutive Banach algebra A such that for all $a \in A$ the C^*-identity*

$$\|aa^*\| = \|a\|^2 \tag{1}$$

holds.

A *morphism* of C^*-algebras is an algebra homomorphism $f : A \longrightarrow B$ which preserves the $*$ structure, i.e.

$$f(a^*) = f(a)^*, \quad \text{for all } a \in A.$$

The C^*-identity (1) puts C^*-algebras in a unique place among all Banach algebras, comparable to the unique position enjoyed by Hilbert spaces among all Banach spaces. Many facts which are true for C^*-algebras are not necessarily true for an arbitrary Banach algebra. For example, one can show, using the spectral radius formula $\rho(a) = \text{Lim } \|a^n\|^{\frac{1}{n}}$ coupled with the C^*-identity, that the norm of a C^*-algebra is unique. In fact it can be shown that a morphism of C^*-algebras is automatically *contractive* in the sense that for all $a \in A$, $\|f(a)\| \leq \|a\|$. In particular they are always continuous. It follows that if $(A, \| \|_1)$ and $(A, \| \|_2)$ are both C^*-algebras then

$$\|a\|_1 = \|a\|_2,$$

for all $a \in A$. Note also that a morphism of C^*-algebras is an *isomorphism* if and only if it is one to one and onto. Isomorphisms of C^*-algebras are necessarily *isometric*.

Example 2.1. let X be a locally compact Hausdorff space and let $A = C_0(X)$ denote the algebra of continuous complex valued functions on X vanishing at infinity. Equipped with the sup norm and the involution defined by complex conjugation, A is easily seen to be a commutative C^*-algebra. It is unital if and only if X is compact, in which case A will be denoted by $C(X)$. By a fundamental theorem of Gelfand and Naimark, to be recalled below, any commutative C^*-algebra is of the form $C_0(X)$ for a canonically defined locally compact Hausdorff space X.

Example 2.2. The algebra $A = \mathcal{L}(H)$ of all bounded linear operators on a complex Hilbert space H endowed with the operator norm and the usual adjoint operation is a C^*-algebra. The crucial C^*-identity $\|aa^*\| = \|a\|^2$ is easily checked. When H is finite dimensional of dimension n we obtain the matrix algebra $A = M_n(\mathbb{C})$. A direct sum of matrix algebras

$$A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

is a C^*-algebra as well. It can be shown that any finite dimensional C^*-algebra is unital and is a direct sum of matrix algebras [7, 49]. In other words, finite dimensional C^*-algebras are semisimple.

Any norm closed subalgebra of $\mathcal{L}(H)$ which is also closed under the adjoint map is clearly a C^*-algebra. A nice example is the algebra $\mathcal{K}(H)$ of *compact operators* on H. By definition, a bounded operator $T : H \to H$ is called compact if it is the norm limit of a sequence of finite rank operators. By the second fundamental theorem of Gelfand and Naimark, to be discussed below, any C^*-algebra is realized as a subalgebra of $\mathcal{L}(H)$ for some Hilbert space H.

Let A be an algebra. A *character* of A is a nonzero multiplicative linear map

$$\chi : A \to \mathbb{C}.$$

If A is unital then necessarily $\chi(1) = 1$. Let $\widehat{A}$ denote the set of characters of A. It is also known as the *maximal spectrum* of A. If A is a Banach algebra it can be shown that any character of A is automatically continuous and has norm one. We can thus endow $\widehat{A}$ with the weak* topology inherited from A^*, the continuous dual of A. The unit ball of A^* is compact in the weak* topology and one can deduce from this fact that $\widehat{A}$ is a locally compact Hausdorff space. It is compact if and only if A is unital.

When A is a unital Banach algebra there is a one to one correspondence between characters of A and the set of maximal ideals of A: to a character χ we associate its kernel, which is a maximal ideal, and to a maximal ideal I one associates the character $\chi : A \to A/I \simeq \mathbb{C}$. If A is a C^*-algebra a character is necessarily a C^* morphism.

An arbitrary C^*-algebra may well have no characters at all. This happens, for example, for simple C^*-algebras, i.e. C^*-algebras with no nontrivial closed two-sided ideal; a simple example of these are matrix algebras $M_n(\mathbb{C})$. A commutative C^*-algebra, however, has plenty of characters to the extent that characters separate points of A and in fact completely characterize it, as we shall see.

Example 2.3. Let X be a locally compact Hausdorff space. For any $x \in X$ we have the *evaluation character*

$$\chi = \mathrm{ev}_x : C_0(X) \longrightarrow \mathbb{C}, \quad \mathrm{ev}_x(f) = f(x).$$

It is easy to see that all characters of $C_0(X)$ are of this form and that the map

$$X \to \widehat{C_0(X)}, \qquad x \mapsto \mathrm{ev}_x$$

is a homeomorphism.

For any commutative Banach algebra A, the *Gelfand transform*

$$\Gamma : A \to C_0(\widehat{A})$$

is defined by $\Gamma(a) = \hat{a}$, where

$$\hat{a}(\chi) = \chi(a).$$

It is a norm contractive algebra homomorphism, as can be easily seen. In general Γ need not be injective or surjective, though its image separates the points of the spectrum. The kernel of Γ is the *nilradical* of A consisting of nilpotent elements of A.

The paper of Gelfand and Naimark [61] is the birthplace of the theory of C^*-algebras. Together with Murray-von Neumann's series of papers on von Neumann algebras [103], they form the foundation stone of operator algebras. The following two fundamental results on the structure of C^*-algebras are proved in this paper. The first result is the foundation for the belief that noncommutative C^*-algebras can be regarded as noncommutative locally compact Hausdorff spaces.

Theorem 2.1. *(Gelfand-Naimark [61]) a) For any commutative C^*-algebra A with spectrum $\widehat{A}$ the Gelfand transform*

$$A \longrightarrow C_0(\widehat{A}), \quad a \mapsto \hat{a},$$

defines an isomorphism of C^*-algebras.
b) Any C^*-algebra is isomorphic to a C^*-subalgebra of the algebra $\mathcal{L}(H)$ of bounded operators on a Hilbert space H.

Using part a), it is easy to see that the functors

$$X \rightsquigarrow C_0(X), \qquad A \rightsquigarrow \widehat{A}$$

define an equivalence between the category of locally compact Hausdorff spaces and proper continuous maps and the opposite of the category of commutative C^*-algebras and proper C^*-morphisms. Under this correspondence compact Hausdorff spaces correspond to unital commutative C^*-algebras. We can therefore think of the opposite of the category of C^*-algebras as the category of locally compact noncommutative spaces.

Exercise 2.1. *Let X be a compact Hausdorff space and $x_0 \in X$. To test your understanding of the Gelfand-Naimark theorem, give a completely C^*-algebraic definition of the fundamental group $\pi_1(X, x_0)$.*

2.2 GNS, KMS, and the flow of time

We briefly indicate the proof of part b) of Theorem (2.1) in the unital case. It is based on the notion of *state* of a C^*-algebra and the accompanying 'left regular representation', called the GNS (Gelfand-Naimark-Segal) construction. We then look at the KMS (Kubo-Martin-Schwinger) condition characterizing the equilibrium states in quantum statistical mechanics and the time evolution defined by a state. In the context of von Neumann algebras, a fundamental result of Connes states that the time evolution is unique up to inner automorphisms.

The concept of state is the noncommutative analogue of Borel probability measure. A state of a unital C^*-algebra A is a positive normalized linear functional $\varphi : A \mapsto \mathbb{C}$:

$$\varphi(a^*a) \geq 0 \quad \forall a \in A, \quad \text{and} \quad \varphi(1) = 1.$$

The *expectation value* of an element (an 'observable') $a \in A$, when the system is in the state φ, is defined by $\varphi(a)$. This terminology is motivated by states in statistical mechanics where one abandons the idea of describing the state of a system by a point in the phase space. Instead, the only reasonable question to ask is the probability of finding the system within a certain region in the phase space. This probability is of course given by a probability measure μ. Then the expected value of an observable $f : M \to \mathbb{R}$, if the system is in the state μ, is $\int f d\mu$.

Similarly, in quantum statistical mechanics the idea of describing the quantum states of a system by a vector (or ray) in a Hilbert space is abandoned and one instead uses a *density matrix*, i.e. a trace class positive operator p with $\mathrm{Tr}(p) = 1$. The expectation value of an observable a, if the system is in the state p, is given by $\mathrm{Tr}(ap)$.

A state is called *pure* if it is not a non-trivial convex combination of two other states. This corresponds to point masses in the classical case and to vector states in the quantum case.

Example 2.4. By the Riesz representation theorem, there is a one to one correspondence between states on $A = C(X)$ and Borel probability measures on X, given by

$$\varphi(f) = \int_X f d\mu.$$

φ is pure if and only if μ is a Dirac mass at some point $x \in X$.

Example 2.5. For $A = M_n(\mathbb{C})$, there is a one to one correspondence between states φ of A and positive matrices p with $\mathrm{Tr}(p) = 1$ (p is called a *density matrix*). It is given by

$$\varphi(a) = \mathrm{Tr}\,(ap).$$

φ is pure if and only if p is of rank 1.

Example 2.6. For a different example, assume $\pi : A \to \mathcal{L}(H)$ is a *representation* of A on a Hilbert space H. This simply means π is a morphism of C^*-algebras. Given any unit vector $v \in H$, we can define a state on A by

$$\varphi(a) = \langle \pi(a)v,\, v\rangle.$$

Such states are called *vector states.* As a consequence of the GNS construction, to be described next, one knows that any state is a vector state in the corresponding GNS representation.

Let φ be a positive linear functional on A. Then

$$\langle a,\, b\rangle := \varphi(b^*a)$$

is a positive semi-definite bilinear form A and hence satisfies the *Cauchy-Schwarz* inequality. That is, for all a, b in A we have

$$|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b).$$

Let

$$N = \{a \in A; \quad \varphi(a^*a) = 0\}.$$

It is easy to see, using the above Cauchy-Schwarz inequality, that N is a closed left ideal of A and

$$\langle a + N,\; b + N\rangle := \langle a,\, b\rangle,$$

is a positive definite inner product on the quotient space A/N. Let H_φ denote the Hilbert space completion of A/N. The *GNS representation*

$$\pi_\varphi : A \longrightarrow \mathcal{L}(H_\varphi)$$

is, by definition, the unique extension of the left regular representation $A \times A/N \to A/N,\ (a,\ b+N) \mapsto ab+N$. Let $v = 1+N$. Notice that we have

$$\varphi(a) = \langle (\pi_\varphi a)(v),\ v\rangle,$$

for all $a \in A$. This shows that the state φ can be recovered from the GNS representation as a vector state.

Example 2.7. For $A = C(X)$ and $\varphi(f) = \int_X f d\mu$, where μ is a Borel probability measure on X, we have $H_\varphi = L^2(X,\, \mu)$. The GNS representation is the representation of $C(X)$ by multiplication operators on $L^2(X,\, \mu)$. In particular, when μ is the Dirac mass at a point $x \in X$, we have $H_\varphi \simeq \mathbb{C}$ and $\pi_\varphi(f) = f(x)$.

The GNS representation (π_φ, H_φ) may fail to be faithful. It can be shown that it is irreducible if and only if φ is a pure state [7, 49]. To construct a faithful representation, and hence an embedding of A into the algebra of bounded operators on a Hilbert space, one first shows that there are enough pure states on A. The proof of the following result is based on the Hahn-Banach and Krein-Milman theorems.

Lemma 2.1. *For any positive element a of A, there exists a pure state φ on A such that $\varphi(a) = \|a\|$.*

Using the GNS representation associated to φ, we can then construct, for any $a \in A$, an irreducible representation π of A such that $\|\pi(a)\| = |\varphi(a)| = \|a\|$.

We can now prove the second theorem of Gelfand and Naimark.

Theorem 2.2. *Every C^*-algebra is isomorphic to a C^*-subalgebra of the algebra of bounded operators on a Hilbert space.*

Proof. Let $\pi = \sum_{\varphi \in \mathcal{S}(A)} \pi_\varphi$ denote the direct sum of all GNS representations for all states of A. By the above remark π is faithful. ☐

Exercise 2.2. *Identify the GNS representation in the case where $A = M_n(\mathbb{C})$ and φ is the normalized trace on A. For which state is the standard representation on $\mathbb{C}^n$ obtained?*

In the remainder of this section we look at the Kubo-Martin-Schwinger (KMS) equilibrium condition for states and some of its consequences. KMS states replace the Gibbs equilibrium states for interacting systems with infinite degrees of freedom. See [12] for an introduction to quantum statistical mechanics; see also [33] and the forthcoming book of Connes and Marcolli [36] for relations between quantum statistical mechanics, number theory and noncommutative geometry. For relations with Tomita-Takesaki theory and Connes' classification of factors the best reference is Connes' book [24].

A *C^*-dynamical system* is a triple (A, G, σ) consisting of a C^*-algebra A, a locally compact group G and a continuous action

$$\sigma : G \longrightarrow \mathrm{Aut}\,(A)$$

of G on A, where $\mathrm{Aut}\,(A)$ denotes the group of C^*-automorphisms of A. The correct continuity condition for σ is *strong continuity* in the sense that for all $a \in A$ the map $g \mapsto \sigma_g(a)$ from $G \to A$ should be continuous. Of particular interest is the case $G = \mathbb{R}$ representing a quantum mechanical system evolving in time. For example, by Stone's theorem one knows that one- parameter groups of automorphisms of $A = \mathcal{L}(\mathcal{H})$ are of the form

$$\sigma_t(a) = e^{itH} a e^{-itH},$$

where H, the *Hamiltonian* of the system, is a self- adjoint, in general unbounded, operator on $\mathcal{H}$. Assuming the operator $e^{-\beta H}$ is trace class, the corresponding *Gibbs equilibrium state* at inverse temperature $\beta = \frac{1}{kT} > 0$ is the state

$$\varphi(a) = \frac{1}{Z(\beta)}\,\mathrm{Tr}\,(a e^{-\beta H}), \tag{2}$$

where the *partition function* Z is defined by

$$Z(\beta) = \,\mathrm{Tr}\,(e^{-\beta H}).$$

According to Feynman [56], formula (2) for the Gibbs equilibrium state (and its classical analogue) is the apex of statistical mechanics. It should however be added that (2) is not powerful enough to deal with interacting systems with an infinite number of degrees of freedom (cf. the first chapter of Connes' book [24] for an example), and in general should be replaced by the KMS equilibrium condition.

Let $(A,\ \sigma_t)$ be a C^*-dynamical system evolving in time. A state $\varphi : A \to \mathbb{C}$ is called a *KMS state at inverse temperature* $\beta > 0$ if for all $a, b \in A$ there exists a function $F_{a,b}(z)$ which is continuous and bounded on the closed strip

$0 \leq \operatorname{Im} z \leq \beta$ in the complex plane and holomorphic in the interior such that for all $t \in \mathbb{R}$

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \qquad \text{and} \qquad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a).$$

Let $\mathcal{A} \subset A$ denote the set of *analytic vectors* of σ_t consisting of those elements $a \in A$ such that $t \mapsto \sigma_t(a)$ extends to a holomorphic function on $\mathbb{C}$. One shows that $\mathcal{A}$ is a dense $*$-subalgebra of A. Now the KMS condition is equivalent to a *twisted trace property* for φ: for all analytic vectors $a, b \in \mathcal{A}$ we have

$$\varphi(ba) = \varphi(a\sigma_{i\beta}(b)).$$

Notice that the automorphism $\sigma_{i\beta}$ obtained by analytically continuing σ_t to imaginary time (in fact imaginary temperature!) is only densely defined.

Example 2.8. Any Gibbs state is a KMS state as can be easily checked.

Example 2.9. *(Hecke algebras, Bost-Connes and Connes-Marcolli systems [9, 33])* A subgroup Γ_0 of a group Γ is called *almost normal* if every left coset $\gamma\Gamma_0$ is a *finite* union of right cosets. In this case we say (Γ, Γ_0) is a *Hecke pair*. Let $L(\gamma)$ denote the number of distinct right cosets $\Gamma_0\gamma_i$ in the decomposition

$$\gamma\Gamma_0 = \cup_i \Gamma_0\gamma_i,$$

and let $R(\gamma) = L(\gamma^{-1})$.

The rational *Hecke algebra* $\mathcal{A}_{\mathbb{Q}} = \mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ of a Hecke pair (Γ, Γ_0) consists of functions with finite support

$$f : \Gamma_0 \setminus \Gamma \to \mathbb{Q}$$

which are right Γ_0-invariant, i.e. $f(\gamma\gamma_0) = f(\gamma)$ for all $\gamma \in \Gamma$ and $\gamma_0 \in \Gamma_0$. Under the convolution product

$$(f_1 * f_2)(\gamma) := \sum_{\Gamma_0\setminus\Gamma} f_1(\gamma\gamma_1^{-1})f_2(\gamma_1),$$

$\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ is an associative unital algebra. Its complexification

$$\mathcal{A}_{\mathbb{C}} = \mathcal{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$$

is a $*$-algebra with an involution given by

$$f^*(\gamma) := \overline{f(\gamma^{-1})}.$$

Notice that if Γ_0 is normal in Γ then one obtains the group algebra of the quotient group Γ/Γ_0. We refer to [9, 33] for the C^*-completion of $\mathcal{A}_\mathbb{C}$, which is similar to the C^*-completion of group algebras. There is a one-parameter group of automorphisms of this Hecke algebra (and its C^*-completion) defined by

$$(\sigma_t f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)} \right)^{-it} f(\gamma).$$

Let P^+ denote the subgroup of the "$ax+b$" group with $a > 0$. The corresponding C^*-algebra for the Hecke pair (Γ_0, Γ) where $\Gamma = P^+_\mathbb{Q}$ and $\Gamma_0 = P^+_\mathbb{Z}$ is the Bost-Connes C^*-algebra. We refer to [33] for a description of the Connes-Marcolli system. One feature of these systems is that their partition functions are expressible in terms of zeta and L-functions of number fields.

Given a state φ on a C^*-algebra A one may ask if there is a one-parameter group of automorphisms of A for which φ is a KMS state at inverse temperature $\beta = 1$. Thanks to Tomita's theory (cf. [24, 7]) one knows that the answer is positive if A is a von Neumann algebra which we will denote by M now. The corresponding automorphism group σ_t^φ, called the *modular automorphism group*, is uniquely defined subject to the condition $\varphi\sigma_t^\varphi = \varphi$ for all $t \in \mathbb{R}$.

A von Neumann algebra typically carries many states. One of the first achievements of Connes, which set his grand classification program of von Neumann algebras in motion, was his proof that the modular automorphism group is unique up to inner automorphisms. More precisely he showed that for any other state ψ on M there is a continuous map u from $\mathbb{R}$ to the group of unitaries of M such that

$$\sigma_t^\varphi(x) = u_t \sigma_t^\psi(x) u_t^{-1} \quad \text{and} \quad u_{t+s} = u_t \sigma_s^\varphi u_s.$$

It follows that the modular automorphism group is independent, up to inner automorphisms, of the state (or weight) and if $\mathrm{Out}\,(M)$ denotes the quotient of the group of automorphisms of M by inner automorphisms, any von Neumann algebra has a god-given dynamical system

$$\sigma : \mathbb{R} \to \mathrm{Out}\,(M)$$

attached to it. This is a purely non-abelian phenomenon as the modular automorphism group is trivial for abelian von Neumann algebras as well for type II factors. For type III factors it turns out the the modular automorphism group possesses a complete set of invariants for the isomorphism type of the algebra in the injective case. This is the beginning of Connes' grand classification theorems for von Neumann algebras, for which we refer the reader to his book [24] and references therein.

2.3 From groups to noncommutative spaces

Many interesting C^*-algebras are defined as group C^*-algebras or as crossed product C^*-algebras. Group C^*-algebras are completions of group algebras with respect to certain pre C^*-norms. To illustrate some of the general ideas of noncommutative geometry and noncommutative index theory, we shall sketch Connes' proof of 'connectedness' of the group C^*-algebra of free groups.

Example 2.10. *(group C^*-algebras)* To any locally compact topological group G one can associate two C^*-algebras, the *full* and the *reduced* group C^*-algebras of G, denoted by $C^*(G)$ and $C_r^*(G)$, respectively. There is a 1-1 correspondence between unitary representations of G and the representations of $C^*(G)$ and a 1-1 correspondence between unitary representations of G which are equivalent to a sub-representation of its left regular representation and representations of the reduced group C^*-algebra $C_r^*(G)$. Both algebras are completions of the convolution algebra of G (under different norms). There is always a surjective C^*-morphism $C^*(G) \to C_r^*(G)$ which is injective if and only if the group G is *amenable*. It is known that abelian, solvable, as well as compact groups are amenable while, for example, non-abelian free groups are non-amenable. Here we consider only discrete groups.

Let Γ be a discrete group and let $H = \ell^2(\Gamma)$ denote the Hilbert space of square summable functions on Γ. It has an orthonormal basis consisting of delta functions $\{\delta_g\}$, $g \in \Gamma$. The *left regular representation* of Γ is the unitary representation

$$\pi : \Gamma \longrightarrow \mathcal{L}(\ell^2(\Gamma))$$

defined by

$$(\pi g) f(h) = f(g^{-1}h).$$

Let $\mathbb{C}\Gamma$ be the group algebra of Γ. There is a unique linear extension of π to an (injective) $*$-algebra homomorphism

$$\pi : \mathbb{C}\Gamma \longrightarrow \mathcal{L}(H), \qquad \pi(\sum a_g g) = \sum a_g \pi(g).$$

The *reduced group C^*-algebra* of Γ, denoted by $C_r^*\Gamma$, is the norm closure of $\pi(\mathbb{C}\Gamma)$ in $\mathcal{L}(H)$. It is obviously a unital C^*-algebra. The *canonical trace* τ on $C_r^*\Gamma$ is defined by

$$\tau(a) = \langle a\delta_e,\ \delta_e \rangle,$$

for all $a \in C_r^*\Gamma$. Notice that on $\mathbb{C}\Gamma$ we have $\tau(\sum a_g g) = a_e$. It is easily seen that τ is *positive* and *faithful* in the sense that $\tau(a^*a) \geq 0$ for all a with equality holding only for $a = 0$.

The *full* group C^*-algebra of Γ is the norm completion of $\mathbb{C}\Gamma$ under the norm

$$\|f\| = \sup\,\{\|\pi(f)\|; \pi \text{ is a } *\text{-representation of } \mathbb{C}\Gamma\},$$

where by a $*$-representation we mean a $*$-representation on a Hilbert space. Note that $\|f\|$ is finite since for $f = \sum_{g\in\Gamma} a_g g$ (finite sum) and any $*$-representation π we have

$$\|\pi(f)\| \leq \sum \|\pi(a_g g)\| \leq \sum |a_g| \|\pi(g)\| \leq \sum |a_g|.$$

By its very definition it is clear that there is a 1-1 correspondence between unitary representations of Γ and C^* representations of $C^*\Gamma$. Since the identity map $\mathrm{id} : (\mathbb{C}\Gamma, \|\,\|) \to (\mathbb{C}\Gamma, \|\,\|_r)$ is continuous, we obtain a surjective C^*-algebra homomorphism

$$C^*\Gamma \longrightarrow C_r^*\Gamma.$$

It is known that this map is an isomorphism if and only if Γ is an amenable group [7, 49, 57].

Example 2.11. By Fourier transform, or the Gelfand-Naimark theorem, we have an algebra isomorphism

$$C_r^*\mathbb{Z}^n \simeq C(\mathbb{T}^n).$$

Under this isomorphism, the canonical trace τ is identified with the Haar measure on the torus $\mathbb{T}^n$. More generally, for any abelian group Γ let $\widehat{\Gamma} = \mathrm{Hom}\,(\Gamma, \mathbb{T})$ be the group of unitary characters of Γ. It is a compact group which is in fact homeomorphic to the space of characters of the commutative C^*-algebra $C_r^*\Gamma$. Thus the Gelfand transform defines an algebra isomorphism

$$C_r^*\Gamma \simeq C(\widehat{\Gamma}) \tag{3}$$

Again the canonical trace τ on the left hand side is identified with the Haar measure on $C(\widehat{\Gamma})$.

In general one should think of the group C^*-algebra of a group Γ as the "algebra of coordinates" on the noncommutative space representing the unitary dual of Γ. Note that by the above example this is fully justified in the commutative case. In the noncommutative case, the unitary dual is a badly behaved space in general but the noncommutative dual is a perfectly legitimate noncommutative space (see the unitary dual of the infinite dihedral group in [24, 34] and its noncommutative replacement).

Example 2.12. For a finite group Γ the group C^*-algebra coincides with the group algebra of Γ. From basic representation theory we know that the group algebra $\mathbb{C}\Gamma$ decomposes as a sum of matrix algebras

$$C^*\Gamma \simeq \mathbb{C}\Gamma \simeq \oplus M_{n_i}(\mathbb{C}),$$

where the summation is over the set of conjugacy classes of Γ.

Example 2.13. *(A noncommutative 'connected' space)* A *projection* in an $*$-algebra is a selfadjoint idempotent, i.e. an element e satisfying

$$e^2 = e = e^*.$$

It is clear that a compact space X is connected if and only if $C(X)$ has no non-trivial projections. Let us agree to call a noncommutative space represented by a C^*-algebra A 'connected' if A has no non-trivial projections. The *Kadison conjecture* states that the reduced group C^*-algebra of a torsion-free discrete group is connected. This conjecture is still open although it has now been verified for various classes of groups [120]. The validity of the conjecture is known to follow from the surjectivity of the *Baum-Connes assembly map*, which is an equivariant index map

$$\mu : K_*^{\Gamma}(\underline{E\Gamma}) \longrightarrow K_*(C_r^*(\Gamma)).$$

This means that if there are enough 'elliptic operators' on the classifying space for proper actions of Γ then one can prove an integrality theorem for values of $\tau(e)$, which then immediately implies the conjecture (e is a projection and τ is the canonical trace on $C_r^*(\Gamma)$). This principle is best described in the example below, due to Connes, where the validity of the conjecture for free groups F_n is established [22]. Notice that the conjecture is obviously true for finitely generated torsion-free abelian groups $\mathbb{Z}^n$ since by Pontryagin duality $C^*(\mathbb{Z}^n) \simeq C(\mathbb{T}^n)$ and $\mathbb{T}^n$ is clearly connected.

Let $\tau : C_r^*(F_2) \to \mathbb{C}$ be the canonical trace. Since τ is positive and faithful, if we can show that, for a projection e, $\tau(e)$ is an integer then we can deduce that $e = 0$ or $e = 1$. The proof of the *integrality* of $\tau(e)$ is remarkably similar to integrality theorems for characteristic numbers in topology proved through index theory. In fact we will show that there is a Fredholm operator F_e^+ with the property that

$$\tau(e) = \text{index}\,(F_e^+),$$

which clearly implies the integrality of $\tau(e)$. For $p \in [1, , \infty)$, let $\mathcal{L}^p(H) \subset \mathcal{L}(H)$ denote the *Schatten ideal* of p-summable compact operators on H.

The proper context for noncommutative index theory is the following [22]:

Definition 2.2. *A p-summable Fredholm module over an algebra $\mathcal{A}$ is a pair (H, F) where*
1. $H = H^+ \oplus H^-$ is a $\mathbb{Z}/2$-graded Hilbert space with grading operator ε,
2. H is a left even A-module,
3. $F \in \mathcal{L}(H)$ is an odd operator with $F^2 = I$ and for all $a \in \mathcal{A}$ one has

$$[F, a] = Fa - aF \in \mathcal{L}^p(H). \tag{4}$$

We say that (H, F) is a Fredholm module over $\mathcal{A}$ if instead of (4) *we have:*

$$[F, a] \in \mathcal{K}(H) \tag{5}$$

for all $a \in \mathcal{A}$.

The p-summability condition (4) singles out ‘smooth subalgebras’ of $\mathcal{A}$: the higher the summability order p is the smoother $a \in \mathcal{A}$ is. This principle is easily corroborated in the commutative case. The smoother a function is the more rapidly decreasing its Fourier coefficients are. That is why if $\mathcal{A}$ is a C^*-algebra, the natural condition to consider is the ‘compact resolvent’ condition (5) instead of (4). In general any Fredholm module over a C^*-algebra A defines a series of subalgebras of ‘smooth functions’ in A which are closed under holomorphic functional calculus and have the same K-theory as A (cf. Section 3.1 for an explanation of these terms).

For $A = C^\infty(M)$, M a closed n-dimensional smooth manifold, p-summable Fredholm modules for $p > n$ are defined using elliptic operators D acting between sections of vector bundles E^+ and E^- on M. Then one lets $F = \frac{D}{|D|}$ be the phase of D (assuming D is injective), and H^+ and H^- the Hilbert spaces of square integrable sections of E^+ and E^-, respectively. The algebra of continuous (respec. smooth) functions on M acts by multiplication on sections and the resulting pair (H, F) is a Fredholm module (resp. p-summable Fredholm module for $p > n$).

Now let (H, F) be a 1-summable Fredholm module over an algebra $\mathcal{A}$. Its *character* is the linear functional $\mathrm{Ch}\,(H, F) : \mathcal{A} \to \mathbb{C}$ defined by

$$\mathrm{Ch}\,(H, F)(a) = \frac{1}{2}\mathrm{Tr}\,(\varepsilon F[F, a]),$$

which is finite for all $a \in \mathcal{A}$ thanks to the 1-summability condition on (H, F). As a good exercise the reader should check that $\mathrm{Ch}\,(H, F)$ is in fact a trace on $\mathcal{A}$.

The second ingredient that we need is an index formula. Let $e \in \mathcal{A}$ be an idempotent. Let $F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$, $H_1 = eH^+$, $H_2 = eH^{-1}$ and $P' : H_1 \to H_2$ and let $Q' : H_2 \to H_1$ be the restrictions of P and Q to H_1 and H_2, respectively. The trace condition $[F, e] \in \mathcal{L}^1(H)$ is equivalent to $P'Q' - 1_{H_2} \in \mathcal{L}^1(H_2)$ and $Q'P' - 1_{H_1} \in \mathcal{L}^1(H_1)$, which of course imply that the operator

$$F_e^+ := P' \tag{6}$$

is Fredholm and its Fredholm index is given by

$$\begin{aligned} \mathrm{Index}\, F_e^+ &= \mathrm{Tr}(I - Q'P') - \mathrm{Tr}(I - P'Q') = \frac{1}{2}\mathrm{Tr}(\varepsilon F[F, e]) \\ &= \mathrm{Ch}\,(H, F)(e). \end{aligned}$$

We see that we are done provided we can construct a 1-summable Fredholm module over a dense and closed under holomorphic functional calculus subalgebra $\mathcal{A}$ of $A = C_r^*(F_2)$ such that for any projection $e \in \mathcal{A}$,

$$\tau(e) = \mathrm{Ch}(H, F)(e).$$

It is known that a group is free if and only if it acts freely on a tree. Let then T be a tree with a free action of F_2 and let T^0 and T^1 denote the set of vertices and edges of T respectively. Let $H^+ = \ell^2(T^0)$, $H^- = \ell^2(T^1) \oplus \mathbb{C}$ with orthonormal basis denoted by ε_q. The action of F_2 on T induce an action of $C_r^*(F_2)$ on H^+ and H^- respectively. Fixing a vertex $p \in T^0$ we can define a one to one correspondence $\varphi : T^0 - p \to T^1$ by sending $q \in T^0 - p$ to the edge containing q and lying between p and q. This defines a unitary operator $P : H^+ \to H^-$ by

$$P(\varepsilon_q) = \varepsilon_{\varphi(q)} \quad \text{and} \quad P(\varepsilon_p) = (0, 1).$$

The compact resolvent condition (36) is a consequence of *almost equivariance* of φ in the sense that for any $q \in T^0 - p$, $\varphi(gq) = g\varphi(q)$ for all but a finite number of g, which is not difficult to prove.

Exercise 2.3. *Prove the last statement. To see what is going on start with $\Gamma = \mathbb{Z}$ and go through all the steps in the proof.*

Let $\mathcal{A} \subset A$ denote the subalgebra of all $a \in A$ such that

$$[F,\, a] \in \mathcal{L}^1(H).$$

Clearly $(H,\, F)$ is a 1-summable Fredholm module over $\mathcal{A}$. It can be shown that $\mathcal{A}$ is stable under holomorphic functional calculus in A (see Section 3.1). It still remains to be checked that for all $a \in \mathcal{A}$ we have $\mathrm{Ch}\,(H, F)(a) = \tau(a)$, which we leave as an exercise.

Example 2.14. *(crossed product algebras)* Let $\alpha : G \to \mathrm{Aut}\,(A)$ be an action of a group G by algebra automorphisms on an algebra A. The action of g on a will be denoted by $g(a)$. The (algebraic) *crossed product algebra* $A \rtimes G$ is the algebra generated by the two subalgebras A and $\mathbb{C}G$ subject to relations

$$gag^{-1} = g(a),$$

for all g in G and $a \in A$. Formally we define $A \rtimes G = A \otimes \mathbb{C}G$ with product given by

$$(a \otimes g)(b \otimes h) = ag(b) \otimes gh.$$

One checks that this is an associative product and that the two definitions are in fact the same.

In many cases $A = C(X)$ or $A = C^\infty(X)$ is an algebra of functions on a space X and G acts on X by homeomorphisms or diffeomorphisms. Then there is an induced action of G on A defined by $(gf)(x) = f(g^{-1}(x))$. One of the key ideas of noncommutative geometry is Connes' dictum that in such situations the crossed product algebra $C(X) \rtimes G$ replaces the algebra of functions on the quotient space X/G (see Section 4.1 for more on this).

Example 2.15. Let $G = \mathbb{Z}_n$ be the cyclic group of order n acting by translations on $X = \{1, 2, \cdots, n\}$. The crossed product algebra $C(X) \rtimes G$ is the algebra generated by elements U and V subject to the relations

$$U^n = 1, \quad V^n = 1, \quad UVU^{-1} = \lambda V,$$

where $\lambda = e^{\frac{2\pi i}{n}}$. Here U is a generator of $\mathbb{Z}_n$, V is a generator of $\hat{\mathbb{Z}}_n$ and we have used the isomorphism $C(X) \simeq \mathbb{C}\hat{\mathbb{Z}}_n$.

We have an isomorphism

$$C(X) \rtimes \mathbb{Z}_n \simeq M_n(\mathbb{C}). \tag{7}$$

To see this consider the $n \times n$ matrices

$$u = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & \bar{\lambda} & 0 & & 0 \\ 0 & 0 & \bar{\lambda}^2 & & 0 \\ & & & & \\ 0 & \cdots & \cdots & 0 & \bar{\lambda}^{n-1} \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & \cdots & 0 \\ & & 0 & 1 & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix}$$

They clearly satisfy the relations

$$u^n = 1, \quad v^n = 1, \quad uvu^{-1} = \lambda v.$$

Moreover one checks that the matrices $u^p v^q$ for $1 \leq p, q \leq n$ are linearly independent, which shows that the algebra generated by u and v in $M_n(\mathbb{C})$ is all of $M_n(\mathbb{C})$. The isomorphism (7) is now defined by

$$U \mapsto u, \quad V \mapsto v.$$

This isomorphism can be easily generalized:

Exercise 2.4. *Let $\mathbb{Z}_n$ act on an algebra A. Show that the dual group $\hat{\mathbb{Z}}_n$ acts on $A \rtimes \mathbb{Z}_n$ and that we have an isomorphism*

$$A \rtimes \mathbb{Z}_n \rtimes \hat{\mathbb{Z}}_n \simeq M_n(A).$$

Extend this to actions of finite abelian groups.

The above isomorphism is a discrete analogue of Takai duality for C^*-crossed products of actions of locally compact abelian groups (cf. [7])

$$A \rtimes G \rtimes \hat{G} \simeq A \otimes \mathcal{K}(L^2(G)),$$

where $\mathcal{K}$ is the algebra of compact operators.

The above purely algebraic theory can be extended to a C^*-algebraic context. For any locally compact topological group G acting by C^*-automorphisms on a C^*-algebra A, one defines the reduced $A \rtimes_r G$ and the full $A \rtimes G$ crossed product C^*-algebras. Assuming $G = \Gamma$ is a discrete group, to define the crossed product let $\pi : A \to \mathcal{L}(H)$ be a faithful representation of A (the definition turns out to be independent of the choice of π). Consider the Hilbert space $\ell^2(\Gamma, H)$ of square summable functions $\xi : \Gamma \to H$ and define a representation $\rho : A \rtimes \Gamma \to \mathcal{L}(H)$ by

$$(\rho(x)\xi)(g) = \sum_h \pi(g^{-1}(x(h)))\xi(h^{-1}g)$$

for all $x \in A \rtimes \Gamma$, $\xi \in H$, and $g \in \Gamma$. It is an injective $*$-algebra homomorphism and the reduced C^*-crossed product algebra $A \rtimes_r \Gamma$ is defined to be the norm completion of the image of ρ.

Example 2.16. The crossed product C^*-algebra

$$A_\theta = C(S^1) \rtimes \mathbb{Z}$$

is known as the *noncommutative torus*. Here $\mathbb{Z}$ acts on S^1 by rotation through the angle $2\pi\theta$.

2.4 Continuous fields of C^*-algebras

Let X be a locally compact Hausdorff topological space and for each $x \in X$ let a C^*-algebra A_x be given. Let also $\Gamma \subset \coprod A_x$ be a $*$-subalgebra. We say this data defines a *continuous field* of C^*-algebras over X if
1) for all $x \in X$, the set $\{s(x);\, s \in \Gamma\}$ is dense in A_x,
2) for all $s \in \Gamma$, $x \to \|s(x)\|$ is continuous on X,
3) Γ is locally uniformly closed, i.e. for any section $s \in \coprod A_x$, if for any $x \in X$ we can approximate s around x arbitrarily closely by elements of Γ, then $s \in \Gamma$.

To any continuous field of C^*-algebras over a locally compact space X we associate the C^* algebra of its *continuous sections* vanishing at infinity (a section $s : X \to \coprod A_x$ is called continuous if $x \to \|s(x)\|$ is continuous).

Examples 2.1. i) A rather trivial example is the C^*-algebra

$$A = \Gamma(X, \operatorname{End}(E))$$

of continuous sections of the endomorphism bundle of a vector bundle E over a compact space X. A nice example of this is the noncommutative torus A_θ for $\theta = \frac{p}{q}$ a rational number. As we shall see

$$A_{\frac{p}{q}} \simeq \Gamma(\mathbb{T}^2, \operatorname{End}(E))$$

is the algebra of continuous sections of the endomorphism bundle of a vector bundle of rank q over the torus $\mathbb{T}^2$. Notice that in general $C(X) \subset Z(A)$ is a subalgebra of the center of A. As we shall see $Z(A_{\frac{p}{q}}) \simeq C(\mathbb{T}^2)$.
ii) A less trivial example is the field over the interval $[0, 1]$, where $A_x = M_2(\mathbb{C})$ for $0 < x < 1$ and $A_0 = A_1 = \mathbb{C} \oplus \mathbb{C}$.
iii) The noncommutative tori A_θ, $0 \leq \theta \leq 1$, can be put together to form a continuous field of C^*-algebras over the circle (see [7]).

Exercise 2.5. *An Azumaya algebra is the algebra of continuous sections of a (locally trivial) bundle of finite dimensional full matrix algebras over a space X. Give an example of an Azumaya algebra which is not of the type $\Gamma(End(E))$ for some vector bundle E.*

Of particular interest are C^*-algebras of continuous sections of a locally trivial bundle of algebras with fibers the algebra $\mathcal{K}(H)$ of compact operators and structure group $\operatorname{Aut}\mathcal{K}(H)$. When H is infinite dimensional such bundles can be completely classified in terms of their *Dixmier-Douady invariant* $\delta(E) \in H^3(X, \mathbb{Z})$ [51]. (cf. [108] for a modern detailed account). The main reason for this is that any automorphism of $\mathcal{K}(H)$ is inner. In fact for any Hilbert space H there is an exact sequence of topological groups

$$1 \longrightarrow U(1) \longrightarrow U(H) \xrightarrow{\text{Ad}} \operatorname{Aut}\mathcal{K}(H) \longrightarrow 1, \tag{8}$$

where the unitary group $U(H)$ has its strong operator topology and the automorphism group $\operatorname{Aut}\mathcal{K}(H)$ is taken with its norm topology. It is rather easy to see that if H is infinite dimensional then $U(H)$ is contractible. (A much harder theorem of Kuiper states that it is also contractible under the norm topology, but this is not needed here.)

Let X be a locally compact Hausdorff and paracompact space. Locally trivial bundles over X with fibers $\mathcal{K}(H)$ and structure group $\operatorname{Aut}(\mathcal{K}(H))$ are classified by their classifying class in the Čech cohomology group $H^1(X, \operatorname{Aut}(\mathcal{K}(H))$. Now, since the middle term in (8) is contractible, it stands to reason to expect that

$$H^1(X, \operatorname{Aut}\mathcal{K}(H)) \simeq H^2(X, U(1)) \simeq H^3(X, \mathbb{Z}).$$

The first isomorphism is actually not so obvious because (8) is an exact sequence of nonabelian groups, but it is true (cf. [108] for a proof). The resulting class

$$\delta(E) \in H^3(X, \mathbb{Z})$$

is a complete isomorphism invariant of such bundles.

Continuous fields of C^*-algebras are often constructed by crossed product algebras. Given a crossed product algebra $A \rtimes_\alpha \mathbb{R}$, by rescaling the action α_t to α_{st}, we obtain a one-parameter family of crossed product algebras $A \rtimes_{\alpha^s} \mathbb{R}$ over $[0, \infty)$ whose fiber at 0 is $A \otimes C_0(\mathbb{R})$. For example, the translation action of $\mathbb{R}$ on itself will give a field of C^*-algebras whose fiber at 0 is $C_0(\mathbb{R}) \otimes C_0(\mathbb{R}) \simeq C_0(\mathbb{R}^2)$ and at $t > 0$ is $C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R} \simeq \mathcal{K}(L^2(\mathbb{R}))$.

Example 2.17. Let H be the discrete *Heisenberg group* of upper triangular matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

with integer entries. It is a solvable, and hence amenable, group which means that the reduced and full group C^*-algebras of H coincide. The group C^*-algebra $C^*(H)$ can also be defined as the C^*-algebra generated by three unitaries U, V, W subject to relations

$$UV = WVU, \quad UW = WU, \quad VW = WV. \tag{9}$$

They correspond to elements

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that W is central and in fact it generates the center $Z(A) \simeq C(S^1)$. For each $e^{2\pi i\theta} \in S^1$, we have a surjective C^*-morphism

$$f_\theta : C^*(H) \longrightarrow A_\theta$$

by sending $U \mapsto U$, $V \mapsto V$ and $W \mapsto e^{2\pi i\theta}1$. This is clear from (9). Thus, roughly speaking, the Heisenberg group C^*-algebra $C^*(H)$ can be viewed as a 'noncommutative bundle' over the circle with noncommutative tori A_θ as fibers. Notice, however, that this bundle is not a locally trivial bundle of algebras as different fibers are not isomorphic to each other. One can show in fact $C^*(H)$ is the C^*-algebra of continuous sections of a field of C^*-algebras over the circle whose fiber at $e^{2\pi i\theta}$ is the noncommutative torus A_θ (cf. [7, 57] and references therein).

2.5 Noncommutative tori

These algebras can be defined in a variety of ways, e.g. as the C^*-algebra of the Kronecker foliation of the two-torus by lines of constant slope $dy = \theta dx$, as the crossed product algebra $C(S^1) \rtimes \mathbb{Z}$ associated to the automorphism of the circle through rotation by an angle $2\pi\theta$, as strict deformation quantization, or by generators and relations as we do here.

Let $\theta \in \mathbb{R}$ and $\lambda = e^{2\pi i\theta}$. The *noncommutative torus* A_θ is the *universal* unital C^*-algebra generated by unitaries U and V subject to the relation

$$UV = \lambda VU \tag{10}$$

The universality property here means that given any unital C^*-algebra B with two unitaries u and v satisfying $uv = \lambda vu$, there exists a unique unital C^* morphism $A_\theta \to B$ sending $U \to u$ and $V \to v$.

Unlike the purely algebraic case where any set of generators and relations automatically defines a universal algebra, this is not the case for universal C^*-algebras. Care must be applied in defining a norm satisfying the C^*-identity, and in general the universal problem does not have a solution (cf. [7] for more on this). For the noncommutative torus we proceed as follows. Consider the unitary operators $U, V : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$(Uf)(n) = e^{2\pi i\theta n} f(n), \qquad (Vf)(n) = f(n-1).$$

They satisfy $UV = \lambda VU$. Let A_θ be the unital C^*-subalgebra of $\mathcal{L}(\ell^2(\mathbb{Z}))$ generated by U and V. To check the universality condition, it suffices to check that the operators $U^m V^n$, $m, n \in \mathbb{Z}$, are linearly independent, which we leave as an exercise.

Using the Fourier isomorphism $\ell^2(\mathbb{Z}) \simeq L^2(S^1)$, we see that A_θ can also be described as the C^*-subalgebra of $\mathcal{L}(L^2(S^1))$ generated by the operators $(Uf)(z) = f(\lambda z)$ and $(Vf)(z) = zf(z)$ for all $f \in L^2(S^1)$. This presentation also makes it clear that we have an isomorphism

$$A_\theta \simeq C(S^1) \rtimes \mathbb{Z}$$

where $\mathbb{Z}$ acts by rotation though the angle $2\pi\theta$.

Since the Haar measure on S^1 is invariant under rotations, the formula

$$\tau(\sum f_i V^i) = \int_{S^1} f_0$$

defines a faithful positive normalized trace

$$\tau : A_\theta \to \mathbb{C}.$$

This means that τ is a trace and

$$\tau(aa^*) > 0, \qquad \tau(1) = 1,$$

for all $a \neq 0$. On the dense subalgebra of A_θ generated by U and V we have

$$\tau(\sum a_{mn} U^m V^n) = a_{00}. \tag{11}$$

Exercise 2.6. *Show that if θ is irrational then there is a unique trace, given by* (11)*, on the subalgebra of A_θ generated by U and V. For rational values of θ show that there are uncountably many traces.*

The map $U \mapsto V$ and $V \mapsto U$ defines an isomorphism $A_\theta \simeq A_{1-\theta}$. This shows that we can restrict to $\theta \in [0, \frac{1}{2})$. It is known that in this range A_{θ_1} is isomorphic to A_{θ_2} if and only if $\theta_1 = \theta_2$. Notice that A_θ is commutative if and only if θ is an integer, and simple Fourier theory shows that A_0 is isomorphic to the algebra $C(\mathbb{T}^2)$ of continuous functions on the 2-torus. For irrational θ, A_θ is known to be a simple C^*-algebra, i.e. it has no proper closed two-sided ideal. In particular it has no finite dimensional representations [7, 49, 57].

For $\theta = \frac{p}{q}$ a rational number, A_θ has a finite dimensional representation. To see this consider the unitary $q \times q$ matrices

$$u = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & \cdots & 0 & \lambda^{q-1} \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}$$

They satisfy the relations

$$uv = \lambda vu, \qquad u^q = v^q = 1. \tag{12}$$

It follows that there is a unique C^* map from $A_\theta \to M_q(\mathbb{C})$ sending $U \to u$ and $V \to v$. This, of course, implies that A_θ is not simple for θ a rational number. In fact it is known that in this case A_θ is Morita equivalent to the commutative algebra $C(\mathbb{T}^2)$, as we shall see shortly.

Exercise 2.7. *Show that the C^*-algebra generated by relations* (12) *is isomorphic to $M_q(\mathbb{C})$. (hint: it suffices to show that the matrices $u^i v^j$, $1 \le i, j \le q$, are linearly independent.)*

Assuming $\theta = \frac{p}{q}$ is rational, we have $U^q V = V U^q$ and $V^q U = U V^q$, which show that U^q and V^q are in the center of A_θ. In fact it can be shown that they generate the center and we have

$$Z(A_\theta) \simeq C(\mathbb{T}^2).$$

One shows that there is a (flat) vector bundle E of rank q over $\mathbb{T}^2$ such that $A_{\frac{p}{q}}$ is isomorphic to the algebra of continuous sections of the endomorphism bundle of E,

$$A_{\frac{p}{q}} \simeq \Gamma(\mathbb{T}^2, \mathrm{End}\,(E)).$$

Exercise 2.8. *Show that if θ is irrational then $Z(A_\theta) = \mathbb{C}1$.*

There is a dense $*$-subalgebra $\mathcal{A}_\theta \subset A_\theta$ that deserves to be called the algebra of *'smooth functions'* on the noncommutative torus (see Section 3.1 for more on the meaning of the word smooth in noncommutative geometry). By definition $\mathcal{A}_\theta$ consists of elements of the form

$$\sum_{(m,n)\in\mathbb{Z}^2} a_{mn} U^m V^n$$

where $(a_{mn}) \in \mathcal{S}(\mathbb{Z}^2)$ is a rapid decay Schwartz class function, i.e. for all $k \geq 1$

$$\operatorname*{Sup}_{m,n} |a_{mn}|(1 + |m| + |n|)^k < \infty.$$

Exercise 2.9. *Show that $\mathcal{A}_0 \simeq C^\infty(\mathbb{T}^2)$.*

A *derivation* or *infinitesimal automorphism* of an algebra $\mathcal{A}$ is a linear map $\delta : \mathcal{A} \to \mathcal{A}$ satisfying $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. For the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on a manifold M one checks that there is a one to one correspondence between derivations on $\mathcal{A}$ and vector fields on M. For this reason derivations are usually considered as noncommutative analogues of vector fields. The fundamental derivations of the noncommutative torus, δ_1, $\delta_2 : \mathcal{A}_\theta \to \mathcal{A}_\theta$ are the derivations uniquely defined by their values on generators of the algebra

$$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \text{and} \quad \delta_2(U) = 0, \quad \delta_2(V) = V,$$

or equivalently

$$\delta_1\Big(\sum_{(m,n)\in\mathbb{Z}^2} a_{mn} U^m V^n\Big) = \sum_{(m,n)\in\mathbb{Z}^2} m a_{mn} U^m V^n,$$

and similarly for δ_2.

The invariance property of the Haar measure for the torus has a noncommutative counterpart. It is easy to see that the canonical trace $\tau : \mathcal{A}_\theta \to \mathbb{C}$ is invariant under δ_1 and δ_2, i.e. $\tau(\delta_i(a)) = 0$ for $i = 1, 2$ and all $a \in \mathcal{A}_\theta$.

Exercise 2.10. *Find all derivations of $\mathcal{A}_\theta$.*

Remark 1. *Although they are deformations of Hopf algebras (in fact deformations of a group), noncommutative tori are not Hopf algebras. They are however in some sense Hopf algebroids. Since they are groupoid algebras, this is not surprising.*

The definition of noncommutative tori can be extended to higher dimensions. Let Θ be a real skew-symmetric $n \times n$ matrix and let $\lambda_{jk} = e^{2\pi i \Theta_{jk}}$. The *noncommutative torus* A_Θ is the universal unital C^*-algebra generated by unitaries $U_1, U_2, \cdots, U_n$ with

$$U_k U_j = \lambda_{jk} U_j U_k$$

for all $1 \leq j, k \leq n$. Alternatively, A_Θ is the universal C^*-algebra generated by unitaries U_l, $l \in \mathbb{Z}^n$ with

$$U_l U_m = e^{\pi i \langle l, \Theta m \rangle} U_{l+m},$$

for all $l, m \in \mathbb{Z}^n$. From this it follows hat for any $B \in GL(n, \mathbb{Z})$ we have $A_{B\Theta B^t} \simeq A_\Theta$. The definitions of the invariant trace, smooth subalgebras, and fundamental derivations are easily extended to higher dimensions. The isomorphism types and Morita equivalence classes of higher dimensional noncommutative tori, as well as finite projective modules over them has been extensively studied in recent years.

3 Beyond C^*-algebras

To plunge deeper into noncommutative geometry one must employ various classes of noncommutative algebras that are not C^*-algebras. They include dense subalgebras of C^*-algebras which are *stable under holomorphic functional calculus*, e.g. algebras of smooth functions on noncommutative tori. Another class consists of *almost commutative algebras*. They have a *Poisson algebra* as their *semiclassical limit* and include algebras of differential operators and enveloping algebras. They appear in questions of deformation quantization and its applications.

3.1 Algebras stable under holomorphic functional calculus

We are going to describe a situation where many features of the embedding $C^\infty(M) \subset C(M)$ of the algebra of smooth functions on a closed manifold into the algebra of continuous functions is captured and extended to the noncommutative world.

Let A be a unital Banach algebra and let f be a holomorphic function defined on a neighborhood of $\mathrm{sp}\,(a)$, the spectrum of $a \in A$. Let

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z1 - a)^{-1} dz, \tag{13}$$

where the contour γ goes around the spectrum of a only once (counter clockwise). The integral is independent of the choice of the contour and it can be shown that for a fixed a, the map $f \mapsto f(a)$ is a unital algebra map from the algebra of holomorphic functions on a neighborhood of $\mathrm{sp}\,(a)$ to A. It is called the *holomorphic functional calculus*. If f happens to be holomorphic in a disc containing $\mathrm{sp}(a)$ with power series expansion $f(z) = \sum c_i z^i$, then one shows, using the Cauchy integral formula, that $f(a) = \sum c_i a^i$. If A is a C^*-algebra and a is a normal element then, thanks to the Gelfand-Naimark theorem, we have the much more powerful *continuous functional calculus* from $C(\mathrm{sp}\,(a)) \to A$. It extends the holomorphic functional calculus (see below).

Definition 3.1. *Let $B \subset A$ be a unital subalgebra of a unital Banach algebra A. We say B is stable under holomorphic functional calculus if for all $a \in B$ and all holomorphic functions on $sp\,(a)$, we have $f(a) \in B$.*

Example 3.1. 1) The algebra $C^\infty(M)$ of smooth functions on a closed smooth manifold M is stable under holomorphic functional calculus in $C(M)$. The same can be said about the algebra $C^k(M)$ of k-times differentiable functions. The algebra $\mathbb{C}[X]$ of polynomial functions is not stable under holomorphic functional calculus in $C[0,1]$.
2) The smooth noncommutative torus $\mathcal{A}_\theta \subset A_\theta$ is stable under holomorphic functional calculus.

Let $\mathrm{sp}_B\,(a)$ denote the spectrum of $a \in B$ with respect to the subalgebra B. Clearly $\mathrm{sp}_A\,(a) \subset \mathrm{sp}_B\,(a)$ but the reverse inclusion holds if and only if invertibility in A implies invertibility in B. A good example to keep in mind is $\mathbb{C}[x] \subset C[0,1]$. It is easy to see that if B is stable under holomorphic functional calculus, then we have the *spectral permanence property*

$$\mathrm{sp}_A\,(a) \subset \mathrm{sp}_B\,(a).$$

Conversely, under some conditions on the subalgebra B the above spectral permanence property implies that B is stable under holomorphic functional calculus. In fact, in this case for all $z \in \mathrm{sp}_A\,(a)$, $(z1 - a)^{-1} \in B$ and if there is a suitable topology in B, stronger than the topology induced from A, in which B is complete, one can then show that the integral (13) converges in B. We give two instances where this technique works.

Let (H, F) be a Fredholm module over a Banach algebra A and assume that the action of A on H is continuous.

Proposition 3.1. *([24]) For each $p \in [1, \infty)$, the subalgebra*

$$\mathcal{A} = \{a \in A; \quad [F,\, a] \in \mathcal{L}^p(H)\}$$

is stable under holomorphic functional calculus.

Exercise 3.1. *(smooth compact operators). Let $\mathcal{K}^\infty \subset \mathcal{K}$ be the algebra of infinite matrices (a_{ij}) with rapid decay coefficients. Show that $\mathcal{K}^\infty$ is stable under holomorphic functional calculus in the algebra of compact operators $\mathcal{K}$.*

Another source of examples are *smooth vectors* of Lie group actions. Let G be a Lie group acting continuously on a Banach algebra A. An element $a \in A$ is called *smooth* if the map $G \to A$ sending $g \mapsto g(a)$ is smooth. It can be shown that smooth vectors form a dense subalgebra of A which is stable under holomorphic functional calculus.

Example 3.2. The formulas

$$U \mapsto \lambda_1 U, \qquad V \mapsto \lambda_2 V,$$

where $(\lambda_1, \lambda_2) \in \mathbb{T}^2$, define an action of the two-torus $\mathbb{T}^2$ on the noncommutative torus A_θ. Its set of smooth vectors can be shown to coincide with the smooth noncommutative torus $\mathcal{A}_\theta$ [17].

For applications to K-theory and density theorems, the following result is crucial [116].

Proposition 3.2. *If B is a dense subalgebra of a Banach algebra A which is stable under holomorphic functional calculus then so is $M_n(B)$ in $M_n(A)$ for all $n \geq 0$.*

Now let $e \in A$ be an idempotent in A. For any $\epsilon > 0$ there is an idempotent $e' \in B$ such that $\|e - e'\| < \epsilon$. In fact, since B is dense in A we can first approximate it by an element $g \in B$. Since $\mathrm{sp}\,(e) \subset \{0, 1\}$, $\mathrm{sp}\,(g)$ is concentrated around 0 and 1. Let f be a holomorphic function which is identically equal to 1 around 0 and 1. Then

$$e' = f(g) = \frac{1}{2\pi i} \int_\gamma f(z)(z1 - e)^{-1} dz,$$

is an idempotent in B which is close to e. In particular $[e] = [f(g)]$ in $K_0(A)$ (see Section 5.1). Thanks to the above Proposition, we can repeat this argument for $M_n(B) \subset M_n(A)$ for all n. It follows that if B is dense is A and is stable under holomorphic functional calculus, the natural embedding $B \to A$ induces an isomorphism $K_0(B) \simeq K_0(A)$ in K-theory (cf. also the article of J. B. Bost [8] where a more general density theorem along these lines is proved).

Example 3.3. *(Toeplitz algebras)* The original Toeplitz algebra $\mathcal{T}$ is defined as the universal unital C^*-algebra generated by an *isometry*, i.e. an element S with

$$S^*S = I.$$

It can be concretely realized as the C^*-subalgebra of $\mathcal{L}(\ell^2(\mathbb{N}))$ generated by the unilateral forward shift operator $S(e_i) = e_{i+1}$, $i = 0, 1, \cdots$. Since the algebra $C(S^1)$ of continuous functions on the circle is the universal algebra defined by a unitary u, the map $S \mapsto u$ defines a C^*-algebra surjection

$$\sigma : \mathcal{T} \longrightarrow C(S^1),$$

called the *symbol map*. It is an example of the symbol map for pseudodifferential operators of order zero over a closed manifold (see below).

The rank one projection $I - SS^*$ is in the kernel of σ. Since the closed ideal generated by $I - SS^*$ is the ideal $\mathcal{K}$ of compact operators, we have $\mathcal{K} \subset \operatorname{Ker}\sigma$. With some more work one shows that in fact $\mathcal{K} = \operatorname{Ker}\sigma$ and therefore we have a short exact sequence of C^*-algebras, called the *Toeplitz extension* (due to Coburn, cf. [53, 57])

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0. \qquad (14)$$

There is an alternative description of the Toeplitz algebra and extension (14) that makes its relation with pseudodifferential operators and index theory more transparent. Let $H = L^2(S^1)^+$ denote the Hilbert space of square integrable functions on the circle whose negative Fourier coefficients vanish and let $P : L^2(S^1) \to L^2(S^1)^+$ denote the canonical projection. Any continuous function $f \in C(S^1)$ defines a *Toeplitz operator*

$$T_f : L^2(S^1)^+ \to L^2(S^1)^+, \qquad T_f(g) = P(gf).$$

It can be shown that the C^*-algebra generated by the set of Toeplitz operators $\{T_f;\ f \in C(S^1)\}$ is isomorphic to the Toeplitz algebra $\mathcal{T}$. The relation

$$T_f T_g - T_{fg} \in \mathcal{K}(H)$$

shows that any Toeplitz operator T can be written as

$$T = T_f + K$$

where K is a compact operator. In fact this decomposition is unique and gives another definition of the symbol map σ by $\sigma(T_f + K) = f$. It is also clear from extension (14) that a Toeplitz operator T is Fredholm if and only if its symbol $\sigma(T)$ is an invertible function on S^1.

The algebra generated by Toeplitz operators T_f for $f \in C^\infty(S^1)$ is called the *smooth Toeplitz algebra* $\mathcal{T}^\infty \subset \mathcal{T}$. Similar to (14) we have an extension

$$0 \longrightarrow \mathcal{K}^\infty \longrightarrow \mathcal{T}^\infty \xrightarrow{\sigma} C^\infty(S^1) \longrightarrow 0. \tag{15}$$

Exercise 3.2. *Show that $\mathcal{T}^\infty$ is stable under holomorphic functional calculus in $\mathcal{T}$.*

Exercise 3.3. *Show that*

$$\varphi(A,\, B) = Tr([A,\, B])$$

defines a cyclic 1-cocycle on $\mathcal{T}^\infty$. If f is a smooth non-vanishing function on the circle, show that

$$index(T_f) = \varphi\,(T_f,\, T_{f^{-1}}).$$

The Toeplitz extension (14) has a grand generalization. On any closed smooth manifold M, a (scalar) pseudodifferential operator D of order zero defines a bounded linear map $D : L^2(M) \to L^2(M)$ and its principal symbol $\sigma(D)$ is a continuous function on $S^*(M)$, the unit cosphere bundle of M (cf. [89]). Let $\Psi^0(M) \subset \mathcal{L}(L^2(M))$ denote the C^*-algebra generated by all pseudodifferential operators of order zero on M. We then have a short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(L^2(M)) \longrightarrow \Psi^0(M) \xrightarrow{\sigma} C(S^*M) \longrightarrow 0.$$

For $M = S^1$, the cosphere bundle splits as the disjoint union of two copies of S^1 and the above sequence is the direct sum of two identical copies, each of which is isomorphic to the Toeplitz extension (14).

3.2 Almost commutative and Poisson algebras

Let A be a unital complex algebra. We say A is a *filtered algebra* if it has an increasing filtration $F^i(A) \subset F^{i+1}(A)$, $i = 0, 1, 2, \cdots$, with $F^0(A) = \mathbb{C}1$, $F^i(A)F^j(A) \subset F^{i+j}(A)$ for all i, j and $\cup_i F^i(A) = A$. Let $F^{-1}(A) = 0$. The *associated graded algebra* of a filtered algebra is the graded algebra

$$\mathrm{Gr}\,(A) = \bigoplus_{i \geq 0} \frac{F^i(A)}{F^{i-1}(A)}.$$

Definition 3.2. *An almost commutative algebra is a filtered algebra whose associated graded algebra $Gr(A)$ is commutative.*

Being almost commutative is equivalent to the commutator condition

$$[F^i(A),\ F^j(A)] \subset F^{i+j-1}(A),$$

for all i, j. As we shall see Weyl algebras and more generally algebras of differential operators on a smooth manifold, and universal enveloping algebras are examples of almost commutative algebras.

Let A be an almost commutative algebra. The original Lie bracket $[x, y] = xy - yx$ on A induces a Lie bracket $\{\ \}$ on $Gr(A)$ via the formula

$$\{x + F^i,\ y + F^j\} := [x,\ y] + F^{i+j-2}.$$

Notice that by the almost commutativity assumption, $[x,\ y]$ is in $F^{i+j-1}(A)$. The induced Lie bracket on $\mathrm{Gr}\,(A)$ is compatible with its multiplication in the sense that for all $a \in \mathrm{Gr}\,(A)$, the map $b \mapsto \{a,\, b\}$ is a derivation. The algebra $\mathrm{Gr}\,(A)$ is called the *semiclassical limit* of the almost commutative algebra A. It is an example of a Poisson algebra as we recall next. The quotient map

$$\sigma : A \to \mathrm{Gr}\,(A)$$

is called the *principal symbol map.*

Any splitting $q : \mathrm{Gr}\,(A) \to A$ of this map can be regarded as a 'naive quantization map'. Linear splittings always exist but they are hardly interesting. One usually demands more. For example one wants q to be a Lie algebra map in the sense that

$$q\{a,\ b\} = [q(a),\ q(b)] \tag{16}$$

for all a, b in $\mathrm{Gr}\,(A)$. This is one form of *Dirac's quantization rule* going back to [50]. *No-go theorems*, e.g. Groenvald-van Hove's (cf. [66] for a discussion and precise statements), state that, under reasonable non-degeneracy conditions, this is almost never possible. The remedy is to have (16) satisfied only in an asymptotic sense. As we shall discuss later in this section, this can be done in different ways either in the context of formal deformation quantization [4, 84] or through strict C^*-algebraic deformation quantization [113].

The notion of a Poisson algebra captures the structure of semiclassical limits.

Definition 3.3. *Let P be a commutative algebra. A Poisson structure on P is a Lie algebra bracket $(a, b) \mapsto \{a, b\}$ on A such that for any $a \in A$, the map $b \mapsto \{a, b\} : A \to A$ is a derivation of A. That is, for all b, c in A we have*

$$\{a, bc\} = \{a, b\}c + b\{a, c\}.$$

In geometric examples (see below) the vector field defined by this derivation is called the *Hamiltonian vector field* of the *Hamiltonian function a*.

Definition 3.4. *A Poisson algebra is a pair $(P, \{, \})$ where P is a commutative algebra and $\{\ ,\ \}$ is a Poisson structure on P.*

We saw that the semiclassical limit $P = \mathrm{Gr}\,(A)$ of any almost commutative algebra A is a Poisson algebra. Conversely, given a Poisson algebra P one may ask if it is the semiclassical limit of an almost commutative algebra. This is one form of the problem of quantization of Poisson algebras, the answer to which for general Poisson algebras is negative. We give a few concrete examples of Poisson algebras.

Example 3.4. A *Poisson manifold* is a manifold M whose algebra of smooth functions $A = C^\infty(M)$ is a Poisson algebra (we should also assume that the bracket $\{\ ,\ \}$ is continuous in the Fréchet topology of A). It is not difficult to see that all Poisson structures on A are of the form

$$\{f,\ g\} := \langle df \wedge dg,\ \pi\rangle,$$

where $\pi \in C^\infty(\bigwedge^2(TM))$ is a smooth 2-vector field on M. This bracket clearly satisfies the Leibniz rule in each variable and one checks that it satisfies the Jacobi identity if and only if $[\pi, \pi] = 0$, where the *Schouten bracket* is defined in local coordinates by

$$[\pi, \pi] = \sum_{l=1}^{n} (\pi_{lj} \frac{\partial \pi_{ik}}{\partial x_l} + \pi_{li} \frac{\partial \pi_{kj}}{\partial x_l} + \pi_{lk} \frac{\partial \pi_{ji}}{\partial x_1}) = 0.$$

The Poisson bracket in local coordinates is given by

$$\{f,\ g\} = \sum_{ij} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Symplectic manifolds are the simplest examples of Poisson manifolds. They correspond to non-degenerate Poisson structures. Given a symplectic form ω, the associated Poisson bracket is given by

$$\{f,\ g\} = \omega(X_f, X_g),$$

where the vector field X_f is the symplectic dual of df.

Let $C^\infty_{\text{poly}}(T^*M)$ be the algebra of smooth functions on T^*M which are polynomial in the cotangent direction. It is a Poisson algebra under the natural symplectic structure of T^*M. This Poisson algebra is the semiclassical limit of the algebra of differential operators on M, as we will see in the next example.

Example 3.5. *(Differential operators on commutative algebras).* Let A be a commutative unital algebra. Let $\mathcal{D}^0(A)$ denote the set of *differential operators of order zero* on A, i.e. A-linear maps from $A \to A$, and for $n \geq 1$, let $\mathcal{D}^n(A)$ be the set of all operators D in $\text{End}_{\mathbb{C}}(A)$ such that for any $a \in A$, $[D,\ a] \in \mathcal{D}^{n-1}(A)$. The set

$$\mathcal{D}(A) = \bigcup_{n \geq 0} \mathcal{D}^n(A)$$

is a subalgebra of $\text{End}_{\mathbb{C}}(A)$, called the *algebra of differential operators* on A. It is an almost commutative algebra under the filtration given by $\mathcal{D}^n$, $n \geq 0$. Elements of $\mathcal{D}^n(A)$ are called differential operators of order n. For example, a linear map $D : A \to A$ is a differential operator of order one if and only if it is of the form $D = \delta + a$, where δ is a derivation on A and $a \in A$.

For general A, the semiclassical limit $\text{Gr}\,(\mathcal{D}(A))$ and its Poisson structure are not easily identified except for coordinate rings of smooth affine varieties or algebras of smooth functions on a manifold. In this case a differential operator D of order n is locally given by

$$D = \sum_{|I| \leq k} a_I(x) \partial^I$$

where $I = (i_1, \cdots, i_n)$ is a multi-index and $\partial^I = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_n}$ is a mixed partial derivative. This expression depends on the local coordinates but its leading terms of total degree n have an invariant meaning provided we replace $\partial_i \mapsto \xi_i \in T^*M$. For $\xi \in T^*_x M$, let

$$\sigma_p(D)(x, \xi) := \sum_{|I| = k} a_I(x) \xi^I$$

Then the function $\sigma_p(D) : T^*M \to \mathbb{C}$, called the *principal symbol* of D, is invariantly defined and belongs to $C^\infty_{\text{poly}}(T^*M)$. The algebra $C^\infty_{\text{poly}}(T^*M)$ inherits a canonical Poisson structure as a subalgebra of the Poisson algebra $C^\infty(T^*M)$ and we have the following

Proposition 3.3. *The principal symbol map induces an isomorphism of Poisson algebras*

$$\sigma_p : Gr\,\mathcal{D}(C^\infty(M)) \overset{\simeq}{\to} C^\infty_{poly}(T^*M).$$

See [16] for a proof of this or, even better, try to prove it yourself by proving it for Weyl algebras first.

Example 3.6. *(The Weyl algebra)* Let $A_1 := \mathcal{D}\,\mathbb{C}[X]$ be the *Weyl algebra* of differential operators on the line. Alternatively, A_1 can be described as the unital complex algebra defined by generators x and p with

$$px - xp = 1.$$

The map $x \mapsto x$, $p \mapsto \frac{d}{dx}$ defines the isomorphism. Physicists prefer to write the defining relation as the *canonical commutation relation* $pq - qp = \frac{h}{2\pi i}1$, where h is Planck's constant and p and q represent momentum and position operators. This is not without merit because we can then let $h \to 0$ and obtain the commutative algebra of polynomials in p and q as the semiclassical limit. Also, i is necessary if we want to consider p and q as selfadjoint operators (why?).

Any element of A_1 has a unique expression as a differential operator with polynomial coefficients $\sum a_i(x)\frac{d^i}{dx^i}$ where the standard filtration is by degree of the differential operator. We have an algebra isomorphism $Gr(A_1) \simeq \mathbb{C}[x, y]$ under which the (principal) symbol map is given by

$$\sigma(\sum_{i=0}^{n} a_i(x)\frac{d^i}{dx^i}) = a_n(x)y^n.$$

The induced Poisson bracket on $\mathbb{C}[x, y]$ is the classical Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

In general, the Weyl algebra A_n is the algebra of differential operators on $\mathbb{C}[x_1, \cdots, x_n]$. Alternatively it can be defined as the universal algebra defined by $2n$ generators $x_1, \cdots x_n, p_1, \cdots, p_n$ with

$$[p_i, x_i] = \delta_{ij}, \quad \text{and} \quad [p_i, p_j] = [x_i, x_j] = 0$$

for all i, j. Notice that $A_n \simeq A_1 \otimes \cdots \otimes A_1$ (n factors).

A lot is known about Weyl algebras and a lot remains to be known, including the Dixmier conjecture about the automorphisms of A_n and its relation with the Jacobian conjecture recently studied by Kontsevich in [5]. The Hochschild and cyclic cohomology of A_n is computed in [60].

Exercise 3.4. . *Show that A_1 is a simple algebra, i.e. it has no non-trivial two-sided ideals; that any derivation of A_1 is inner; and that $[A_1, A_1] = A_1$, i.e. there are no non-trivial traces on A_1. Prove all of this for A_n. Any derivation of A_1 is inner; is it true that any automorphism of A_1 is inner?*

Exercise 3.5. *Let $A = \mathbb{C}[x]/(x^2)$ be the algebra of dual numbers. Describe its algebra of differential operators.*

Example 3.7. *(Universal enveloping algebras)* Let $U(\mathfrak{g})$ denote the *enveloping algebra* of a Lie algebra $\mathfrak{g}$. By definition, $U(\mathfrak{g})$ is the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$. For $p \geq 0$, let $F^p(U(\mathfrak{g}))$ be the subspace generated by tensors of degree at most p. This turns $U(\mathfrak{g})$ into a filtered algebra and the Poincaré-Birkhoff-Witt theorem asserts that its associated graded algebra is canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$. The algebra isomorphism is induced by the *symmetrization map* $s : S(\mathfrak{g}) \to \mathrm{Gr}\,(U(\mathfrak{g}))$, defined by

$$s(X_1 X_2 \cdots X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(p)}.$$

Note that $S(\mathfrak{g})$ is the algebra of polynomial functions on the dual space $\mathfrak{g}^*$ which is a Poisson manifold under the bracket

$$\{f\,,\, g\}(X) = [Df(X), \;\; Dg(X)]$$

for all $f, g \in C^\infty(\mathfrak{g}^*)$ and $X \in \mathfrak{g}^*$. Here we have used the canonical isomorphism $\mathfrak{g} \simeq \mathfrak{g}^{**}$, to regard the differential $Df(X) \in \mathfrak{g}^{**}$ as an element of $\mathfrak{g}$. The induced Poisson structure on polynomial functions coincides with the Poisson structure in $\mathrm{Gr}\,(U(\mathfrak{g}))$.

Example 3.8. *(algebra of formal pseudodifferential operators on the circle)* This algebra is obtained by formally inverting the differentiation operator $\partial := \frac{d}{dx}$ and then completing the resulting algebra. A formal pseudodifferential operator on the circle is an expression of the form $\sum_{-\infty}^{n} a_i(x)\partial^i$, where each $a_i(x)$ is a Laurent polynomial. The multiplication in uniquely defined by the rules $\partial x - x\partial = 1$ and $\partial\partial^{-1} = \partial^{-1}\partial = 1$. We denote the resulting algebra by Ψ_1. The *Adler-Manin trace* on Ψ_1 [95], also called the *noncommutative residue*, is defined by

$$\mathrm{Tr}\,(\sum_{-\infty}^{n} a_i(x)\partial^i) = \mathrm{Res}\,(a_{-1}(x);\ 0) = \frac{1}{2\pi i}\int_{S^1} a_{-1}(x).$$

This is a trace on Ψ_1. In fact one can show that $\Psi_1/[\Psi_1,\ \Psi_1]$ is one-dimensional which means that any trace on Ψ_1 is a multiple of Tr. Notice that for the Weyl algebra A_1 we have $[A_1,\ A_1] = A_1$.

Another interesting difference between Ψ_1 and A_1 is that Ψ_1 admits non-inner derivations (see exercise below). The algebra Ψ_1 has a nice generalization to algebras of pseudodifferential operators in higher dimensions. The appropriate extension of the above trace is the *noncommutative residue* of Guillemin and Wodzicki (cf. [122]. See also [24] for relations with the Dixmier trace and its place in noncommutative Riemannian geometry).

Exercise 3.6. *Unlike the algebra of differential operators, Ψ_1 admits non-inner derivations. Clearly $log\,\partial \notin \Psi_1$, but show that for any $a \in \Psi_1$, we have $[log\,\partial,\ a] \in \Psi_1$ and therefore the map*

$$a \mapsto \delta(a) := [log\,\partial,\ a]$$

defines a non-inner derivation of Ψ_1 [87]. The corresponding Lie algebra 2-cocycle

$$\varphi(a, b) = Tr(a[log\,\partial,\ b])$$

is the Radul cocycle.

3.3 Deformation theory

In this section we shall freely use results about Hochschild cohomology from Section 6.5. In the last Section we saw one way to formalize the idea of quantization through the notion of an almost commutative algebra and its semiclassical limit which is a Poisson algebra. A closely related notion is *formal deformation quantization*, or *star products* [4, 84]. Let $(A, \{\ ,\ \})$ be a Poisson algebra and let $A[[h]]$ be the algebra of formal power series over A. A formal deformation of A is an associative $\mathbb{C}[[h]]$-linear multiplication

$$*_h : A[[h]] \otimes A[[h]] \to A[[h]]$$

such that $*_0$ is the original multiplication and for all a, b in A,

$$\frac{a *_h b - b *_h a}{h} \to \{a, b\}$$

as $h \to 0$. Writing

$$a *_h b = B_0(a,b) + hB_1(a,b) + h^2 B_2(a,b) + \cdots$$

where $B_i : A \otimes A \to A$ are Hochschild 2-cochains on A with values in A, we see that the initial conditions on $*_h$ are equivalent to

$$B_0(a,b) = ab, \quad \text{and} \quad B_1(a,b) - B_1(b,a) = \{a,\, b\}.$$

The associativity condition on $*_h$ is equivalent to an infinite system of equations involving the cochains B_i. They are given by

$$B_0 \circ B_n + B_1 \circ B_{n-1} + \cdots + B_n \circ B_0 = 0, \quad \text{for all}\, n \geq 0,$$

or equivalently

$$\sum_{i=1}^{n-1} B_i \circ B_{n-i} = \delta B_n. \tag{17}$$

Here, the *Gerstenhaber* $\circ$ *product* of 2-cochains $f,\, g : A \otimes A \to A$ is defined as the 3-cochain

$$f \circ g(a,b,c) = f(g(a,b),c) - f(a, g(b,c)).$$

Notice that a 2-cochain f defines an associative product if and only if $f \circ f = 0$. Also notice that the Hochschild coboundary δf can be written as $\delta f = -m \circ f - f \circ m$, where $m : A \otimes A \to A$ is the multiplication of A. These observations lead to the associativity equations (17).

To solve these equations starting with $B_0 = m$, by antisymmetrizing we can always assume that B_1 is antisymmetric and hence we can assume $B_1 = \frac{1}{2}\{\ ,\ \}$. Having found $B_0, B_1, \cdots B_{n-1}$, we can find a B_n satisfying (17) if and only if the cocycle $\sum_{i=1}^{n-1} B_i \circ B_{n-i}$ is a coboundary, i.e. its class in $H^3(A,A)$ should vanish. The upshot is that the third Hochschild cohomology $H^3(A,A)$ is the *space of obstructions* for the deformation quantization problem. In particular if $H^3(A,A) = 0$ then any Poisson bracket on A can be deformed. Notice, however, that this is only a sufficient condition and by no means is necessary, as will be shown below.

In the most interesting examples, e.g. for $A = C^\infty(M)$, $H^3(A,A) \neq 0$. To see this consider the differential graded Lie algebra $(C(A,\, A),\, [\ ,\ ],\, \delta)$ of continuous Hochschild cochains on A, and the differential graded Lie algebra, with zero differential, $(\bigwedge(TM),\, [\ ,\ ],\, 0)$ of polyvector fields on M. The bracket in the first is the Gerstenhaber bracket and in the second is the Schouten bracket of polyvector fields. By a theorem of Connes (see the resolution in Lemma 44 in [22]), we know that the *antisymmetrization map*

$$\alpha : (C^\infty(\bigwedge TM),\, 0) \to (C(A,\, A),\, \delta)$$

sending a polyvector field $X_1 \wedge \cdots \wedge X_k$ to the functional φ defined by

$$\varphi(f^1, \cdots, f^k) = df^1(X_1)df^2(X_2)\cdots df^k(X_k)$$

is a quasi-isomorphism of differential graded algebras. In particular, it induces an isomorphism of graded commutative algebras

$$H^k(A, A) \simeq C^\infty(\bigwedge^k TM).$$

The map α, however, is not a morphism of Lie algebras and that is where the real difficulty of deforming a Poisson structure is hidden. The *formality theorem* of M. Kontsevich [84] states that as a differential graded Lie algebra, $(C(A, A), \delta, [\ ,\ ])$ is formal in the sense that it is quasi-isomorphic to its cohomology. Equivalently, it means that one can correct the map α, by adding an infinite number of terms, to a morphism of L_∞ algebras. This shows that the original deformation problem of Poisson structures can be transferred to $(C^\infty(\bigwedge TM),\, 0)$ where it is unobstructed since the differential in the latter DGL is zero. We give a couple of simple examples where deformations can be explicitly constructed.

Example 3.9. The simplest non-trivial Poisson manifold is the dual $\mathfrak{g}^*$ of a finite dimensional Lie algebra $\mathfrak{g}$. Let $U_h(\mathfrak{g}) = T(\mathfrak{g})/I$, where the ideal I is generated by

$$x \otimes y - y \otimes x - h[x, y],\ \ x, y \in \mathfrak{g}.$$

By the Poincaré-Birkhoff-Witt theorem, the antisymmetrization map $\alpha_h : S(\mathfrak{g}) \to U_h(\mathfrak{g})$ is a linear isomorphism. We can define a $*$-product on $S(\mathfrak{g})$ by

$$f *_h g = \alpha_h^{-1}(\alpha_h(f)\alpha_h(g)) = \sum_{n=0}^{\infty} h^n B_n(f, g).$$

With some work one can show that B_n are bidifferential operators and hence the formula extends to a $*$-product on $C^\infty(\mathfrak{g}^*)$.

Example 3.10. *(Moyal-Weyl quantization)* Consider the algebra generated by x and y with relation $xy - yx = \frac{h}{i}1$. Iterated application of the Leibniz rule gives the formula for the product

$$f *_h g = \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{-ih}{2})^n B_n(f, g),$$

where $B_0(f,g) = fg,\ B_1(f,g) = \{f,\ g\}$ is the standard Poison bracket, and for $n \geq 2$,

$$B_n(f,\,g) = (-1)^n \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) (\partial_x^k \partial_y^{n-k} f)(\partial_x^{n-k} \partial_y^k g).$$

Notice that this formula makes sense for $f,\, g \in C^\infty(\mathbb{R}^2)$ and defines a deformation of this algebra with its standard Poisson structure. This can be extended to arbitrary constant Poisson structures

$$\{f,\ g\} = \sum \pi^{ij} \partial_i f\, \partial_j g.$$

The Weyl-Moyal $*$ product is then given by

$$f *_h g = \exp(-i\frac{h}{2} \sum \pi^{ij} \partial_i \wedge \partial_j)(f,\ g).$$

Remark 2. As we mentioned before, deformation quantization has its origins in Dirac's quantization rule in quantum mechanics [50]. The original idea was to replace the classical observables (functions) by quantum observables (operators) in such a way that Poisson brackets of functions correspond to commutators of operators. It was, however, soon realized that a rigid interpretation of Dirac's rule is impossible and it must be understood only in an asymptotic sense. In fact there are some well known 'no go-theorems' that state that under reasonable non-degeneracy assumptions this rigid notion of quantization is not possible (cf. [66] and references therein). One approach, as in [4], formulates quantization of a classical system as formal deformation quantization of a Poisson structure manifold. We should mention that the algebraic underpinnings of deformation theory of (associative and Lie) algebras and the relevance of Hochschild cohomology goes back to Gerstenhaber's papers [63].

No discussion of deformation quantizations is complete without discussing Rieffel's deformation quantization [113]. Roughly speaking, one demands that formal power series of formal deformation theory should actually be convergent. More precisely, let $(M, \{\ ,\ \})$ be a Poisson manifold. A *strict deformation quantization* of the Poisson algebra $\mathcal{A} = C^\infty(M)$ is a family of pre C^*-algebra structures $(*_h,\ \|\ \|_h)$ on $\mathcal{A}$ for $h \geq 0$ such that the family forms a continuous field of C^*-algebras on $[0,\ \infty)$ (in particular for any $f \in \mathcal{A},\quad h \mapsto \|f\|_h$ is continuous) and for all $f, g \in \mathcal{A}$,

$$\| \frac{f *_h g - g *_h f}{ih} \|_h \to \{f,\ g\}$$

as $h \to 0$. We therefore have a family of C^*-algebras A_h obtained by completing $\mathcal{A}$ with respect to the norm $\| \|_h$.

Example 3.11. *(noncommutative tori)* In [111] it is shown that the family of noncommutative tori A_θ form a strict deformation quantization of the Poisson algebra of smooth functions on the 2-torus. This in fact appears as a special case of a more general result. Let α be a smooth action of $\mathbb{R}^n$ on $\mathcal{A} = C^\infty(M)$. Let X_i denote the infinitesimal generators for this action. Each skew-symmetric $n \times n$ matrix J defines a Poisson bracket on $\mathcal{A}$ by

$$\{f, g\} = \sum J_{ij} X_i(f) X_j(g).$$

For each $h \in \mathbb{R}$, define a new product $*_h$ on $\mathcal{A}$ by

$$f *_h g = \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{hJu}(f) \alpha_v(g) e^{2\pi i u \cdot v} du dv.$$

The $*$ structure is by conjugation and is undeformed (see [111] for the definition of $\|f\|_h$). For $\mathcal{A} = C^\infty(\mathbb{T}^2)$ with the natural $\mathbb{R}^2$ action one obtains A_θ.

Remark 3. . Does any Poisson manifold admit a strict deformation quantization? This question is still open (even for symplectic manifolds). In [112], Rieffel shows that the canonical symplectic structure on the 2-sphere admits no $SO(3)$-invariant strict deformation quantization. An intriguing idea promoted by Connes and Marcolli in [34] is that existence of a strict deformation quantization of a Poisson manifold should be regarded as an *integrability condition* for the formal deformation quantization. There is a clear analogy with the case of formal and convergent power series solutions of differential equations around singular points. They ask for a possible 'theory of ambiguity', i.e. a cohomology theory that could capture the difference between the two cases.

4 Sources of noncommutative spaces

At present we can identify at least four methods by which noncommutative spaces are constructed:
i) noncommutative quotients;
ii) algebraic and C^*-algebraic deformations;
iii) Hopf algebras and quantum groups;
iv) cohomological constructions.

It should be stressed that these are by no means mutually exclusive; there are intimate relations between these sources and sometimes a noncommutative space can be described by several methods, as is the case with noncommutative tori. The majority of examples, by far, fall into the first category. We won't discuss the last idea, advanced by Connes and Landi [32] and Connes and Dubois-Violette [26]. Very briefly the idea is that if one writes the conditions for the Chern character of an idempotent in cyclic homology to be trivial on the level of chains, then one obtains interesting examples of algebras such as noncommutative spheres and spherical manifolds, Grassmannians, and Yang-Mills algebras.

4.1 Noncommutative quotients

The quotient space $X/\sim$ of a Hausdorff space may easily fail to be Hausdorff and may even be an indiscrete topological space with only two open sets. This happens, for example, when at least one equivalence class is dense in X. Similarly the quotient of a smooth manifold may become singular and fail to be smooth.

The method of *noncommutative quotients* as advanced by Connes in [24] allows one to replace "bad quotients" by nice noncommutative spaces, represented by noncommutative algebras. In general these noncommutative algebras are defined as *groupoid algebras*. In some cases, like quotients by group actions, the noncommutative quotient can be defined as a crossed product algebra too, but in general the use of *groupoids* seem to be unavoidable.

An equivalence relation is usually obtained from a much richer structure by forgetting part of this structure. For example, it may arise from an action of a group G on X where $x \sim y$ if and only if $gx = y$ for some g in G (*orbit equivalence*). Note that there may be, in general, many g with this property. That is x may be identifiable with y in more than one way. Of course when we form the equivalence relation this extra information is lost. The key idea in dealing with bad quotients in Connes' theory is to keep track of this extra information, by first forming a groupoid.

Now Connes' dictum in forming noncommutative quotients can be summarized as follows:

$$\textbf{quotient data} \rightsquigarrow \textbf{groupoid} \rightsquigarrow \textbf{groupoid algebra},$$

where the noncommutative quotient is defined to be the groupoid algebra itself.

Definition 4.1. *A groupoid is a small category in which every morphism is an isomorphism.*

The set of objects of a groupoid $\mathcal{G}$ shall be denoted by $\mathcal{G}^{(0)}$. Every morphism has a *source*, *target* and an *inverse*. They define maps

$$s : \mathcal{G} \longrightarrow \mathcal{G}^{(0)}, \quad t : \mathcal{G} \longrightarrow \mathcal{G}^{(0)}, \quad i : \mathcal{G} \longrightarrow \mathcal{G}.$$

Composition $\gamma_1 \circ \gamma_2$ of morphisms γ_1 and γ_2 is only defined if $s(\gamma_1) = t(\gamma_2)$. Composition defines a map

$$\circ : \mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2);\ s(\gamma_1) = t(\gamma_2)\} \longrightarrow \mathcal{G},$$

which is associative in an obvious sense.

Examples 4.1. i) Every group G defines a groupoid $\mathcal{G}$ with one object $*$ and

$$\mathrm{Hom}_{\mathcal{G}}(*, *) = G.$$

The composition of morphisms is simply by group multiplication.

ii) Let X be a G-set. We define a groupoid $\mathcal{G} = X \rtimes G$, called the *transformation groupoid* of the action, as follows. Let obj $\mathcal{G} = X$, and

$$\mathrm{Hom}_{\mathcal{G}}(x, y) = \{g \in G;\ gx = y\}.$$

Composition of morphisms is defined via group multiplication. It is easily checked that $\mathcal{G}$ is a groupoid. Its set of morphisms can be identified as

$$\mathcal{G} \simeq X \times G,$$

where the composition of morphisms is given by

$$(gx, h) \circ (x, g) = (x, hg).$$

iii) Let $\sim$ denote an equivalence relation on a set X. We define a groupoid $\mathcal{G}$, called the *graph of* $\sim$, as follows. Let obj $\mathcal{G} = X$, and let $\mathrm{Hom}_{\mathcal{G}}(x, y)$ be a one element set if $x \sim y$, and be the empty set otherwise.

Note that the set of morphisms of $\mathcal{G}$ is identified with the graph of the relation $\sim$ in the usual sense:

$$\mathcal{G} = \{(x, y);\ x \sim y\} \subset X \times X.$$

The *groupoid algebra* of a groupoid $\mathcal{G}$ is the algebra of functions

$$\mathbb{C}\mathcal{G} = \{f : \mathcal{G} \to \mathbb{C};\ f \text{ has finite support}\},$$

with finite support on $\mathcal{G}$. Under the *convolution product*

$$(fg)(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2),$$

and the involution

$$f^*(\gamma) = \overline{f(\gamma^{-1})}$$

it is a $*$-algebra.

Examples 4.2. i) Let $\mathcal{G}$ be the groupoid defined by a group G. Then clearly the groupoid algebra $\mathbb{C}\mathcal{G}$ is isomorphic to the group algebra $\mathbb{C}G$.
ii) If $\mathcal{G} = X \rtimes G$ is a transformation groupoid then we have an algebra isomorphism

$$\mathbb{C}\mathcal{G} \simeq C(X) \rtimes G.$$

iiI) Let $\mathcal{G}$ be the *groupoid of pairs* on a set of n elements, i.e.

$$\mathcal{G} = \{(i, j);\ i, j = 1, \cdots n\}$$

with composition given by

$$(l, k) \circ (j, i) = (l, i) \quad \text{if } k = j.$$

(Composition is not defined otherwise). We have

$$\mathbb{C}\mathcal{G} \simeq M_n(\mathbb{C}).$$

Indeed, it is easily checked that the map

$$(i, j) \mapsto E_{i,j},$$

where $E_{i,j}$ denote the matrix units, is an algebra isomorphism. This is an extremely important example. In fact, as Connes points out in [24], the matrices in Heisenberg's matrix quantum mechanics [69] were arrived at by a similar procedure.

Exercise 4.1. *Show that the groupoid algebra of a finite groupoid (finite set of objects and finite set of morphisms) can be decomposed as a direct sum of tensor products of group algebras and matrix algebras.*

As the above exercise shows, one cannot get very far with just discrete groupoids and soon one needs to work with topological and smooth groupoids associated to, say, continuous actions of topological groups and to foliations.

A (locally compact) *topological groupoid* is a groupoid such that its set of morphisms $\mathcal{G}$ and set of objects $\mathcal{G}^{(0)}$ are (locally compact) topological spaces, and its composition, source, target and inverse maps are continuous. An *étale groupoid* is a locally compact groupoid such that the fibers of its target map $\mathcal{G}^x = t^{-1}(x), \quad x \in \mathcal{G}^{(0)}$, are discrete.

A *smooth groupoid*, also known as *Lie groupoid*, is a groupoid such that $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are smooth manifolds, the inclusion $\mathcal{G}^{(0)} \to \mathcal{G}$ as well as the maps s, t, i and the composition map $\circ$ are smooth, and s and t are submersions. This last condition will guarantee that the domain of the composition map $\mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2); s(\gamma_1) = t(\gamma_2)\}$ is a smooth manifold.

To define the convolution algebra of a topological groupoid and its C^*-completions, we need an analogue of Haar measure for groupoids. A *Haar measure* on a locally compact groupoid $\mathcal{G}$ is a family of measures μ^x on each t-fiber $\mathcal{G}^x$. The family is supposed to be continuous and left invariant in an obvious sense (cf. [109], unlike locally compact topological groups, groupoids need not have an invariant Haar measure). Given a Haar measure, we can then define, for functions with compact support $f, g \in C_c(\mathcal{G})$ their convolution product

$$(f * g)(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) = \int_{\mathcal{G}^{t(\gamma)}} f(\gamma_1) g(\gamma_1^{-1}\gamma) d\mu^{t(\gamma)}. \qquad (18)$$

This turns $C_c(\mathcal{G})$ into a $*$-algebra. The involution is defined by $f^*(\gamma) = \overline{f(\gamma^{-1})}$. For each fiber $\mathcal{G}^x$, we have an $*$-representation π_x of $C_c(\mathcal{G})$ on the Hilbert space $L^2(\mathcal{G}^x, \mu^x)$ defined by

$$(\pi_x f)(\xi)(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) \xi(\gamma_2) = \int_{\mathcal{G}^{t(\gamma)}} f(\gamma_1) \xi(\gamma_1^{-1}\gamma) d\mu^{t(\gamma)}.$$

We can then define a pre C^*-norm on $C_c(\mathcal{G})$ by

$$\|f\| := \sup \{\|\pi_x(f)\|;\, x \in \mathcal{G}^0\}.$$

The completion of $C_c(\mathcal{G})$ under this norm is the *reduced C^*-algebra* of the groupoid $\mathcal{G}$ and will be denoted by $C_r^*(\mathcal{G})$.

There are two special cases that are particularly important and convenient to work with: étale and smooth groupoids. Notice that for an étale groupoid each fiber is a discrete set and with counting measure on each fiber we obtain a Haar measure. The convolution product is then given by

$$(f * g)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2) = \sum_{\mathcal{G}^{t(\gamma)}} f(\gamma_1)g(\gamma_1^{-1}\gamma).$$

Notice that for each γ this is a finite sum since the support of f is compact and hence contains only finitely many points of each fiber.

A second interesting case where one can do away with Haar measures is smooth groupoids. Let $C_c^\infty(\mathcal{G}, \Omega^{\frac{1}{2}})$ be the space of sections, with compact support, of the line bundle of half-densities on a smooth groupoid $\mathcal{G}$. Since the product of two half-densities is a 1-density which has a well defined integral, the integral (18) for $f, g \in C_c^\infty(\mathcal{G}, \Omega^{\frac{1}{2}})$ is well defined and we obtain the smooth convolution algebra $C_c^\infty(\mathcal{G})$.

Examples 4.3. 1. Let Γ be a discrete group acting by homeomorphisms on a locally compact space X. Then the transformation groupoid $X \rtimes \Gamma$ is an étale groupoid and the groupoid algebra recovers the crossed product algebra:

$$C_c(\mathcal{G}) \simeq C_c(X) \rtimes \Gamma \quad \text{and} \quad C_r^*(\mathcal{G}) \simeq C_0(X) \rtimes_r \Gamma.$$

For $X = S^1$ and $\Gamma = \mathbb{Z}$ acting through rotation by an angle θ, we recover the noncommutative torus as a groupoid algebra, which is one among many incarnations of A_θ.

2. Let X be a locally compact space with a Borel probability measure μ and $\mathcal{G}$ be the groupoid of pairs on X. Then for $f, g \in C_c(X \times X)$ the convolution product (18) reduces to

$$(f * g)(x, z) = \int_X f(x, y)g(y, z)d\mu(y),$$

which is reminiscent of matrix multiplication or products of integral operators. In fact the map $T : C_c(X \times X) \to \mathcal{K}(L^2(X, \mu))$ sending f to the integral operator

$$(Tf)(g)(x) = \int_X f(x, y)g(y)d\mu(y)$$

is clearly an algebra map and can be shown to be 1-1 and onto.

On the other extreme, if $\mathcal{G}$ is the groupoid of the discrete equivalence relation on a locally compact space X, also known as the *groupoid of pairs*, then clearly $C_c(\mathcal{G}) \simeq C_c(X)$.

Example 4.1. *(Non-Hausdorff manifolds)* Let

$$X = S^1 \times 0 \cup S^1 \times 1$$

be the disjoint union of two copies of the circle. We identify $(x, 0) \sim (x, 1)$ for all $x \neq 1$ in S^1. The quotient space $X/\sim$ is a non-Hausdorff manifold. The groupoid of the equivalence relation $\sim$

$$\mathcal{G} \, = \, \{(x, y) \, \subset X \times X; \, x \sim y\,\}$$

is a smooth étale groupoid. Its smooth groupoid algebra is given by

$$C^\infty(\mathcal{G}) = \, \{f \in C^\infty(S^1, \, M_2(\mathbb{C})); \;\, f(1) \text{ is diagonal}\,\}$$

There are many interesting examples of noncommutative quotients that we did not discuss here but are of much interest in noncommutative geometry. They include: foliation algebras, the space of Penrose tilings, the adèle space and the space of $\mathbb{Q}$-lattices in number theory. They can all be defined as groupoid algebras and variations thereof. We refer to Connes-Marcolli's article in this volume as well as [33, 34, 36, 24] for a proper introduction.

Exercise 4.2. *Show that the Hecke algebras $\mathcal{H}(\Gamma, \Gamma_0)$ defined in Example 2.7 are groupoid algebras.*

The following result of M. Rieffel [110] clarifies the relation between the classical quotients and noncommutative quotients for group actions:

Theorem 4.1. *Assume G acts freely and properly on a locally compact Hausdorff space X. Then we have a strong Morita equivalence between the C^*-algebras $C_0(X/G)$ and $C_0(X) \rtimes_r G$.*

4.2 Hopf algebras and quantum groups

Many examples of noncommutative spaces are Hopf algebras or quantum groups. They are either the algebra of coordinates of a quantum group, or, dually, the convolution algebra or the enveloping algebra of a quantum group. In this section we shall make no distinction between Hopf algebras and quantum groups and use these words interchangeably. The theory of Hopf algebras (as well as Hopf spaces) was born in the paper of H. Hopf in his celebrated computation of the rational cohomology of compact connected Lie groups [73]. This line of investigation eventually led to the Cartier-Milnor-Moore theorem [99] characterizing connected cocommutative Hopf algebras as enveloping algebras of Lie algebras.

A purely algebraic theory, with motivations independent from algebraic topology, was created by Sweedler in the 1960's. This line of investigation took a big leap forward with the work of Drinfeld and Jimbo resulting in quantizing all classical Lie groups and Lie algebras [54].

In a different direction, immediately after Pontryagin's duality theorem for locally compact abelian groups, attempts were made to extend it to noncommutative groups. The Tannaka-Krein duality theorem was an important first step. Note that the dual, in any sense of the word, of a noncommutative group is necessarily not a group and one is naturally interested in extending the category of groups to a larger category which is closed under duality and hopefully is even equivalent to its second dual. Hopf von Neumann algebras of G.I. Kac and Vainerman achieve this in the measure theoretic world of von Neumann algebras [55]. The theory of *locally compact quantum groups* of Kustermans and Vaes [88] which was developed much later achieves this goal in the category of C^*-algebras. An important step towards this program was the theory of compact quantum groups of S. L. Woronowicz (cf. [123] for a survey). We refer to [77, 83, 94, 96, 97, 119] for the general theory of Hopf algebras and quantum groups.

The first serious interaction between Hopf algebras and noncommutative geometry started in earnest in the paper of Connes and Moscovici on transverse index theory [38] (cf. also [39, 40, 41] for further developments). In this paper a noncommutative and non-cocommutative Hopf algebra appears as the quantum symmetries of the noncommutative space of codimension one foliations. The same Hopf algebra was later shown to act on the noncommutative space of modular Hecke algebras [42].

To understand the definition of a Hopf algebra, let us see what kind of extra structure exists on the algebra of functions on a group. For simplicity, let G be a finite group, though this is by no means necessary, and let $H = C(G)$ be the algebra of functions from $G \to \mathbb{C}$. The multiplication $m : G \times G \to G$, inversion $i : G \to G$, and unit element $e \in G$, once dualized, define unital algebra maps

$$\Delta : H \to H \otimes H, \quad S : H \to H, \quad \epsilon : H \to \mathbb{C},$$

by the formulas

$$\Delta f(x, y) = f(xy), \quad Sf(x) = f(x^{-1}), \quad \varepsilon(f) = f(e),$$

where we have identified $C(G \times G)$ with $C(G) \otimes C(G)$. Let also

$$m : H \otimes H \to H, \qquad \eta : \mathbb{C} \to H$$

denote the multiplication and unit maps of the algebra H. The associativity, inverse, and unit axioms for groups are dualized and in fact are easily seen to be equivalent to the following *coassociativity, antipode, and counit axioms* for H:

$$\begin{aligned} (\Delta \otimes I)\Delta &= (I \otimes \Delta)\Delta, \\ (\varepsilon \otimes I)\Delta &= (I \otimes \varepsilon)\Delta = I, \\ m(S \otimes I) &= m(I \otimes S) = \eta\varepsilon. \end{aligned}$$

Definition 4.2. *A unital algebra (H, m, η) endowed with unital algebra homomorphisms $\Delta : H \to H \otimes H$, $\varepsilon : H \to \mathbb{C}$ and a linear map $S : H \to H$ satisfying the above equations is called a Hopf algebra.*

It can be shown that the antipode S is unique and is an anti-algebra map. It is also an anti-coalgebra map.

Example 4.2. Commutative or cocommutative Hopf algebras are closely related to groups and Lie algebras. We give a few examples to indicate this connection.

i) Let Γ be a discrete group (it need not be finite) and $H = \mathbb{C}\Gamma$ the group algebra of Γ. Let

$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \varepsilon(g) = 1,$$

for all $g \in \Gamma$ and linearly extend them to H. Then it is easy to check that $(H, \Delta, \varepsilon, S)$ is a cocommutative Hopf algebra. It is commutative if and only if Γ is commutative.

ii) Let $\mathfrak{g}$ be a Lie algebra and $H = U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Using the universal property of $U(\mathfrak{g})$ one checks that there are uniquely defined algebra homomorphisms $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $\varepsilon : U(\mathfrak{g}) \to \mathbb{C}$ and an anti-algebra map $S : U(\mathfrak{g}) \to U(\mathfrak{g})$, determined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad \text{and} \quad S(X) = -X,$$

for all $X \in \mathfrak{g}$. One then checks easily that $(U(\mathfrak{g}), \Delta, \varepsilon, S)$ is a cocommutative Hopf algebra. It is commutative if and only if $\mathfrak{g}$ is an abelian Lie algebra, in which case $U(\mathfrak{g}) = S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$.

iii) (Compact groups) Let G be a compact topological group. A continuous function $f : G \to \mathbb{C}$ is called *representable* if the set of left translations of f by all elements of G forms a finite dimensional subspace of $C(G)$. It is easy to see that the set of representable functions, $H = \mathrm{Rep}\,(G)$, is a subalgebra of $C(G)$. Let $m : G \times G \to G$ denote the multiplication of G and $m^* : C(G \times G) \to C(G)$, $m^*f(x,y) = f(xy)$, denote its dual map. One checks that if f is representable, then

$$m^*f \in \mathrm{Rep}\,(G) \otimes \mathrm{Rep}\,(G) \subset C(G \times G).$$

Let e denote the identity of G. The formulas

$$\Delta f = m^*f, \quad \varepsilon f = f(e), \quad \text{and} \;\; (Sf)(g) = f(g^{-1}),$$

define a Hopf algebra structure on *Rep*(G). Alternatively, one can describe $\mathrm{Rep}\,(G)$ as the linear span of *matrix coefficients* of all finite dimensional complex representations of G. By the *Peter-Weyl Theorem,* $\mathrm{Rep}(G)$ is a dense subalgebra of $C(G)$. This algebra is finitely generated (as an algebra) if and only if G is a Lie group.

iv) (Affine group schemes) The coordinate ring $H = \mathbb{C}[G]$ of an affine algebraic group G is a commutative Hopf algebra. The maps Δ, ε, and S are the duals of the multiplication, unit, and inversion maps of G, respectively. More generally, an *affine group scheme* over a commutative ring k is a commutative Hopf algebra over k. Given such a Hopf algebra H, it is easy to see that for any commutative k-algebra A, the set $\mathrm{Hom}_{Alg}\,(H, A)$ is a group under convolution product and $A \mapsto \mathrm{Hom}_{Alg}\,(H, A)$ is a functor from the category $ComAlg_k$ of commutative k-algebras to the category of groups. Conversely, any *representable functor* $ComAlg_k \to Groups$ is represented by a, unique up to isomorphism, commutative k-Hopf algebra. Thus the category of affine group schemes is equivalent to the category of representable functors $ComAlg_k \to Groups$.

Example 4.3. *(compact quantum groups)* A prototypical example is Woronowicz's $SU_q(2)$, for $0 < q \leq 1$. As a C^*-algebra it is the unital C^*-algebra generated by α and β subject to the relations

$$\beta\beta^* = \beta^*\beta,\;\; \alpha\beta = q\beta\alpha,\;\; \alpha\beta^* = q\beta^*\alpha,\;\; \alpha\alpha^* + q^2\beta^*\beta = \alpha^*\alpha + \beta^*\beta = I.$$

Notice that these relations amount to saying that

$$U = \begin{pmatrix} \alpha & q\beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

is unitary, i.e. $UU^* = U^*U = I$. Its coproduct and antipode are defined by

$$\Delta \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

$$S(\alpha) = \alpha^*, \;\; S(\beta) = -q^{-1}\beta^*, \;\; S(\beta^*) = -q\beta, \;\; S(\alpha^*) = \alpha.$$

Notice that the coproduct is only defined on the algebra $\mathcal{O}(SU_q(2))$ of matrix elements on the quantum group, and its extension to $C(SU_q(2))$ lands in the completed tensor product

$$\Delta : C(SU_q(2)) \longrightarrow C(SU_q(2)) \hat{\otimes} C(SU_q(2)).$$

At $q = 1$ we obtain the algebra of continuous functions on $SU(2)$. We refer to [88] for a survey of compact and locally compact quantum groups.

Example 4.4. (Connes-Moscovici Hopf algebras) A very important example for noncommutative geometry and its applications to transverse geometry and number theory is the family of *Connes-Moscovici Hopf algebras* $\mathcal{H}_n$ for $n \geq 0$ [39, 40, 41]. They are deformations of the group $G = \text{Diff}\,(\mathbb{R}^n)$ of diffeomorphisms of $\mathbb{R}^n$ and can also be thought of as deformations of the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$. These algebras appeared for the first time as quantum symmetries of transverse frame bundles of codimension n foliations. We briefly treat the case $n = 1$ here. The main features of $\mathcal{H}_1$ stem from the fact that the group $G = \text{Diff}\,(\mathbb{R}^n)$ has a *factorization* of the form

$$G = G_1 G_2,$$

where G_1 is the subgroup of diffeomorphisms φ that satisfy

$$\varphi(0) = 0, \quad \varphi'(0) = 1,$$

and G_2 is the $ax+b$- group of affine diffeomorphisms. We introduce two Hopf algebras corresponding to G_1 and G_2 respectively. Let F denote the Hopf algebra of polynomial functions on the pro-unipotent group G_1. It can also be defined as the *continuous dual* of the enveloping algebra of the Lie algebra of G_1. It is a commutative Hopf algebra generated by the Connes-Moscovici coordinate functions δ_n, $n = 1, 2, \ldots$, defined by

$$\delta_n(\varphi) = \frac{d^n}{dt^n}(\log(\varphi'(t))|_{t=0}.$$

The second Hopf algebra, U, is the universal enveloping algebra of the Lie algebra $\mathfrak{g}_2$ of the $ax+b$-group. It has generators X and Y and one relation $[X,Y]=X$.

The factorization $G=G_1G_2$ defines a *matched pair* of Hopf algebras consisting of F and U. More precisely, The Hopf algebra F has a right U-module algebra structure defined by

$$\delta_n(X)=-\delta_{n+1}, \quad \text{ and } \ \delta_n(Y)=-n\delta_n.$$

The Hopf algebra U, on the other hand, has a left F-comodule coalgebra structure via

$$X\mapsto 1\otimes X+\delta_1\otimes X, \quad \text{ and } \ Y\mapsto 1\otimes Y.$$

One can check that they are a matched pair of Hopf algebras in the sense of G.I. Kac and Majid [94] and the resulting bicrossed product Hopf algebra

$$\mathcal{H}_1=F\bowtie U$$

is the Connes-Moscovici Hopf algebra $\mathcal{H}_1$. (See [39] for a slightly different approach and fine points of the proof.)

Thus $\mathcal{H}_1$ is the universal Hopf algebra generated by $\{X,\,Y,\,\delta_n;\ n=1,2,\cdots\}$ with relations

$$[Y,\,X]=X,\ \ [Y,\,\delta_n]=n\delta_n,\ \ [X,\,\delta_n]=\delta_{n+1},\ \ [\delta_k,\,\delta_l]=0,$$

$$\Delta Y=Y\otimes 1+1\otimes Y,\quad \Delta\delta_1=\delta_1\otimes 1+1\otimes\delta_1,$$

$$\Delta X=X\otimes 1+1\otimes X+\delta_1\otimes Y,$$

$$S(Y)=-Y,\ \ S(X)=-X+\delta_1Y,\ \ S(\delta_1)=-\delta_1,$$

for $n,k,l=1,2,\cdots$.

Another recent point of interaction between Hopf algebras and noncommutative geometry is the work of Connes and Kreimer in renormalization schemes of quantum field theory. We refer to [29, 30, 31, 34, 36, 35] and references therein for this fascinating new subject.

5 Topological K-theory

The topological K-theory of spaces and its main theorem, *the Bott periodicity theorem*, can be extended to noncommutative Banach algebras. Of all topological invariants of spaces, K-theory has the distinct feature that it is the easiest to extend to noncommutative spaces. Moreover, on a large class of C^*-algebras the theory can be characterized by a few simple axioms. In the next section we take up the question of Chern character in noncommutative geometry. It is to address this and similar questions that cyclic cohomology and Connes' Chern character map enter the game.

K-theory was first introduced by Grothendieck in 1958 in his extension of the Riemann-Roch theorem to algebraic varieties. The isomorphism classes of bounded complexes of coherent sheaves on a variety X form an abelian monoid and the group that they generated was called $K_0(X)$. Soon after, Atiyah and Hirzebruch realized that in a similar fashion complex vector bundles over a compact space X define a group $K^0(X)$ and, moreover, using standard methods of algebraic topology, one obtains a *generalized cohomology theory* for spaces in this way. Bott's periodicity theorem for homotopy groups of stable unitary groups immediately implies that the new functor is 2-periodic. By the mid-1970's it was clear to operator algebraists that topological K-theory and Bott periodicity theorem can be extended to all Banach algebras. Our references for this section include [6, 57, 24].

5.1 The K_0 functor

Since the definition of $K_0(A)$ depends only on the underlying ring structure of A and makes sense for any ring, we shall define $K_0(A)$ for any ring A. Let A be a unital noncommutative ring. A right A-module P is called *projective* if it is a direct summand of a free module, i.e. there exists a right A-module Q such that

$$P \oplus Q \simeq A^I.$$

Equivalently, P is projective if and only if any short exact sequence of right A-modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

splits. Let $\mathcal{P}(A)$ denote the set of isomorphism classes of finitely generated projective right A-modules. Under the operation of direct sum, $\mathcal{P}(A)$ is an abelian monoid. The group $K_0(A)$ is, by definition, the universal group generated by the monoid $\mathcal{P}(A)$. Thus elements of $K_0(A)$ can be written as $[P]-[Q]$ for P, $Q \in \mathcal{P}(A)$, with $[P]-[Q]=[P']-[Q']$ if and only if there is an $R \in \mathcal{P}(A)$ such that $P \oplus Q' \oplus R \simeq P' \oplus Q \oplus R$.

A unital ring homomorphism $f : A \to B$ defines a map (*base change*) $f_* : \mathcal{P}(A) \to \mathcal{P}(B)$ by

$$f_*(P) = P \otimes_A B$$

where the left A-module structure on B is induced by f. This map is clearly additive and hence induces an additive map

$$f_* : K_0(A) \to K_0(B).$$

This shows that $A \to K_0(A)$ is a functor.

We need to define K_0 of non-unital rings. Let A^+ be the *unitization* of a non-unital ring A. By definition, $A^+ = A \oplus \mathbb{Z}$ with multiplication $(a, m)(b, n) = (ab + na + mb, mn)$ and unit element $(0, 1)$. A non-unital ring map $f : A \to B$ clearly induces a unital ring map $f^+ : A^+ \to B^+$ by $f^+(a, n) = (f(a), n)$. The canonical morphism $A^+ \to \mathbb{Z}$, sending $(a, n) \to n$, is unital and we define $K_0(A)$ as the kernel of the induced map $K_0(A^+) \to K_0(\mathbb{Z})$. If A is already unital then the surjection $A^+ \to \mathbb{Z}$ splits and one shows that the two definitions coincide.

The first important result about K_0 is its *half-exactness* (cf., for example, [6] for a proof): for any exact sequence of rings

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0,$$

the induced sequence

$$K_0(I) \longrightarrow K_0(A) \longrightarrow K_0(A/I) \tag{19}$$

is *exact in the middle*. Simple examples show that exactness at the left and right ends can fail and in fact the extent to which they fail is measured by higher K-groups as we define them in the next section.

Remark 4. *When A is commutative, the tensor product $P \otimes_A Q$ of A-modules is well defined and is an A-module again. It is finite and projective if P and Q are finite and projective. This operation turns $K_0(A)$ into a commutative ring. In general, for noncommutative rings no such multiplicative structure exists on $K_0(A)$.*

There is an alternative description of $K_0(A)$ in terms of idempotents in matrix algebras over A that is often convenient. An idempotent $e \in M_n(A)$ defines a right A-module map

$$e : A^n \longrightarrow A^n$$

by left multiplication by e. Let $P_e = eA^n$ be the image of e. The relation

$$A^n = eA^n \oplus (1 - e)A^n$$

shows that P_e is a finite projective right A-module. Different idempotents can define isomorphic modules. This happens, for example, if e and f are *equivalent idempotents* (sometimes called *similar*) in the sense that

$$e = ufu^{-1}$$

for some invertible $u \in GL(n, A)$. Let $M(A) = \cup M_n(A)$ be the direct limit of matrix algebras $M_n(A)$ under the embeddings $M_n(A) \to M_{n+1}(A)$ defined by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Similarly let $GL(A)$ be the direct limit of the groups $GL(n, A)$. It acts on $M(A)$ by conjugation.

Definition 5.1. *Two idempotents $e \in M_k(A)$ and $f \in M_l(A)$ are called stably equivalent if their images in $M(A)$ are equivalent under the action of $GL(A)$.*

The following is easy to prove and answers our original question:

Lemma 5.1. *The projective modules P_e and P_f are isomorphic if and only if the idempotents e and f are stably equivalent.*

Let $\text{Idem}(M(A))/GL(A)$ denote the set of stable equivalence classes of idempotents over A. This is an abelian monoid under the operation

$$(e,\, f) \mapsto e \oplus f := \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

It is clear that any finite projective module is of the type P_e for some idempotent e. In fact writing $P \oplus Q \simeq A^n$, one can let e be the idempotent corresponding to the projection map $(p, q) \mapsto (p, 0)$. These observations prove the following lemma:

Lemma 5.2. *For any unital ring A, the map $e \mapsto P_e$ defines an isomorphism of monoids*

$$\mathit{Idem}(M(A))/GL(A) \simeq \mathcal{P}(A).$$

Given an idempotent $e = (e_{ij}) \in M_n(A)$, its image under a homomorphism $f : A \to B$ is the idempotent $f_*(e) = (f(e_{ij}))$. This is our formula for $f_* : K_0(A) \to K_0(B)$ in the idempotent picture of K-theory.

For a Banach algebra A, $K_0(A)$ can be described in terms of connected components of the space of idempotents of $M(A)$ under its inductive limit topology (a subset $V \subset M(A)$ is open in the inductive limit topology if and only if $V \cap M_n(A)$ is open for all n). It is based on the following important observation: Let e and f be idempotents in a unital Banach algebra A and assume $\|e-f\| < 1/\|2e-1\|$. Then $e \sim f$. In fact with

$$v = (2e-1)(2f-1)+1$$

and $u = \frac{1}{2}v$, we have $ueu^{-1} = f$. To see that u is invertible note that $\|u-1\| < 1$. One consequence of this fact is that if e and f are in the same path component of the space of idempotents in A, then they are equivalent. As a result we have, for any Banach algebra A, an isomorphism of monoids

$$\mathcal{P}(A) \simeq \pi_0(\text{Idem}\,(M(A))),$$

where π_0 is the functor of path components.

For C^*-algebras, instead of idempotents it suffices to consider only the *projections*. A projection is a self-adjoint idempotent ($p^2 = p = p^*$). The reason is that every idempotent in a C^*-algebra is similar to a projection [6]: let e be an idempotent and set $z = 1+(e-e^*)(e^*-e)$. Then z is invertible and positive and one shows that $p = ee^*z^{-1}$ is a projection and is similar to e.

Exercise 5.1. *Show that the set of projections of a C^*-algebra is homotopy equivalent (in fact a retraction) of the set of idempotents.*

Let $\text{Proj}(M(A))$ denote the space of projections in $M(A)$. We have established isomorphisms of monoids

$$\mathcal{P}(A) \simeq \pi_0(\text{Idem}\,(M(A))) \simeq \pi_0(\text{Proj}\,(M(A))).$$

From the above homotopic interpretation of K_0 for Banach algebras, its *homotopy invariance* and *continuity* easily follows. Let $f, g : A \to B$ be continuous homomorphisms between Banach algebras. They are called *homotopic* if there exists a continuous homomorphism

$$F : A \to C([0,1],\, B)$$

such that $f = e_0F$ and $g = e_1F$, where e_0, $e_1 : C([0,1], B) \to B$ are the evaluations at 0 and 1 maps. Now by our definition of K_0 via π_0, it is clear that $e_{0*} = e_{1*} : K_0(C([0,\,1],\, B)) \to K_0(B)$ and hence

$$f_* = g_* : K_0(A) \to K_0(B),$$

which shows that K_0 is homotopy invariant.

In a similar way one can also show that K_0 preserves direct limits of Banach algebras: if $A = \varinjlim (A_i, f_{ij})$ is an inductive limit of Banach algebras then $K_0(A) = \varinjlim (K_0(A_i), f_{ij*})$. This property is referred to as *continuity* of K_0.

In addition to its homotopy invariance and continuity, we collect a couple of other properties of K_0 which hold for all rings:

- *Morita Invariance*: if A and B are Morita equivalent unital rings then $K_0(A) \simeq K_0(B)$. This is clear since Morita equivalent rings, by definition, have equivalent categories of modules and the equivalence can be shown to preserve the categories of finite projective modules. Therefore $\mathcal{P}(A) \simeq \mathcal{P}(B)$.

- *Additivity*: $K_0(A \oplus B) \simeq K_0(A) \oplus K_0(B)$ for unital rings A and B. This is a consequence of $\mathcal{P}(A \oplus B) \simeq \mathcal{P}(A) \oplus \mathcal{P}(B)$, which is easy to prove.

Example 5.1. *(commutative algebras)* For $A = C_0(X)$ we have

$$K_0(C_0(X)) \simeq K^0(X), \tag{20}$$

where K^0 is the topological K-theory of spaces. The reason for this is the *Swan theorem* [118] (cf. also Serre [117] for the corresponding result in the context of affine varieties), according to which for any compact Hausdorff space X the category of finite projective $C(X)$-modules is equivalent to the category of complex vector bundles on X. The equivalence is via the *global section functor*. Given a vector bundle $p : E \to X$, let

$$P = \Gamma(E) = \{s : X \to E;\ ps = \mathrm{id}_X\}$$

be the set of all continuous global sections of E. It is clear that under fiberwise scalar multiplication and addition, P is a $C(X)$-module. Using the local triviality of E and a partition of unity one shows that there is a vector bundle F on X such that $E \oplus F$ is a trivial bundle, or, equivalently,

$$P \oplus Q \simeq A^n,$$

where Q is the module of global sections of F. This shows that P is finite and projective. It is not difficult to show that all finite projective modules are obtained in this way and Γ is an equivalence of categories (see exercise below). Now the rest of the proof of (20) is clear since $K^0(X)$ is, by definition, the universal group defined by the monoid of complex vector bundles on X.

Exercise 5.2. *Given a finite projective $C(X)$-module P, let Q be a $C(X)$-module such that $P \oplus Q \simeq A^n$, for some integer n. Let $e : A^n \to A^n$ be the right A-linear projection map $(p, q) \mapsto (p, 0)$. It is obviously an idempotent in $M_n(C(X))$. Define the vector bundle E to be the image of e:*

$$E = \{(x, v);\ e(x)v = v, \text{for all } x \in X,\ v \in \mathbb{C}^n\} \subset X \times \mathbb{C}^n.$$

Now it is easily seen that $\Gamma(E) \simeq P$. With some more work it is shown that the functor Γ is full and faithful and hence defines an equivalence of categories [118].

Motivated by the Serre-Swan theorem, one usually thinks of finite projective modules over noncommutative algebras as *noncommutative vector bundles.*

Example 5.2. Here is a nice example of a projection in $M_2(C_0(\mathbb{R}^2)^+)$. Let

$$p = \frac{1}{1 + |z|^2} \begin{pmatrix} |z|^2 & z \\ \bar{z} & 1 \end{pmatrix}$$

It does not define an element of $K_0(C_0(\mathbb{R}^2))$ since $p(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$. The difference

$$\beta = p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is however the generator of $K_0(C_0(\mathbb{R}^2)) \simeq \mathbb{Z}$. This is a consequence of the Bott periodicity theorem that we recall later in this section. β is called the *Bott generator* and p the *Bott projection*. Now we have $C(S^2) = C_0(\mathbb{R}^2)^+$. Let [1] denote the class of the trivial line bundle on S^2. It follows that [1] and β form a basis for $K_0(C(S^2)) \simeq K^0(S^2) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Examples 5.1. i) $K_0(\mathbb{C}) \simeq \mathbb{Z}$. In fact any finite projective module over $\mathbb{C}$ is simply a finite dimensional complex vector space whose isomorphism class is determined by its dimension. This shows that $\mathcal{P}(\mathbb{C}) \simeq \mathbb{N}$, from which our claim follows.
ii) By Morita invariance, we then have $K_0(M_n(\mathbb{C})) \simeq \mathbb{Z}$.
iii) The algebra of compact operators $\mathcal{K} = \mathcal{K}(H)$ on a separable Hilbert space is the direct limit of matrix algebras. Using the continuity of K_0 we conclude that $K_0(\mathcal{K}) \simeq \mathbb{Z}$.
iv) On the other hand, $K_0(\mathcal{L}) = 0$ where $\mathcal{L} = \mathcal{L}(H)$ is the algebra of bounded operators on an infinite dimensional Hilbert space. To prove this let $e \in M_n(\mathcal{L}) = \mathcal{L}(H^n)$ be an idempotent. The idempotents $e \oplus I$ and $0 \oplus I$ are

equivalent in $M_{2n}(\mathcal{L}) = \mathcal{L}(H^{2n})$ since both have infinite dimensional range. This shows that $[e] = 0$. Notice that for commutative algebras $A = C(X)$, $K_0(A)$ always contains a copy of $\mathbb{Z}$.

Consider the exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L}(H) \longrightarrow \mathcal{C} \longrightarrow 0,$$

where $\mathcal{C} := \mathcal{L}(H)/\mathcal{K}$ is called the *Calkin algebra*. From the above example we see that the corresponding K_0 sequence (19) fails to be exact on the left.

Example 5.3. *(The trace map)* Let A be a unital algebra, V be a vector space and $\tau : A \to V$ be a trace on A. Then τ induces an additive map

$$\tau : K_0(A) \to V$$

as follows. Given an idempotent $e = (e_{ij}) \in M_k(A)$, let

$$\tau([e]) = \sum_{i=1}^{k} \tau(e_{ii}). \tag{21}$$

Using the trace property of τ, one has $\tau(ueu^{-1}) = \tau(e)$. The additivity property $\tau(e \oplus f) = \tau(e) + \tau(f)$ is clear. This shows that (21) is a well defined map on K-theory.

Alternatively, given a finite projective A-module P, let

$$\tau([P]) = \mathrm{Tr}\,(\mathrm{id}_P)$$

where

$$\mathrm{Tr} : \mathrm{End}_A\,(P) \simeq P^* \otimes_A P \to V,$$

the *Hattori-Stallings trace*, is the natural extension of τ defined by

$$\mathrm{Tr}\,(f \otimes \xi) = \tau(f(\xi))$$

for all $f \in P^* = \mathrm{Hom}_A(P, A)$ and $\xi \in P$. This is the simplest example of a pairing between cyclic cohomology and K-theory (Connes' Chern character), to be defined in full generality later in these notes. Notice that if $\tau([e]) \neq 0$ then we can conclude that $[e] \neq 0$ in $K_0(A)$. This is often very useful in applications.

Example 5.4. *(Hopf line bundle on quantum spheres)* Let $0 < q \leq 1$ be a real number. The algebra $C(S_q^2)$ of functions on the *standard Podleś quantum sphere* S_q^2 is, by definition, the unital C^*-algebra generated by elements a and b with relations

$$aa^* + q^{-4}b^2 = 1,\ a^*a + b^2 = 1,\ ab = q^{-2}ba,\ a^*b = q^2ba^*.$$

The quantum analogue of the Dirac (or Hopf) monopole line bundle over S^2 is given by the following projection in $M_2(C(S_q^2))$ [14]:

$$\mathbf{e}_q = \frac{1}{2}\begin{bmatrix} 1+q^{-2}b & qa \\ q^{-1}a^* & 1-b \end{bmatrix}.$$

It can be directly checked that $\mathbf{e}_q^2 = \mathbf{e}_q = \mathbf{e}_q^*$. For $q = 1$, $C(S_1^2) = C(S^2)$ and the corresponding projection defines the Hopf line bundle on S^2. We refer to the article of Landi and van Suijlekom in this volume [91] for a survey of noncommutative bundles and instantons in general.

Example 5.5. *($K_0(A_\theta)$)* We shall see later in this Section, using the Pimsner-Voiculescu exact sequence, that

$$K_0(A_\theta) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

One generator is the class of the trivial idempotent [1]. Notice that $[1] \neq 0$ because $\tau(1) = 1 \neq 0$, where $\tau : A_\theta \to \mathbb{C}$ is the canonical trace. When θ is irrational a second generator is given by the *Powers-Rieffel projection* $p \in A_\theta$. The projection p is of the form

$$p = U_2^*\, g(U_1)^* + f(U_1) + g(U_1)\, U_2, \tag{22}$$

where $f, g \in C^\infty(S^1)$. By $g(U_1)$ we mean of course $\sum \hat{g}_n U_1^n$ where $\hat{g}_n$ are the Fourier coefficients of g. To fulfill the projection condition $p^2 = p = p^*$, f and g must satisfy certain relations (cf. [24]) one of which implies that $\int_0^1 f(t)dt = \theta$. There are many such solutions but their corresponding projections are all homotopic and hence define the same class in $K_0(A_\theta)$. Now in (22), the only contribution to the trace $\tau(p)$ comes from the constant term of the middle term and hence

$$\tau(p) = \int_0^1 f(t)dt = \theta.$$

It follows that the range of the trace map $\tau : K_0(A_\theta) \to \mathbb{C}$ is in fact the subgroup $\mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R}$.

Example 5.6. *(Relation with Fredholm operators)* The space of Fredholm operators, under the norm topology, is a *classifying space* for the K-theory of spaces. Let $[X,\ \mathcal{F}]$ denote the set of homotopy classes of continuous maps from a compact space X to the space of Fredholm operators $\mathcal{F}$ on an infinite dimensional Hilbert space. Such continuous maps should be thought of as families of Fredholm operators parameterized by X. By a theorem of Atiyah and Jänich (cf. [3] for a proof and a generalization) there exists a well defined *index map* index : $[X,\ \mathcal{F}] \to K^0(X)$ which induces an isomorphism

$$\text{index} : [X,\, \mathcal{F}] \simeq K^0(X). \tag{23}$$

Its definition is as follows. Given a Fredholm family $T : X \to \mathcal{F}$, if $\dim \mathrm{Ker}(T_x)$ and $\dim \mathrm{Coker}(T_x)$ are locally constant functions of x, then the family of finite dimensional subspaces $\mathrm{Ker}(T_x)$ and $\mathrm{Coker}(T_x)$, $x \in X$, define vector bundles denoted Ker (T) and Coker (T) on X, and their difference

$$\text{index}\,(T) := \mathrm{Ker}\,(T) - \mathrm{Coker}\,(T)$$

is the K-theory class associated to T. For a general family the dimensions of the subspaces $\mathrm{Ker}(T_x)$ and $\mathrm{Coker}(T_x)$ may be discontinuous, but one shows that it can always be continuously deformed to a family where these dimensions are continuous .

The isomorphism (23) is fundamental. For example, the index of a family of elliptic operators which are fiberwise elliptic, by this result, is an element of the K-theory of the base manifold (and not an integer). In noncommutative geometry, for example in transverse index theory on foliated manifolds, the parameterizing space X is highly singular and is replaced by a noncommutative algebra A. The above analytic index map (23), with values in $K_0(A)$, still can be defined and its identification is one of the major problems of noncommutative index theory [24].

5.2 The higher K-functors

Starting with K_1, algebraic and topological K-theory begin to differ from each other. In this section we shall first briefly indicate the definition of algebraic K_1 of rings and then define, for Banach algebras, a sequence of functors K_n for $n \geq 1$.

For a unital ring A, let $GL(A)$ be the *direct limit* of groups $GL(n, A)$ of invertible $n \times n$ matrices over A where the direct system $GL(n, A) \hookrightarrow GL(n+1, A)$ is defined by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$. The algebraic K_1 of A is defined as the abelianization of $GL(A)$:

$$K_1^{\text{alg}}(A) := GL(A)/[GL(A),\, GL(A)],$$

where $[\,,\,]$ denotes the commutator subgroup.

Applied to $A = C(X)$, this definition does not reproduce the topological $K^1(X)$. For example for $A = \mathbb{C} = C(\text{pt})$ we have $K_1^{\text{alg}}(\mathbb{C}) \simeq \mathbb{C}^\times$ where the isomorphism is induced by the determinant map

$$\det :\ GL(\mathbb{C}) \to \mathbb{C}^\times,$$

while $K^1(\text{pt}) = 0$. It turns out that, to obtain the right result, one should divide $GL(A)$ by a bigger subgroup, i.e. by the *closure* of its commutator subgroup. This works for all Banach algebras and will give the right definition of topological K_1. A better approach however is to define the higher K groups in terms of K_0 and the *suspension functor*.

The *suspension* of a Banach algebra A is the Banach algebra

$$SA = C_0(\mathbb{R},\ A)$$

of continuous functions from $\mathbb{R}$ to A vanishing at infinity. Notice that for $A = C(X)$, SA is isomorphic to the algebra of continuous functions on $X \times [0,1]$ vanishing on $X \times \{0,1\}$. It follows that $SA^+ \simeq C(\Sigma X)$, where ΣX is the *suspension* of X obtained by collapsing $X \times \{0,1\}$ to a point in $X \times [0,1]$.

Definition 5.2. *The higher topological K groups of a Banach algebra A are the K_0 groups of the iterated suspensions of A:*

$$K_n(A) = K_0(S^n A), \qquad n \geq 1.$$

This is a bit too abstract. It is better to think of higher K groups of A as higher homotopy groups of $GL(A)$. To do this we need the following Lemma. Let $GL^\circ(n, A)$ denote the connected component of the identity in $GL(n, A)$.

Lemma 5.3. *i) Let $f : A \to B$ be a surjective unital homomorphism of unital Banach algebras. Then $f : GL^\circ(1, A) \to GL^\circ(1, B)$ is surjective.*
ii) For any $u \in GL(n, A)$, $\quad \text{diag}\,(u, u^{-1}) \in GL^\circ(2n,\, A)$.

To prove the first statement notice that the group generated by the exponentials e^y, $y \in B$, coincides with $GL^\circ(1, B)$. Now since f is surjective we have $e^y = e^{f(x)} = f(e^x)$ which implies that any product of exponentials is in the image of f. To prove the second statement we can use the path

$$z_t = \operatorname{diag}(u,\, u^{-1}) u_t \operatorname{diag}(u,\, u^{-1})\, u_t^{-1},$$

where

$$u_t = \begin{pmatrix} cos\frac{\pi}{2}t & -sin\frac{\pi}{2}t \\ sin\frac{\pi}{2}t & cos\frac{\pi}{2}t \end{pmatrix},$$

connecting $\operatorname{diag}(u,\, u^{-1})$ to $\operatorname{diag}(1,\, 1)$.

We can now show that

$$K_1(A) \simeq \pi_0(GL(A)), \tag{24}$$

the group of connected components of $GL(A)$. To see this let $u \in GL(n, A)$. Then, by the above lemma, there is a path α_t in $GL(2n,\, A)$ connecting diag $(u,\, u^{-1}) \in GL^0(2n,\, A)$ to I_{2n}. Let $p_n = \operatorname{diag}(I_n,\, 0)$. Then $e_t = \alpha_t p_n \alpha_t^{-1}$ is an idempotent in $(SA)^+$ and the map $[u] \mapsto [e_t] - [p_n]$ implements the isomorphism in (24).

Now since $\pi_n(GL(A)) \simeq \pi_{n-1}(GL(SA))$, using (24) we obtain

$$K_n(A) \simeq \pi_{n-1}(GL(A)). \tag{25}$$

Example 5.7. Let $A = \mathbb{C}$. Then by (25), we have

$$K_n(\mathbb{C}) \simeq \pi_{n-1}(GL(\mathbb{C})) \simeq \pi_{n-1}(U(\mathbb{C})).$$

By the Bott periodicity theorem, the homotopy groups of the stable unitary groups $U(\mathbb{C})$ are periodic, i.e. for all n we have

$$\pi_n(U(\mathbb{C})) \simeq \pi_{n+2}(U(\mathbb{C})),$$

and hence

$$K_{n+2}(\mathbb{C}) \simeq K_n(\mathbb{C}).$$

This is the simplest instance of the general Bott periodicity theorem for the K-theory of Banach algebras to be discussed in the next section.

Example 5.8. For any Banach algebra A, we have a surjection

$$K_1^{alg}(A) \twoheadrightarrow K_1(A).$$

Using (24), this follows from

$$GL^{\circ}(A) = \overline{[GL(A),\, GL(A)]},$$

which we leave as an exercise. For $A = \mathbb{C}$, we have $K_1^{\text{alg}}(\mathbb{C}) \simeq \mathbb{C}^{\times}$ with the isomorphism given by the determinant map $\det : GL(\mathbb{C}) \to \mathbb{C}^{\times}$, while $K_1(\mathbb{C}) = 0$ (see the next example).

Example 5.9. (i) Since $GL(n,\mathbb{C})$ is connected for all n, we have $K_1(\mathbb{C}) = 0$. Similarly, using polar decomposition, one shows that for any von Neumann algebra A, $GL(n,A)$ is connected for all n and hence $K_1(A) = 0$.
(ii) By Morita invariance we have $K_1(M_n(\mathbb{C})) = 0$.
(iii) Since the algebra $\mathcal{K}$ of compact operators is the direct limit of finite matrices, by continuity we have $K_1(\mathcal{K}) = 0$.

Exercise 5.3. Starting from the definitions, show that for $i = 0, 1$

$$K_i(C(S^1)) \simeq \mathbb{Z}.$$

Under these isomorphisms a projection $e : S^1 \to M_n(\mathbb{C})$ is sent to $\mathrm{tr}(e) \in \mathbb{Z}$ and an invertible $u : S^1 \to GL(n,\mathbb{C})$ is sent to the winding number of $\det(u(z))$.

The suspension functor is *exact* in the sense that for any exact sequence of Banach algebras $0 \to I \to A \to A/I \to 0$ the sequence

$$0 \longrightarrow SI \longrightarrow SA \longrightarrow S(A/I) \longrightarrow 0$$

is exact too. Coupled with the half exactness of K_0, we conclude that the sequences

$$K_n(I) \longrightarrow K_n(A) \longrightarrow K_n(A/I)$$

are exact in the middle for all $n \geq 0$.

One can splice these half exact sequences into a long exact sequence

$$\cdots \to K_n(I) \to K_n(A) \to K_n(A/I) \to \cdots \to K_0(I) \to K_0(A) \to K_0(A/I) \tag{26}$$

To do this it suffices to show that there exists a *connecting homomorphism*

$$\partial : K_1(A) \longrightarrow K_0(I)$$

which renders the sequence

$$K_1(I) \to K_1(A) \to K_1(A/I) \xrightarrow{\partial} K_0(I) \to K_0(A) \to K_0(A/I) \tag{27}$$

exact. It is sometimes called the *(generalized) index map* since for the Calkin extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L}(H) \longrightarrow \mathcal{C} \longrightarrow 0 \tag{28}$$

it coincides with the index of Fredholm operators

$$\partial = \mathrm{index} : K_1(\mathcal{C}) \longrightarrow K_0(\mathcal{K}) \simeq \mathbb{Z}.$$

Let $u \in GL_n(A/I)$ and let $w \in GL_{2n}(A)$ be a lift of diag (u, u^{-1}). Define

$$\partial([u]) = [wp_n w^{-1}] - [p_n] \in K_0(I),$$

where the projection $p_n = \text{diag}(I_n, 0)$. It can be shown that this map is well defined and (27) is exact.

Example 5.10. Let A be a unital C^*-algebra. Using polar decomposition one shows that any invertible in $GL(n, A)$ is homotopic to a unitary. Given such a unitary u, we can find a partial isometry v in $M_{2n}(A)$ lifting diag$(u, 1)$. Now the unitary

$$w = \begin{pmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{pmatrix}$$

lifts diag $((u, 1), (u^{-1}, 1))$ and hence

$$\partial([u]) = [wp_{2n}w^{-1}] - [p_{2n}] = [1 - v^*v] - [1 - vv^*].$$

For the Calkin extension (28) this maps sends an invertible in the Calkin algebra $\mathcal{C}$, i.e. a Fredholm operator, to its Fredholm index in $K_0(\mathcal{K}) \simeq \mathbb{Z}$.

Remark 5. *A more conceptual way to get the long exact sequence* (26) *would be to derive it from the homotopy exact sequence of a fibration.*

5.3 Bott periodicity theorem

Homotopy invariance, Morita invariance, additivity and the exact sequence (26) are essential features of topological K-theory. The deepest result of K-theory, however, at least in the commutative case, is the Bott periodicity theorem. It states that there is a natural isomorphism between K_0 and K_2. The isomorphism is given by the *Bott map*

$$\beta : K_0(A) \to K_1(SA).$$

Since $K_1(SA) \simeq \pi_1(GL(A))$ is the homotopy group of the stable general linear group of A, β should somehow turn an idempotent in $M(A)$ into a loop of invertibles in $GL(A)$. We assume A is unital (the general case easily follows). Given an idempotent $e \in M_n(A)$, define a map $u_e : S^1 \to GL(n, A)$ by $u_e(z) = ze + (1 - z)e$. It defines a loop in $GL(A)$ based at 1, whose homotopy class is an element of $\pi_1(GL(A)) \simeq K_1(SA)$. Now the *Bott map* $\beta : K_0(A) \longrightarrow K_1(SA)$ is defined by

$$\beta([e] - [f]) = u_e u_f^{-1}.$$

Notice that, since u_e is a group homomorphism the additivity of β follows.

Theorem 5.1. *(Bott periodicity theorem) For a complex Banach algebra A the Bott map*

$$\beta : K_0(A) \to K_2(A)$$

is a natural isomorphism.

It follows that for all $n \geq 0$,

$$K_n(A) \simeq K_{n+2}(A)$$

and the long exact sequence (26) reduces to a periodic 6-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{i_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/I) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ K_1(A/I) & \xleftarrow[\pi_*]{} & K_1(A) & \xleftarrow[i_*]{} & K_1(I) \end{array}$$

Example 5.11. For $A = \mathbb{C}$ we already knew that $K_0(\mathbb{C}) \simeq \mathbb{Z}$ and $K_1(\mathbb{C}) \simeq 0$. Using Bott periodicity we obtain $K_{2n}(\mathbb{C}) \simeq \mathbb{Z}$, and $K_{2n+1}(\mathbb{C}) \simeq 0$. This is a non-trivial result and in fact of the same magnitude of difficulty as Bott periodicity. Since the spheres S^n are iterated suspensions of a point, we obtain $K^0(S^{2n}) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $K^1(S^{2n}) \simeq 0$, and $K^0(S^{2n+1}) \simeq K^1(S^{2n+1}) \simeq \mathbb{Z}$.

Example 5.12. *(The Toeplitz extension)* The Toeplitz extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0$$

was introduced in Section 3.1. Now the index map $\partial : K_1(C(S^1)) \to K_0(\mathcal{K})$ in the 6-term exact sequence is an isomorphism (this is, more or less, equivalent to the Gohberg-Krein index theorem for Toeplitz operators). Since $K_0(\mathcal{K}) = \mathbb{Z}$, $K_0(\mathcal{K}) = 0$, and $K_i(C(S^1)) = \mathbb{Z}$ for $i = 0, 1$, from the above 6-term exact sequence we deduce that $K_0(\mathcal{T}) \simeq K_1(\mathcal{T}) = \mathbb{Z}$.

5.4 Further results

In the commutative case the properties of homotopy invariance, long exact sequence, and Bott periodicity, suffice to give a good understanding of the K-theory of spaces such as CW complexes. Noncommutative spaces are of course much richer and more complicated. Here we give two further results that have no counterpart in the commutative case. Proofs of both can be found in [6] or in the original articles cited below.

Theorem 5.2. *(Connes' Thom isomorphism [20]). If $\alpha : \mathbb{R} \to Aut(A)$ is a continuous one-parameter group of automorphisms of a C^*-algebra A, then*

$$K_i(A \rtimes_\alpha \mathbb{R}) \simeq K_{1-i}(A), \qquad i = 0, 1.$$

In particular this result shows that the K-theory of $A \rtimes_\alpha \mathbb{R}$ is independent of the action α.

The dimension shift is reminiscent of the dimension shift in the classical Thom isomorphism theorem relating the K-theory with compact support of the total space of a vector bundle with the K-theory of its base. Note that if the action is trivial then the theorem reduces to the Bott periodicity theorem. In fact in this case we have

$$A \rtimes_\alpha \mathbb{R} \simeq A \otimes C_0(\mathbb{R}) \simeq SA.$$

The second result we would like to highlight in this section is the 6-term exact sequence of Pimsner and Voiculescu:

Theorem 5.3. *(Pimsner-Voiculescu exact sequence [106]). For any automorphism $\alpha \in Aut(A)$ of a C^*-algebra A there is a 6-term exact sequence*

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{1-\alpha_*} & K_0(A) & \xrightarrow{i_*} & K_0(A \rtimes_\alpha \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
K_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow[i_*]{} & K_1(A) & \xleftarrow[1-\alpha_*]{} & K_1(A)
\end{array}$$

Example 5.13. *(K-theory of noncommutative tori)* A beautiful application of this result is to the K-theory of the noncommutative torus. We have $A_\theta = C(S^1) \rtimes_\alpha \mathbb{Z}$, where the automorphism α is through rotation by $2\pi\theta$. But α is homotopic to the identity through rotations by $2\pi\theta t$, $t \in [0, 1]$. By homotopy invariance of K_i, we obtain $\alpha_* = 1$. Using $K_i(C(S^1)) \simeq \mathbb{Z}$ for $i = 0, 1$ and a simple diagram chase we conclude from the Pimsner-Voiculescu exact sequence that

$$K_i(A_\theta) \simeq \mathbb{Z} \oplus \mathbb{Z} \quad \text{for } i = 0, 1.$$

Thus it seems that K-theory by itself cannot distinguish the isomorphism class of A_θ for different θ. There is however an extra piece of structure in $K_0(A)$, for A a C^*-algebra, that can be used in this regard. Notice that

$K_0(A)$ is an *ordered group* with its positive cone defined by projections in $M(A)$. Equipped with this extra structure one can then show that $A_{\theta_1} \simeq A_{\theta_2}$ iff $\theta_1 = \theta_2$ or $\theta_1 = 1 - \theta_2$ (cf. [6] and references therein).

5.5 Twisted K-theory

Twisted K-theory has been around for quite some time (cf. Donovon-Karoubi [52], Rosenberg [114]). The recent surge of interest in the subject has to do with both mathematics and high energy physics. In mathematics, a recent result of Freed, Hopkins and Teleman shows that the twisted equivariant K-theory of a compact Lie group is isomorphic to the Verlinde algebra (fusion algebra) of the group [58, 59]. The latter algebra is the algebra of projective representations of the loop group of the group at a fixed *level*. In some semiclassical limits of string theory over a background spacetime X, the strengths of B-fields are elements of $H^3(X, \mathbb{Z})$. When this B-field is non-trivial the topological charges of D-branes are interpreted as elements of twisted K-theory with respect to the twisting defined by B (cf. [10, 11] for a mathematical perspective). A recent comprehensive study of twisted K-theory can be found in Atiyah and Segal's article [3].

The twisting coefficients (local systems) of twisted K-theory are cohomology classes in $H^3(X, \mathbb{Z})$. There are at least two approaches to the subject. One can either extend the definition of K-theory through Fredholm operators and the relation (23) to include twistings as in [3], or one can define the twisted K-theory as the K-theory of a noncommutative algebra as is done in [114]. We shall briefly describe this latter definition.

Let X be a locally compact, Hausdorff, and second countable space. We recall the classification of locally trivial bundles of algebras with fibers isomorphic to the algebra $\mathcal{K} = \mathcal{K}(H)$ of compact operators on an infinite dimensional Hilbert space, and with structure group $\mathrm{Aut}\,(\mathcal{K})$. As we saw in Section 2.4 there is a one to one correspondence between isomorphism classes of such bundles and $H^3(X,\, \mathbb{Z})$. Given such a bundle of algebras $\mathcal{A}$, its Dixmier-Douady invariant

$$\delta(\mathcal{A}) \in H^3(X, \mathbb{Z})$$

is a complete isomorphism invariant of such bundles.

Now given a pair $(X,\, \delta)$ as above, the *twisted K-theory* of X can be defined as the K-theory of the C^*- algebra $A = \Gamma(X,\, \mathcal{A})$ of continuous sections of $\mathcal{A}$ vanishing at infinity:

$$K^i_\delta(X) := K_i(A).$$

There is also an equivariant version of twisted K-theory, denoted by $K_G^\delta(X)$, that is specially important in view of the recent work [58]. The coefficients for this theory are elements of the equivariant cohomology $H_G^3(X, \mathbb{Z})$, where G is a compact Lie group acting on a space X. We refer to [3] for its definition. For simplicity, let G be a compact, connected, simply connected, and simple Lie group. Then the central extensions of the loop group LG of G are characterized by a positive integer k, called the level [107]. For each positive integer k, the positive energy representations of this central extension, up to equivalence, constitute a finite set and we denote by $V_k(G)$ the free abelian group generated by this set. There is a commutative ring structure on this set, corresponding to tensor product of representations.

Now let G act on itself by conjugation ($G = X$). Then the equivariant cohomology $H^3(G, \mathbb{Z})$ is a free group of rank one whose elements we shall denote by integers. The theorem of Freed-Hopkins-Teleman states that, at each level k, the fusion ring $V_k(G)$ is isomorphic to the twisted equivariant K-theory of G:

$$K_G^n(G) \simeq V_k(G),$$

where the integer n can be explicitly defined in terms of $k \geq 0$ and G [58, 59].

Example 5.14. Let $G = SU(2) \simeq S^3$. Then $H^3(G, \mathbb{Z}) \simeq \mathbb{Z}$. For each integer n representing a class in $H^3(G, \mathbb{Z})$ there is a bundle $\mathcal{A}_n$ of algebras of compact operators $\mathcal{K}$ over S^3 obtained by gluing the trivial bundles $S^3_+ \times \mathcal{K}$ and $S^3_- \times \mathcal{K}$ on the upper and lower hemispheres respectively. The gluing is defined by a map $S^2 \to \mathrm{Aut}\,(\mathcal{K})$ of degree n. Let A_n denote the C^*-algebra of continuous sections of $\mathcal{A}_n$. We have then, by definition,

$$K^{i,n}(S^3) = K_i(A_n).$$

The representation ring of $G = SU(2)$ is a polynomial algebra whose generator is the fundamental 2-dimensional representation of G. The twisted equivariant K-theory of $SU(2)$ can be explicitly computed and shown to be a quotient of the representation ring (cf. [58, 59]).

5.6 K-homology

In [2] and a little later and independently in [13], Atiyah and Brown, Douglas and Fillmore proposed theories dual to topological K-theory, using techniques of functional analysis and operator algebras. The cycles for Atiyah's theory are *abstract elliptic operators* (H, F) over $C(X)$ where $H = H^+ \oplus H^-$ is a $\mathbb{Z}_2$-graded Hilbert space, $\pi : C(X) \to \mathcal{L}(H)$ is an even representation of

$C(X)$, and $F : H \to H$ is an even bounded operator with $F^2 - I \in \mathcal{K}(H)$. This data must satisfy the condition

$$[F, \pi(a)] \in \mathcal{K}(H).$$

We see that an abstract elliptic operator is the same as a Fredholm module over $C(X)$ as in definition (4) except that now instead of $F^2 = I$ we have the above condition. The two definitions are however essentially equivalent. In particular the formula

$$\langle (H, F), [e] \rangle = \operatorname{index} F_e^+$$

where the Fredholm operator F_e^+ is defined in (6) defines a pairing between the K-theory of X and abstract elliptic operators on X.

Let $\mathrm{Ext}(A)$ denote the set of isomorphism classes of extensions of a C^*-algebra A of the form

$$0 \longrightarrow \mathcal{K} \longrightarrow E \longrightarrow A \longrightarrow 0.$$

There is a natural operation of addition of extensions which turn $\mathrm{Ext}(A)$ into an abelian monoid. It can be shown that if A is a nuclear C^*-algebra, for example if $A = C(X)$ is commutative, then $\mathrm{Ext}(A)$ is actually a group. There is a pairing

$$K_1(A) \times \mathrm{Ext}(A) \to \mathbb{Z}$$

which is defined as follows. Let δ be the connecting homomorphism in the 6-term exact sequence of an extension $\mathcal{E}$ representing an element of $\mathrm{Ext}\,(A)$, and let $[u] \in K_1\,(A)$. Then

$$\langle [u]\,,\, [\mathcal{E}] \rangle := \delta([u]) \in K_0\,(\mathcal{K}) \simeq \mathbb{Z}.$$

Example 5.15. A simple example of a non-trivial extension is the Toeplitz extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

from Section 3.1. It can be shown that the class of this extension generates the K-homology group $\mathrm{Ext}(C(S^1))$ [13]. A more elaborate example is the pseudodifferential extension of the algebra of functions on the cosphere bundle:

$$0 \longrightarrow \mathcal{K}(L^2(M)) \longrightarrow \Psi^0(M) \xrightarrow{\sigma} C(S^*M) \longrightarrow 0.$$

briefly discussed in Section 3.1. For a comprehensive introduction to K-homology see Higson and Roe's [71].

For noncommutative spaces, the above approach to K-homology works best when the corresponding C^*-algebra is nuclear. For arbitrary C^*-algebras, Kasparov's KK-theory provides a unified approach to both K-theory and K-homology in a bivariant theory (cf. [78] and [6]).

6 Cyclic Cohomology

Let A be an algebra and $e, f \in M_n(A)$ be two idempotents. How can we show that $[e] \neq [f]$ in $K_0(A)$? Here is a simple device that is often helpful in this regard. As we saw in Example (5.3), any trace $\tau : A \longrightarrow \mathbb{C}$ induces an additive map

$$\tau : K_0(A) \longrightarrow \mathbb{C},$$

via the formula

$$\tau([e]) := \sum_{i=1}^{n} e_{ii}.$$

Now if $\tau([e]) \neq \tau([f])$, then, of course, $[e] \neq [f]$.

Exercise 6.1. *Let $A = C(X)$, where X is compact and connected, and let $\tau(f) = f(x)$, for some fixed $x \in X$. Show that if E is a vector bundle on X and $e \in M_n(C(X))$ an idempotent defining E, then $\tau([e]) = dim\,(E_x)$ where E_x is the fibre of E over x. Thus τ simply measures the rank of the vector bundle.*

The topological information hidden in an idempotent is much more subtle than just its 'rank' and in fact traces can only capture zero-dimensional information. To know more about idempotents and K-theory we need higher dimensional analogues of traces. They are called *cyclic cocycles* and their study is the subject of *cyclic cohomology*. As we shall see in this section, cyclic cohomology is the right noncommutative analogue of de Rham homology of currents on smooth manifolds.

Cyclic cohomology was first discovered by Alain Connes [19, 21, 22]. Let us first recall a remarkable subcomplex of the Hochschild complex called the *Connes complex* that was introduced by him for the definition of cyclic cohomology. For an algebra $\mathcal{A}$ let

$$C^n(\mathcal{A}) = \mathrm{Hom}\,(\mathcal{A}^{\otimes(n+1)}, \mathbb{C}), \quad n = 0, 1, \cdots,$$

denote the space of $(n+1)$-linear functionals on $\mathcal{A}$ with values in $\mathbb{C}$. The

Hochschild differential $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ is defined by

$$\begin{aligned}(b\varphi)(a_0, \cdots, a_{n+1}) &= \sum_{i=0}^{n} (-1)^i \varphi(a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\ &\quad +(-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \cdots, a_n).\end{aligned}$$

One checks that $b^2 = 0$. The cohomology of the complex $(C(\mathcal{A}),\, b)$ is, by definition, the Hochschild cohomology of $\mathcal{A}$ (with coefficients in the A-bimodule A^*) and will be denoted by $HH^n(\mathcal{A})$. An n-cochain $\varphi \in C^n(\mathcal{A})$ is called *cyclic* if

$$\varphi(a_n, a_0, \cdots, a_{n-1}) = (-1)^n \varphi(a_0, a_1, \cdots, a_n)$$

for all $a_0, \cdots, a_n$ in $\mathcal{A}$. Though it is not obvious at all, one can check that cyclic cochains form a subcomplex

$$(C_\lambda(\mathcal{A}),\, b) \subset (C(\mathcal{A}),\, b)$$

of the Hochschild complex.

We shall refer to $(C_\lambda(\mathcal{A}),\, b)$ as the *Connes complex* of $\mathcal{A}$. Its cohomology, by definition, is the cyclic cohomology of $\mathcal{A}$ and will be denoted by $HC^n(\mathcal{A})$. We start our introduction to cyclic cohomology by some concrete examples of cyclic cocycles à la Connes.

6.1 Cyclic cocycles

We give an example of a cyclic cocycle. Let M be a closed (i.e. compact without boundary), smooth, oriented, n-manifold. For $f^0, \cdots, f^n \in \mathcal{A} = C^\infty(M)$, let

$$\varphi(f^0, \cdots, f^n) = \int_M f^0 df^1 \cdots df^n.$$

The $(n+1)$-linear cochain

$$\varphi : \mathcal{A} \times \cdots \times \mathcal{A} \to \mathbb{C}$$

has three properties: it is continuous with respect to the natural Fréchet space topology on $\mathcal{A}$; it is a Hochschild cocycle; and it is cyclic. For the cocycle condition, notice that

$$\begin{aligned}(b\varphi)(f^0, \cdots, f^{n+1}) &= \sum_{i=0}^{n} (-1)^i \int_M f^0 df^1 \cdots d(f^i f^{i+1}) \cdots df^{n+1} \\ &\quad +(-1)^{n+1} \int_M f^{n+1} f^0 df^1 \cdots df^n \\ &= 0,\end{aligned}$$

where we only used the Leibniz rule for the de Rham differential d and the graded commutativity of the algebra $(\Omega M,\, d)$ of differential forms on M. The cyclic property of φ

$$\varphi(f^n, f^0, \cdots, f^{n-1}) = (-1)^n \varphi(f^0, \cdots, f^n)$$

is more interesting. In fact since

$$\int_M (f^n df^0 \cdots df^{n-1} - (-1)^n f^0 df^1 \cdots df^n) = \int_M d(f^n f^0 df^1 \cdots df^{n-1}),$$

we see that the cyclic property of φ follows from Stokes' formula

$$\int_M d\omega = 0,$$

which is valid for any $(n-1)$-form ω on a closed manifold M.

A remarkable property of cyclic cocycles is that, unlike de Rham cocycles which make sense only over commutative algebras, they can be defined over any noncommutative algebra and the resulting cohomology theory is the right generalization of de Rham homology of currents on a smooth manifold. Before developing cyclic cohomology any further we give one more example. In the above situation it is clear that if $V \subset M$ is a closed p-dimensional oriented submanifold then the formula

$$\varphi(f^0, \cdots, f^p) = \int_V f^0 df^1 \cdots df^p$$

defines a cyclic p-cocycle on $\mathcal{A}$. We can replace V by *closed currents* on M and obtain more cyclic cocycles.

Recall that a p-dimensional *current* C on M is a continuous linear functional $C : \Omega^p M \to \mathbb{C}$ on the space of p-forms on M. We write $\langle C,\, \omega\rangle$ instead of $C(\omega)$. For example a zero dimensional current on M is just a distribution on M. The differential of a current is defined by $\langle dC,\, \omega\rangle = \langle C,\, d\omega\rangle$ and in this way one obtains the complex of currents on M whose homology is the *de Rham homology* of M.

Exercise 6.2. *Let C be a p-dimensional current on M. Show that the $(p+1)$-linear functional*

$$\varphi_C(f^0, \cdots, f^p) = \langle C\,,\; f^0 df^1 \cdots df^p\rangle$$

is a Hochschild cocycle on $\mathcal{A}$. Show that if C is closed then φ_C is a cyclic p-cocycle on $\mathcal{A}$.

Let $\mathcal{A}$ be an algebra. Define the operators

$$b' : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A}), \quad \text{and} \quad \lambda : C^n(\mathcal{A}) \to C^n(\mathcal{A}),$$

by

$$\begin{aligned}(b'\varphi)(a_0, \cdots, a_{n+1}) &= \sum_{i=0}^{n} (-1)^i \varphi(a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1}), \\ (\lambda\varphi)(a_0, \cdots, a_n) &= (-1)^n \varphi(a_n, a_0, \cdots, a_{n-1}),\end{aligned}$$

By a direct computation one checks that

$$(1 - \lambda)b = b'(1 - \lambda), \qquad b'^2 = 0. \tag{29}$$

Notice that a cochain $\varphi \in C^n$ is cyclic if and only if $(1 - \lambda)\varphi = 0$. Using (29) we obtain

Lemma 6.1. *The space of cyclic cochains is invariant under b, i.e. for all n,*

$$b\, C^n_\lambda(\mathcal{A}) \subset C^{n+1}_\lambda(\mathcal{A}).$$

We therefore have a subcomplex of the Hochschild complex, called the *Connes complex* of $\mathcal{A}$:

$$C^0_\lambda(\mathcal{A}) \xrightarrow{b} C^1_\lambda(\mathcal{A}) \xrightarrow{b} C^2_\lambda(\mathcal{A}) \xrightarrow{b} \cdots \tag{30}$$

The cohomology of this complex is called the *cyclic cohomology* of $\mathcal{A}$ and will be denoted by $HC^n(\mathcal{A})$, $n = 0, 1, 2, \cdots$. A cocycle for cyclic cohomology is called a *cyclic cocycle*. It satisfies the two conditions:

$$(1 - \lambda)\varphi = 0, \quad \textit{and} \quad b\varphi = 0.$$

Examples 6.1. 1. Clearly $HC^0(A) = HH^0(A)$ is the space of traces on A. In particular if A is commutative then $HC^0(\mathcal{A}) \simeq A^*$ is the linear dual of A.

2. Let $\mathcal{A} = C^\infty(M, M_n(\mathbb{C}))$ be the space of smooth matrix valued functions on a closed smooth oriented manifold M. For any closed de Rham p-current on M

$$\varphi_C(f^0, \cdots, f^p) = \langle C\, , Tr(f^0 df^1 \cdots df^p)\rangle$$

is a cyclic p-cocycle on $\mathcal{A}$.

3. Let $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation and $\tau : \mathcal{A} \to \mathbb{C}$ an invariant trace, i.e. $\tau(\delta(a)) = 0$ for all $a \in \mathcal{A}$. Then one checks that

$$\varphi(a_0, a_1) = \tau(a_0 \delta(a_1)) \tag{31}$$

is a cyclic 1-cocycle on $\mathcal{A}$. This example can be generalized. Let δ_1 and δ_2 be a pair of *commuting* derivations which leave a trace τ invariant. Then

$$\varphi(a_0, a_1, a_2) = \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))) \tag{32}$$

is a cyclic 2-cocycle on $\mathcal{A}$.

Here is a concrete example with $\mathcal{A} = \mathcal{A}_\theta$ a smooth noncommutative torus. Let δ_1, $\delta_2 : \mathcal{A}_\theta \to \mathcal{A}_\theta$ be the unique derivations defined by

$$\delta_1(U) = U, \quad \delta_1(V) = 0; \quad \delta_2(U) = 0, \quad \delta_2(V) = V.$$

They commute with each other and preserve the standard trace τ on $\mathcal{A}_\theta$. The resulting cyclic 1-cocycles $\varphi_1(a_0, a_1) = \tau(a_0\delta_1(a_1))$ and $\varphi_1'(a_0, a_1) = \tau(a_0\delta_2(a_1))$ form a basis for the periodic cyclic cohomology $HP^1(\mathcal{A}_\theta)$. Similarly, the corresponding cocycle (32) together with τ form a basis for $HP^0(\mathcal{A}_\theta)$.

Consider the short exact sequence of complexes

$$0 \to C_\lambda \to C \to C/C_\lambda \to 0$$

Its associated long exact sequence is

$$\cdots \longrightarrow HC^n(\mathcal{A}) \longrightarrow HH^n(\mathcal{A}) \longrightarrow H^n(C/C_\lambda) \longrightarrow HC^{n+1}(\mathcal{A}) \longrightarrow \cdots \tag{33}$$

We need to identify the cohomology groups $H^n(C/C_\lambda)$. To this end, consider the short exact sequence

$$0 \longrightarrow C/C_\lambda \xrightarrow{1-\lambda} (C, b') \xrightarrow{N} C_\lambda \longrightarrow 0, \tag{34}$$

where the operator N is defined by

$$N = 1 + \lambda + \lambda^2 + \cdots + \lambda^n : C^n \longrightarrow C^n.$$

The relations

$$N(1-\lambda) = (1-\lambda)N = 0, \quad \text{and} \quad bN = Nb'$$

can be verified and they show that $1 - \lambda$ and N are morphisms of complexes in (34).

Exercise 6.3. *Show that* (34) *is exact (the interesting part is to show that* $Ker\,N \subset Im(1-\lambda)$*).*

Now, assuming A is unital, the middle complex (C, b') in (34) can be shown to be exact with contracting homotopy $s : C^n \to C^{n-1}$ given by

$$(s\varphi)(a_0, \cdots, a_{n-1}) = (-1)^n \varphi(a_0, \cdots, a_{n-1}, 1).$$

It follows that $H^n(C/C_\lambda) \simeq HC^{n-1}(\mathcal{A})$. Using this in (33), we obtain *Connes' long exact* sequence relating Hochschild and cyclic cohomology:

$$\cdots \longrightarrow HC^n(\mathcal{A}) \xrightarrow{I} HH^n(\mathcal{A}) \xrightarrow{B} HC^{n-1}(\mathcal{A}) \xrightarrow{S} HC^{n+1}(\mathcal{A}) \longrightarrow \cdots \tag{35}$$

The operators B and S can be made more explicit by finding the connecting homomorphisms in the above long exact sequences. Remarkably, there is a formula for *Connes' boundary operator* B on the level of cochains given by

$$B = Ns(1-\lambda) = NB_0,$$

where $B_0 : C^n \to C^{n-1}$ is defined by

$$B_0\varphi(a_0, \cdots, a_{n-1}) = \varphi(1, a_0, \cdots, a_{n-1}) - (-1)^n \varphi(a_0, \cdots, a_{n-1}, 1).$$

The operator $S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A})$ is called the *periodicity operator* and is in fact related to Bott periodicity. The *periodic cyclic cohomology* of $\mathcal{A}$ is defined as the direct limit under the operator S of cyclic cohomology groups:

$$HP^i(\mathcal{A}) = \varinjlim HC^{2n+i}(\mathcal{A}), \qquad i = 0,\, 1$$

A typical application of (35) is to extract information about cyclic cohomology from Hochschild cohomology. For example, assume $f : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism such that $f^* : HH^n(\mathcal{B}) \to HH^n(\mathcal{A})$ is an isomorphism for all $n \geq 0$. Then, using the five lemma, we conclude that $f^* : HC^n(\mathcal{B}) \to HC^n(\mathcal{A})$ is an isomorphism for all n. In particular from Morita invariance of Hochschild cohomology one obtains the Morita invariance of cyclic cohomology.

6.2 Connes' spectral sequence

The cyclic complex (30) and the long exact sequence (35), as useful as they are, are not powerful enough for computations. A much deeper relation between Hochschild and cyclic cohomology groups is encoded in Connes' spectral sequence that we recall now. This spectral sequence resembles in many ways the Hodge to de Rham spectral sequence for complex manifolds. About this connection we shall say nothing in these notes but see [74] where a conjecture of Kontsevich and Soibelman about the degeneration of this spectral sequence is proved.

Let $\mathcal{A}$ be a unital algebra. Connes' (b, B)- bicomplex of $\mathcal{A}$ is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
C^2(\mathcal{A}) & \xrightarrow{B} & C^1(\mathcal{A}) & \xrightarrow{B} & C^0(\mathcal{A}) \\
b\uparrow & & b\uparrow & & \\
C^1(\mathcal{A}) & \xrightarrow{B} & C^0(\mathcal{A}) & & \\
b\uparrow & & & & \\
C^0(\mathcal{A}) & & & &
\end{array}$$

Of the three relations

$$b^2 = 0, \quad bB + Bb = 0, \quad B^2 = 0,$$

only the middle relation is not obvious. But this follows from the relations $b's + sb' = 1$, $(1-\lambda)b = b'(1-\lambda)$ and $Nb' = bN$, already used in this section.

Theorem 6.1. *(Connes [22]) The map $\varphi \mapsto (0, \cdots, 0, \varphi)$ is a quasi-isomorphism of complexes*

$$(C_\lambda(\mathcal{A}), b) \to (Tot\mathcal{B}(\mathcal{A}), b + B)$$

This is a consequence of the vanishing of the E^2 term of the second spectral sequence (filtration by columns) of $\mathcal{B}(A)$. To prove this consider the short exact sequence of b-complexes

$$0 \longrightarrow \operatorname{Im} B \longrightarrow \operatorname{Ker} B \longrightarrow \operatorname{Ker} B / \operatorname{Im} B \longrightarrow 0$$

By a hard lemma of Connes ([22], Lemma 41), the induced map

$$H_b(\operatorname{Im} B) \longrightarrow H_b(\operatorname{Ker} B)$$

is an isomorphism. It follows that $H_b(\operatorname{Ker} B / \operatorname{Im} B)$ vanish. To take care of the first column one appeals to the fact that

$$\operatorname{Im} B \simeq Ker(1 - \lambda)$$

is the space of cyclic cochains.

6.3 Topological algebras

There is no difficulty in defining *continuous* analogues of Hochschild and cyclic cohomology groups for Banach algebras. One simply replaces bimodules by Banach bimodules (where the left and right module actions are bounded operators) and cochains by continuous cochains. Since the multiplication of a Banach algebra is a bounded map, all operators including the Hochschild boundary and the cyclic operator extend to this continuous setting. The resulting Hochschild and cyclic theory for Banach and C^*-algebras, however, is hardly useful and tends to vanish in many interesting examples. This is hardly surprising since the definition of any Hochschild and cyclic cocycle of dimension bigger than zero involves differentiating the elements of the algebra in one way or another. This is in sharp contrast with topological K-theory where the right setting is the setting of Banach or C^*-algebras.

Exercise 6.4. *Let X be a compact Hausdorff space. Show that any derivation $\delta : C(X) \longrightarrow C(X)$ is identically zero. (hint: first show that if $f = g^2$ and $g(x) = 0$ for some $x \in X$, then $\delta(f)(x) = 0$.)*

Remark 6. *By results of Connes and Haagerup (cf. [24] and references therein), we know that a C^*-algebra is amenable if and only if it is nuclear. Amenability refers to the property that for all $n \geq 1$,*

$$H^n_{cont}(A, M^*) = 0,$$

for any Banach dual bimodule M^*. In particular, by using Connes' long exact sequence, we find that, for any nuclear C^*-algebra A,

$$HC^{2n}_{cont}(A) = A^*, \quad \text{and} \quad HC^{2n+1}_{cont}(A) = 0,$$

for all $n \geq 0$.

The right class of topological algebras for cyclic cohomology turns out to be the class of *locally convex algebras* [22]. An algebra $\mathcal{A}$ equipped with a locally convex topology is called a locally convex algebra if its multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is jointly continuous. Basic examples of locally convex algebras include the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on a closed manifold and the smooth noncommutative tori $\mathcal{A}_\theta$ and their higher dimensional analogues. The topology of $C^\infty(M)$ is defined by the sequence of seminorms

$$\|f\|_n = \sup |\partial^\alpha f|; \quad |\alpha| \leq n,$$

where the supremum is over a fixed, finite, coordinate cover for M (see the exercise below for the topology of $\mathcal{A}_\theta$).

Given locally convex topological vector spaces V_1 and V_2, their *projective tensor product* is a locally convex space $V_1 \hat{\otimes} V_2$ together with a universal jointly continuous bilinear map $V_1 \otimes V_2 \to V_1 \hat{\otimes} V_2$ [65]. It follows from the universal property that for any locally convex space W, we have a natural isomorphism between continuous bilinear maps $V_1 \times V_2 \to W$ and continuous linear maps $V_1 \hat{\otimes} V_2 \to W$. One of the nice properties of the projective tensor product is that for smooth compact manifolds M and N, the natural map

$$C^\infty(M) \hat{\otimes} C^\infty(N) \to C^\infty(M \times N)$$

is an isomorphism.

A *topological left $\mathcal{A}$-module* is a locally convex topological vector space $\mathcal{M}$ endowed with a continuous left $\mathcal{A}$-module action $\mathcal{A} \times \mathcal{M} \to \mathcal{M}$. A *topological free* left $\mathcal{A}$-module is a module of the type $\mathcal{M} = \mathcal{A} \hat{\otimes} V$ where V is a locally convex space. A *projective* module is a module which is a direct summand in a free module.

Given a locally convex algebra $\mathcal{A}$, let

$$C^n_{\mathrm{cont}}(\mathcal{A}) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}^{\hat{\otimes} n}, \mathbb{C})$$

be the space of continuous $(n+1)$-linear functionals on A and let $C^n_{\mathrm{cont},\,\lambda}(\mathcal{A})$ denote the space of continuous cyclic cochains on $\mathcal{A}$. All the algebraic definitions and results of sections 6.1 and 6.2 extend to this topological setting. In particular one defines topological Hochschild and cyclic cohomology groups of a locally convex algebra. The right class of topological projective and free resolutions are those resolutions that admit a continuous linear splitting. This extra condition is needed when one wants to prove comparison theorems for resolutions. We won't go into details here since this is very well explained in Connes' original article [22].

Exercise 6.5. *The sequence of norms*

$$p_k(a) = \mathit{Sup}\,\{(1 + |n| + |m|)^k |a_{mn}|\}$$

defines a locally convex topology on the smooth noncommutative torus $\mathcal{A}_\theta$. Show that the multiplication of $\mathcal{A}_\theta$ is continuous in this topology.

6.4 The deformation complex

What we called the Hochschild cohomology of $\mathcal{A}$ and denoted by $HH^n(\mathcal{A})$ is in fact the Hochschild cohomology of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-bimodule

$\mathcal{A}^*$. In general, given an $\mathcal{A}$-bimodule $\mathcal{M}$, the Hochschild complex of $\mathcal{A}$ with coefficients in the bimodule $\mathcal{M}$ is the complex

$$C^0(\mathcal{A}, \mathcal{M}) \xrightarrow{\delta} C^1(\mathcal{A}, \mathcal{M}) \xrightarrow{\delta} C^2(\mathcal{A}, \mathcal{M}) \longrightarrow \cdots$$

where $C^0(\mathcal{A}, \mathcal{M}) = \mathcal{M}$ and $C^n(\mathcal{A}, \mathcal{M}) = \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, \mathcal{M})$ is the space of n-linear functionals on $\mathcal{A}$ with values in $\mathcal{M}$. The differential δ is given by

$$\begin{aligned}(\delta\varphi)(a_1, \cdots, a_{n+1}) \quad = \quad & a_1\varphi(a_2, \cdots, a_{n+1}) + \sum_{i=1}^{n}(-1)^{i+1}\varphi(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\ & +(-1)^{n+1}\varphi(a_1, \cdots, a_n)a_{n+1}.\end{aligned}$$

Two special cases are particularly important. For $\mathcal{M} = \mathcal{A}^*$, the linear dual of $\mathcal{A}$ with the bimodule action

$$(afb)(c) = f(bca)$$

for all a, b, c in $\mathcal{A}$ and $f \in \mathcal{A}^*$, we obtain the Hochschild groups $H^n(\mathcal{A}, \mathcal{A}^*) = HH^n(A)$. This is important in cyclic cohomology since as we saw it enters into a long exact sequence with cyclic groups. The second important case is when $\mathcal{M} = \mathcal{A}$ with bimodule structure given by left and right multiplication. The resulting complex $(C(\mathcal{A}, \mathcal{A}), \delta)$ is called the *deformation complex* of $\mathcal{A}$. It is the complex that underlies the deformation theory of associative algebras as studied by Gerstenhaber [62].

There is a much deeper structure hidden in the deformation complex $C(\mathcal{A}, \mathcal{A}), \delta)$ than meets the eye and we will only barely scratch the surface. The first piece of structure is the cup product. The *cup product* $\cup : C^p \times C^q \to C^{p+q}$ is defined by

$$(f \cup g)(a^1, \cdots, a^{p+q}) = f(a^1, \cdots, a^p)g(a^{p+1}, \cdots, a^{p+q}).$$

Notice that $\cup$ is associative and one checks that this product is compatible with the differential δ and hence induces an associative graded product on $H(\mathcal{A}, \mathcal{A}) = \oplus H^n(\mathcal{A}, \mathcal{A})$. What is not so obvious however is that this product is graded commutative for any $\mathcal{A}$ [62].

The second piece of structure on $(C(\mathcal{A}, \mathcal{A}), \delta)$ is a graded Lie bracket. It is based on the Gerstenhaber circle product $\circ : C^p \times C^q \longrightarrow C^{p+q-1}$ defined by

$$(f \circ g)(a_1, \cdots, a_{p+q-1}) = \sum_{i=1}^{p-1}(-1)^{|g|(|f|+i-1)} f(a^1, \cdots, g(a^i, \cdots, a^{i+p}), \cdots, a^{p+q-1}).$$

Notice that $\circ$ is not an associative product. Nevertheless one can show that [62] the corresponding graded bracket $[\,,\,] : C^p \times C^q \to C^{p+q-1}$

$$[f,\, g] = f \circ g - (-1)^{(p-1)(q-1)} g \circ f$$

defines a graded Lie algebra structure on deformation cohomology $H(\mathcal{A},\, \mathcal{A})$. Notice that the Lie algebra grading is now shifted by one.

What is most interesting is that the cup product and the Lie algebra structure are compatible in the sense that $[\,,\,]$ is a graded derivation for the cup product; or in short $(H(\mathcal{A},\, \mathcal{A}),\, \cup,\, [\,,\,])$ is a graded Poisson algebra.

The *fine structure* of the Hochschild cochain complex $(C(\mathcal{A},\, \mathcal{A}),\, \delta)$, e.g. the existence of higher order products and homotopies between them is the subject of many studies in recent years [84, 85, 86]. While it is relatively easy to write down these higher order products in the form of a brace algebra structure on the Hochschild complex, relating them to known geometric structures such as moduli of curves is quite hard.

Remark 7. *The graded Poisson algebra structure on deformation cohomology $H(\mathcal{A},\, \mathcal{A})$ poses a natural question: is $H(\mathcal{A},\, \mathcal{A})$ the semiclassical limit of a quantum cohomology theory for algebras?*

Example 6.1. Let $\mathcal{A} = C^\infty(M)$, where M is a compact n-dimensional manifold. In [22] Connes gives a projective resolution of the topological left $\mathcal{A}\hat{\otimes}\mathcal{A}$-module $\mathcal{A}$,

$$\mathcal{A} \leftarrow \mathcal{M}_0 \leftarrow \mathcal{M}_1 \cdots \leftarrow \mathcal{M}_n \leftarrow 0 \tag{36}$$

where $\mathcal{M}_i$ is the space of smooth sections of the vector bundle $p_1^*(\bigwedge^i TM)$ and $p_1 : M \times M \to M$ is the projection on the second factor. After applying the $\mathrm{Hom}_{\mathcal{A}\hat{\otimes}\mathcal{A}}(-,\, \mathcal{A})$ functor to (36), one obtains a complex with zero differentials, which shows that

$$H^p(\mathcal{A},\, \mathcal{A}) \simeq C^\infty(\bigwedge^p TM), \quad p = 0, 1, \cdots .$$

The latter is the space of polyvector fields on M.

6.5 Cyclic homology

Cyclic cohomology is a contravariant functor on the category of algebras. There is a dual covariant theory called *cyclic homology* that we introduce now. The relation between the two is similar to the relation between currents and differential forms on manifolds.

For each $n \geq 0$, let $C_n(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)}$. Define the operators

$$\begin{aligned}
b : C_n(\mathcal{A}) &\longrightarrow C_{n-1}(\mathcal{A}) \\
b' : C_n(\mathcal{A}) &\longrightarrow C_{n-1}(\mathcal{A}) \\
\lambda : C_n(\mathcal{A}) &\longrightarrow C_n(\mathcal{A}) \\
s : C_n(\mathcal{A}) &\longrightarrow C_{n+1}(\mathcal{A}) \\
N : C_n(\mathcal{A}) &\longrightarrow C_n(\mathcal{A}) \\
B : C_n(\mathcal{A}) &\longrightarrow C_{n+1}(\mathcal{A})
\end{aligned}$$

by

$$\begin{aligned}
b(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\
&+ (-1)^n (a_n a_0 \otimes a_1 \cdots \otimes a_{n-1}) \\
b'(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\
\lambda(a_0 \otimes \cdots \otimes a_n) &= (-1)^n (a_n \otimes a_0 \cdots \otimes a_{n-1}) \\
s(a_0 \otimes \cdots \otimes a_n) &= (-1)^n (a_0 \otimes \cdots \otimes a_n \otimes 1) \\
N &= 1 + \lambda + \lambda^2 + \cdots + \lambda^n \\
B &= (1 - \lambda) s N
\end{aligned}$$

They satisfy the relations

$$b^2 = 0, \qquad b'^2 = 0, \qquad (1-\lambda) b' = b(1-\lambda)$$
$$b'N = Nb, \qquad B^2 = 0, \qquad bB + Bb = 0$$

The complex $(C_\bullet(\mathcal{A}), b)$ is the Hochschild complex of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-bimodule $\mathcal{A}$. The complex

$$C_n^\lambda(\mathcal{A}) := C_n(\mathcal{A}) / \mathrm{Im}(1 - \lambda)$$

is called the *Connes complex* of $\mathcal{A}$ for cyclic homology. Its homology, denoted by $HC_n(\mathcal{A}),\ n = 0, 1, \cdots$, is called the *cyclic homology* of $\mathcal{A}$. It is clear that the space of cyclic cochains is the linear dual of the space of cyclic chains

$$C_\lambda^n(\mathcal{A}) \simeq \mathrm{Hom}\,(C_n^\lambda(\mathcal{A}), \mathbb{C})$$

and

$$HC^n(\mathcal{A}) \simeq HC_n(\mathcal{A})^*.$$

Similar to cyclic cohomology, there is a long exact sequence relating Hochschild and cyclic homologies, and also there is a spectral sequence from Hochschild to cyclic homology. In particular cyclic homology can be computed using the following bicomplex.

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \\
\downarrow b & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & \\
\downarrow b & & & & \\
A & & & &
\end{array}$$

Example 6.2. *(Hochschild-Kostant-Rosenberg and Connes theorems)* Let

$$\mathcal{A} \xrightarrow{d} \Omega^1\mathcal{A} \xrightarrow{d} \Omega^2\mathcal{A} \xrightarrow{d} \cdots$$

denote the de Rham complex of a commutative unital algebra $\mathcal{A}$. By definition $d : \mathcal{A} \to \Omega^1\mathcal{A}$ is a *universal derivation* into a symmetric $\mathcal{A}$-bimodule and $\Omega^n\mathcal{A} := \wedge^n_{\mathcal{A}}\Omega^1\mathcal{A}$ is the k-th exterior power of $\Omega^1\mathcal{A}$ over $\mathcal{A}$. One usually defines $\Omega^1\mathcal{A}$, the *module of Kähler differentials*, as I/I^2, where I is the kernel of the multiplication map $A \otimes A \to A$. d is then defined by

$$d(a) = a \otimes 1 - 1 \otimes a \mod(I^2).$$

The universal derivation d has a unique extension to a graded derivation of degree one on $\Omega\mathcal{A}$, denoted by d.

The *antisymmetrization map*

$$\varepsilon_n : \Omega^n\mathcal{A} \longrightarrow \mathcal{A}^{\otimes(n+1)}, \quad n = 0, 1, 2, \cdots,$$

is defined by

$$\varepsilon_n(a_0 da_1 \wedge \cdots \wedge da_n) = \sum_{\sigma \in S_n} sgn(\sigma) a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},$$

where S_n is the symmetric group of order n . We also have a map

$$\mu_n : \mathcal{A}^{\otimes n} \longrightarrow \Omega^n\mathcal{A}, \quad n = 0, 1, \cdots$$

$$\mu_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 da_1 \wedge \cdots \wedge da_n.$$

One checks that the resulting maps

$$(\Omega\mathcal{A},\, 0) \to (C(\mathcal{A}),\, b), \quad \text{and,} \quad (C(\mathcal{A}),\, b) \to (\Omega\mathcal{A},\, 0)$$

are morphisms of complexes, i.e.

$$b \circ \varepsilon_n = 0, \quad \mu_n \circ b = 0.$$

Moreover, one can easily check that

$$\mu_n \circ \varepsilon_n = n!\, \mathrm{Id}_n.$$

It follows that, for any commutative algebra $\mathcal{A}$, the antisymmetrization map induces an inclusion

$$\varepsilon_n : \Omega^n\mathcal{A} \hookrightarrow HH_n(\mathcal{A}),$$

for all n.

The Hochschild-Kostant-Rosenberg theorem [72] states that if $\mathcal{A}$ is a *regular algebra*, e.g. the algebra of regular functions on a smooth affine variety, then ε_n defines an algebra isomorphism

$$\varepsilon_n : \Omega^n\mathcal{A} \simeq HH_n(\mathcal{A})$$

between Hochschild homology of $\mathcal{A}$ and the algebra of differential forms on $\mathcal{A}$.

To compute the cyclic homology of $\mathcal{A}$, we first show that under the map μ the operator B corresponds to the de Rham differential d. More precisely, for each integer $n \geq 0$ we have a commutative diagram:

$$\begin{array}{ccc} C_n(\mathcal{A}) & \xrightarrow{\mu} & \Omega^n\mathcal{A} \\ \Big\downarrow B & & \Big\downarrow d \\ C_{n+1}(\mathcal{A}) & \xrightarrow{\mu} & \Omega^{n+1}\mathcal{A} \end{array}$$

We have

$$\begin{aligned} \mu B(f_0 \otimes \cdots \otimes f_n) &= \mu \sum_{i=0}^{n} (-1)^{ni}(1 \otimes f_i \otimes \cdots \otimes f_{i-1} - (-1)^n f_i \otimes \cdots f_{i-1} \otimes 1) \\ &= \frac{1}{(n+1)!} \sum_{i=0}^{n} (-1)^{ni} df_i \cdots df_{i-1} \\ &= \frac{1}{(n+1)!}(n+1) df_0 \cdots df_n \\ &= d\mu(f_0 \otimes \cdots \otimes f_n). \end{aligned}$$

It follows that μ defines a morphism of bicomplexes

$$\mathcal{B}(\mathcal{A}) \longrightarrow \Omega(\mathcal{A}),$$

where $\Omega(\mathcal{A})$ is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\Omega^2\mathcal{A} & \xleftarrow{d} & \Omega^1\mathcal{A} & \xleftarrow{d} & \Omega^0\mathcal{A} \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
\Omega^1\mathcal{A} & \xleftarrow{d} & \Omega^0\mathcal{A} & & \\
\downarrow{\scriptstyle 0} & & & & \\
\Omega^0\mathcal{A} & & & &
\end{array}$$

Since μ induces isomorphisms on row homologies, it induces isomorphisms on total homologies as well. Thus we have [22, 92]:

$$HC_n(\mathcal{A}) \simeq \Omega^n\mathcal{A}/\mathrm{Im}\, d \oplus H^{n-2}_{dR}(\mathcal{A}) \oplus \cdots \oplus H^k_{dR}(\mathcal{A}),$$

where $k = 0$ if n is even and $k = 1$ if n is odd.

Using the same map μ acting between the corresponding periodic complexes, one concludes that the periodic cyclic homology of $\mathcal{A}$ is given by

$$HP_k(\mathcal{A}) \simeq \bigoplus_i H^{2i+k}_{dR}(\mathcal{A}), \quad k = 0, 1.$$

By a completely similar method one can compute the *continuous cyclic homology* of the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on a smooth closed manifold M. Here by continuous cyclic homology we mean the homology of the cyclic complex where instead of algebraic tensor products $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$, one uses the topological *projective tensor product* $\mathcal{A}\hat{\otimes} \cdots \hat{\otimes}\mathcal{A}$. The continuous Hochschild homology of $\mathcal{A}$ can be computed using Connes' topological resolution for $\mathcal{A}$ as an $\mathcal{A}$-bimodule as in Example (6.1). The result is

$$HH^{cont}_n(C^\infty(M)) \simeq \Omega^n M$$

with isomorphism induced by the map

$$f_0 \otimes f_1 \otimes \cdots \otimes f_n \mapsto f_0 df_1 \cdots df_n.$$

The rest of the computation of continuous cyclic homology follows the same pattern as in the case of regular algebras above. The end result is [22]:

$$HC_n^{\mathrm{cont}}(C^\infty(M)) \simeq \Omega^n M/\mathrm{Im}\, d \oplus H_{dR}^{n-2}(M) \oplus \cdots \oplus H_{dR}^{k}(M),$$

and

$$HP_k^{cont}(C^\infty(M)) \simeq \bigoplus_i H_{dR}^{2i+k}(M), \quad k = 0, 1.$$

The cyclic (co)homology of (topological) algebras is computed in many cases. We refer to [22] for smooth noncommutative tori, to [15] for group algebras, and to [60, 105] for crossed product algebras. In [1] the spectral sequence for group crossed products has been extended to Hopf algebra crossed products. We refer to [46, 47, 48] for an alternative approach to cyclic (co)homoloy due to Cuntz and Quillen.

6.6 Connes-Chern character

We can now indicate Connes' generalization of the pairing between $K_0(\mathcal{A})$ and traces on $\mathcal{A}$ (Example (5.3)) to a full fledged pairing between K-theory and cyclic cohomology:

$$K_0(\mathcal{A}) \times HC^{2n}(\mathcal{A}) \longrightarrow \mathbb{C}, \tag{37}$$
$$K_1^{\mathrm{alg}}(\mathcal{A}) \times HC^{2n+1}(\mathcal{A}) \longrightarrow \mathbb{C}, \tag{38}$$

These maps are defined for all $n \geq 0$ and are compatible with the S-operation on cyclic cohomology. In the dual setting of cyclic homology, these pairings translate into noncommutative *Connes-Chern characters*

$$\begin{aligned} \mathrm{Ch}_0^{2n} : K_0(\mathcal{A}) &\longrightarrow HC_{2n}(\mathcal{A}), \\ \mathrm{Ch}_1^{2n+1} : K_1^{\mathrm{alg}}(\mathcal{A}) &\longrightarrow HC_{2n+1}(\mathcal{A}), \end{aligned}$$

compatible with the S-operation on cyclic homology. As a consequence of compatibility with the periodicity operator S, we obtain maps

$$\begin{aligned} \mathrm{Ch}_0 : K_0(\mathcal{A}) &\longrightarrow HP_0(\mathcal{A}), \\ \mathrm{Ch}_1 : K_1^{\mathrm{alg}}(\mathcal{A}) &\longrightarrow HP_1(\mathcal{A}). \end{aligned}$$

For $\mathcal{A} = C^\infty(M)$, these maps reduce to the classical Chern character as defined via the connection and curvature formalism of Chern-Weil theory [100].

The definition of these pairings rest on the following three facts [22, 24, 92]:

1) For any $k \geq 1$, the map $\varphi \mapsto \varphi_k$ from $C^n(\mathcal{A}) \to C^n(M_k(\mathcal{A}))$ defined by

$$\varphi_k(m_0 \otimes a_0, m_1 \otimes a_1, \cdots, m_n \otimes a_n) = \mathrm{Tr}\,(m_0 m_1 \cdots m_n)\varphi(a_0, a_1, \cdots, a_n)$$

commutes with the operators b and λ. It follows that if φ is a cyclic cocycle on $\mathcal{A}$, then φ_k is a cyclic cocycle on $M_k(\mathcal{A})$.

2) Inner automorphisms act by the identity on Hochschild and hence on cyclic cohomology.

3) (normalization) The inclusion of *normalized cochains* $C_\lambda^{\mathrm{norm}}(\mathcal{A}) \to C_\lambda(\mathcal{A})$ is a quasi-isomorphism in dimensions $n \geq 1$. A cyclic cochain φ is called *normalized* if $\varphi(a_0, \cdots, a_n) = 0$ if $a_i = 1$ for some i.

Now let $e \in M_k(A)$ be an idempotent and $[\varphi] \in HC^{2n}(\mathcal{A})$. The pairing (37) is defined by the bilinear map

$$\langle [\varphi],\, [e] \rangle = \varphi_k(e, \cdots, e).$$

Let us first check that the value of the pairing depends only on the cyclic cohomology class of φ. It suffices to assume $k = 1$ (why?). Let $\varphi = b\psi$ with $\psi \in C_\lambda^{2n-1}(A)$. Then we have

$$\begin{aligned}
\varphi(e, \cdots, e) &= b\psi(e, \cdots, e) \\
&= \psi(ee, e, \cdots, e) - \psi(e, ee, \cdots, e) + \cdots + (-1)^{2n}\psi(ee, e, \cdots, e) \\
&= \psi(e, \cdots, e) \\
&= 0,
\end{aligned}$$

where the last relation follows from the cyclic property of ψ. The pairing is clearly invariant under the inclusion $M_k(\mathcal{A}) \to M_{k+1}(\mathcal{A})$.

It remains to show that the value of $\langle [\varphi]\,,\, [e] \rangle$, for fixed φ, only depends on the class of $[e] \in K_0(\mathcal{A})$. It suffices to check that for $u \in GL_k(\mathcal{A})$, $\langle [\varphi],\, [e] \rangle = \langle [\varphi],\, [ueu^{-1}] \rangle$. But this is exactly fact 2) above.

Exercise 6.6. *Let $e \in \mathcal{A}$ be an idempotent. Show that*

$$Ch_0^{2k}(e) := (-1)^k \frac{(2k)!}{k!} tr((e - \frac{1}{2}) \otimes e^{\otimes(2k+1)}), \quad k = 0, \cdots, n,$$

defines a cycle in the (b, B)-bicomplex of $\mathcal{A}$. This is the formula for the Connes-Chern character in the bicomplex picture of cyclic homology.

Dually, given a cocycle $\varphi = (\varphi_0, \varphi_2, \cdots, \varphi_2 n)$ in the (b, B)-bicomplex, its pairing with an idempotent $e \in M_k(\mathcal{A})$ is given by

$$\langle [\varphi],\ [e] \rangle = \sum_{k=1}^{n} (-1)^k \frac{k!}{(2k)!} \varphi_{2k}(e - \frac{1}{2}, e, \cdots, e).$$

Given a normalized cyclic cocycle $\varphi \in HC^{2n+1}(\mathcal{A})$ and an invertible $u \in GL(k, A)$, let

$$\langle [\varphi],\ [u] \rangle = \varphi_k(u, u^{-1}, \cdots, u, u^{-1}).$$

It can be shown (cf. [22], Part II) that the above formula defines a pairing between K_1^{alg} and HC^{2n+1}.

Exercise 6.7. *Given an invertible $u \in GL(n, \mathcal{A})$, show that*

$$Ch_1^{2k+1} := (-1)^k k!\, Tr(u^{-1} \otimes u)^{\otimes 2k}$$

defines a cycle in the normalized (b, B)-bicomplex of $\mathcal{A}$. This is the formula for the Connes-Chern character in the (b, B)-bicomplex picture of cyclic homology. Dually, given a normalized (b, B)-cocycle $\varphi = (\varphi_1, \cdots, \varphi_{2n+1})$, the formula

$$\langle [\varphi],\ [u] \rangle = \sum_{k=1}^{n} (-1)^k \varphi_{2k+1}(u, u^{-1}, \cdots, u, u^{-1})$$

defines the pairing between K_1^{alg} and HC^{2n+1}.

It often happens that an element of $K_0(\mathcal{A})$ is represented by a finite projective module and not by an explicit idempotent. It is then important to have a formalism that would give the value of its pairing with cyclic cocycles. This is based on a noncommutative version of Chern-Weil theory developed by Connes in [18, 22] that we sketch next.

Let $\mathcal{E}$ be a finite projective right $\mathcal{A}$-module, $(\Omega,\ d)$ a differential calculus on $\mathcal{A}$ and let $\int : \Omega^{2n} \to \mathbb{C}$ be a closed graded trace representing a cyclic cocycle φ on $\mathcal{A}$. Thanks to its projectivity, $\mathcal{E}$ admits a *connection*, i.e. a degree one map

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega \to \mathcal{E} \otimes_{\mathcal{A}} \Omega$$

which satisfies the graded Leibniz rule

$$\nabla(\xi\omega) = \nabla(\xi)\omega + (-1)^{\deg \xi} \xi d\omega$$

with respect to the right Ω-module structure on $\mathcal{E} \otimes_{\mathcal{A}} \Omega$. The *curvature* of ∇ is the operator ∇^2, which can be easily checked to be Ω-linear,

$$\nabla^2 \in \mathrm{End}_{\Omega}(E \otimes_{\mathcal{A}} \Omega) = \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \otimes \Omega.$$

Now since $\mathcal{E}$ is finite projective over $\mathcal{A}$ it follows that $\mathcal{E} \otimes_{\mathcal{A}} \Omega$ is finite projective over Ω and therefore the trace $\int : \Omega \to \mathbb{C}$ extends to a trace, denoted again by $\int$, on $\mathrm{End}_{\mathcal{A}}(\mathcal{E}) \otimes \Omega$ (cf. formula (21)). The following result of Connes relates the value of the pairing as defined above to its value computed through the Chern-Weil formalism:

$$\langle [\mathcal{E}],\, [\varphi] \rangle = \frac{1}{n!} \int \nabla^n.$$

Example 6.3. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing functions on the real line. The operators $u, v : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined by

$$(uf)(x) = f(x - \theta), \qquad (vf)(x) = e^{2\pi i \theta} f(x)$$

satisfy the relation $uv = e^{2\pi i\theta} vu$ and hence turn $\mathcal{S}(\mathbb{R})$ into a right $\mathcal{A}_\theta$-module via the maps $U \mapsto u$, $V \mapsto v$. We denote this module by $\mathcal{E}_{0,1}$. It is the simplest of a series of modules $\mathcal{E}_{p,q}$ defined by Connes in [18]. It turns out that $\mathcal{E}_{0,1}$ is finite projective, and for the canonical trace τ on $\mathcal{E}_{0,1}$ we have

$$\langle \tau,\, \mathcal{E}_{0,1} \rangle = -\theta.$$

Using the two derivations δ_1, δ_2 as a basis for "invariant vector fields" one can define a differential calculus $\Omega^0 \oplus \Omega^1 \oplus \Omega^2$ on $\mathcal{A}_\theta$ with $\Omega^i = \bigwedge^i \{de_1, de_2\} \otimes \mathcal{A}_\theta$. A connection on $\mathcal{E}$ with respect to this calculus is a pair of operators $\nabla_1,\ \nabla_2 : \mathcal{E} \to \mathcal{E}$ satisfying

$$\nabla_j(\xi a) = (\nabla_j)(\xi a) + \xi \delta_j(a)$$

for all $\xi \in \mathcal{E}_{0,1}$ and $a \in \mathcal{A}_\theta$ and $j = 1, 2$. One can check that the following formula defines a connection on $\mathcal{E}_{0,1}$ [18, 24]:

$$\nabla_1(\xi)(s) = -\frac{s}{\theta}\, \xi(s), \qquad \nabla_2(\xi)(s) = \frac{d\xi}{ds}(s).$$

The curvature of this connection is constant and is given by $\nabla^2 = [\nabla_1, \nabla_2] = \frac{1}{\theta} I \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}_{0,1})$.

Remark 8. *Chern-Weil theory is a theory of characteristic classes for smooth principal G-bundles, where G is a Lie group. The above theory for noncommutative vector bundles should be generalized to noncommutative analogues of principal bundles. A good point to start would be the theory of Hopf-Galois extensions.*

In the remainder of this section we shall briefly introduce Connes' Chern character in K-homology (cf. [22, 24] for a full account). In fact one of the main reasons for introducing cyclic cohomology was to define a Chern character in K-homology [19]. In the even case, let Let (H, F) be an even p-summable Fredholm module over an algebra $\mathcal{A}$ as in definition (2.2). For each even integer $n \geq p-1$, define an n-cochain φ_n on $\mathcal{A}$ by

$$\varphi_n(a_0, \cdots, a_n) = \text{Trace}\,(\varepsilon\, a_0[F,\, a_1] \cdots [F,\, a_n]).$$

The p-summability condition on (H, F) ensures that the above product of commutators is in fact a trace class operator and φ_n is finite for all $n \geq p-1$ (this is obvious if $n > p-1$; in general one has to manipulate the commutators a bit to prove this).

Exercise 6.8. *Show that φ_n is a cyclic cocycle. Also show that if n is odd then $\varphi_n = 0$.*

Although this definition depends on n, it can be shown that the cyclic cocycles φ_n are related to each other by the periodicity operator S,

$$S\varphi_n = \varphi_{n+2},$$

and therefore define an even periodic cyclic cohomology class. This is Connes' Chern character of a K-homology class in the even case. For applications it is important to have an index formula which computes the value of the pairing $\langle [\varphi_n],\, [e] \rangle$ as the index of a Fredholm operator similar to Example 2.11. We refer to [22, 24] for this.

Let A be a nuclear C^*-algebra. In the odd case, smooth p-summable elements of K-homology group $K^1(A)$ can be represented either by smooth Brown-Douglas-Fillmore extensions

$$0 \longrightarrow \mathcal{L}^p \longrightarrow \mathcal{E} \longrightarrow \mathcal{A} \longrightarrow 0$$

or by Kasparov modules. We refer to [22] for the definition of the Connes-Chern character in the odd case. We have already met one example of this though in the case of smooth Toeplitz extension

$$0 \longrightarrow \mathcal{K}^\infty \longrightarrow \mathcal{T}^\infty \longrightarrow C^\infty(S^1) \longrightarrow 0.$$

The map $f \to T_f$, sending a function on the circle to the corresponding Toeplitz operator, is a section for the symbol map σ. The extension is p-summable for all $p \geq 1$. Its Connes character is represented by the cyclic 1-cocycle on $C^\infty(S^1)$ defined by

$$\varphi_1(f,\, g) = \text{Tr}\,([T_f,\, T_g]).$$

6.7 Cyclic modules

Cyclic cohomology of algebras was first defined by Connes through explicit complexes or bicomplexes [19, 22]. Soon after he introduced the notion of *cyclic module* and defined its cyclic cohomology [21]. Later developments proved that this extension was of great significance. Apart from earlier applications, here we have the very recent work [37] in mind where the abelian category of cyclic modules plays the role of the category of motives in noncommutative geometry. Another recent example is the cyclic cohomology of Hopf algebras [39, 40, 67, 68], which cannot be defined as the cyclic cohomology of an algebra or a coalgebra but only as the cyclic cohomology of a cyclic module naturally attached to the given Hopf algebra.

The original motivation of [21] was to define cyclic cohomology of algebras as a derived functor. Since the category of algebras and algebra homomorphisms is not even an additive category (for the simple reason that the sum of two algebra homomorphisms is not an algebra homomorphism in general), the standard (abelian) homological algebra is not applicable. In Connes' approach, the category Λ_k of cyclic k-modules appears as an "abelianization" of the category of k-algebras. Cyclic cohomology is then shown to be the derived functor of the functor of traces, as we explain in this section.

The *simplicial category* Δ is the category whose objects are totally ordered sets

$$[n] = \{0 < 1 < \cdots < n\},$$

for $n = 0, 1, 2, \cdots$. A morphism $f : [n] \to [m]$ is an order preserving, i.e. monotone non-decreasing, map $f : \{0, 1, \cdots, n\} \to \{0, 1, \cdots, m\}$. Of particular interest among the morphisms of Δ are *faces* δ_i and *degeneracies* σ_i,

$$\delta_i : [n-1] \to [n], \quad \sigma_i : [n] \to [n-1], \qquad i = 1, 2, \cdots$$

By definition δ_i is the unique injective morphism missing i and σ_i is the unique surjective morphism identifying i with $i+1$. It can be checked that they satisfy the following *simplicial identities*:

$$\begin{aligned}
\delta_j \delta_i &= \delta_{j-1}\delta_i \quad \text{if} \quad i < j, \\
\sigma_i \sigma_i &= \sigma_i \sigma_i \quad \text{if} \quad i < j, \\
\sigma_i \delta_i &= \begin{cases} \sigma_{j-1}\delta_i & i < j \\ \text{id} & i = j \text{ or } i = j+1 \\ \sigma_j \delta_{i-1} & i > j+1. \end{cases}
\end{aligned}$$

Every morphism of Δ can be uniquely decomposed as a product of faces followed by a product of degeneracies.

The cyclic category Λ has the same set of objects as Δ and in fact contains Δ as a subcategory. An, unfortunately unintuitive, definition of its morphisms is as follows (see [24] for a more intuitive definition in terms of homotopy classes of maps from $S^1 \to S^1$). Morphisms of Λ are generated by simplicial morphisms δ_i, σ_i as above and $\tau_n : [n] \to [n]$ for $n \geq 0$. They are subject to the above simplicial as well as the following extra relations:

$$\begin{aligned}
\tau_n\delta_i &= \delta_{i-1}\tau_{n-1} && 1 \leq i \leq n \\
\tau_n\delta_0 &= \delta_n \\
\tau_n\sigma_i &= \sigma_{i-1}\tau_{n-1} && 1 \leq i \leq n \\
\tau_n\sigma_0 &= \sigma_n\tau_{n+1}^2 \\
\tau_n^{n+1} &= \text{id.}
\end{aligned}$$

A *cyclic object* in a category $\mathcal{C}$ is a functor $\Lambda^{\text{op}} \to \mathcal{C}$. A *cocyclic object* in $\mathcal{C}$ is a functor $\Lambda \to \mathcal{C}$. For any commutative ring k, we denote the category of cyclic k-modules by Λ_k. A morphism of cyclic k-modules is a natural transformation between the corresponding functors. Equivalently, a morphism $f : X \to Y$ consists of a sequence of k-linear maps $f_n : X_n \to Y_n$ compatible with the face, degeneracy, and cyclic operators. It is clear that Λ_k is an abelian category. The kernel and cokernel of a morphism f is defined pointwise: $(\text{Ker} f)_n = \text{Ker} f_n : X_n \to Y_n$ and $(\text{Coker} f)_n = \text{Coker} f_n : X_n \to Y_n$. More generally, if $\mathcal{A}$ is any abelian category then the category $\Lambda\mathcal{A}$ of cyclic objects in $\mathcal{A}$ is itself an abelian category.

Let Alg_k denote the category of unital k-algebras and unital algebra homomorphisms. There is a functor

$$\natural : Alg_k \longrightarrow \Lambda_k,$$

defined as follows. To an algebra A, we associate the cyclic module $A^\natural$ defined by $A_n^\natural = A^{\otimes(n+1)}, n \geq 0$, with face, degeneracy and cyclic operators given by

$$\begin{aligned}
\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\
\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \\
\sigma_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n \\
\tau_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n \otimes a_0 \cdots \otimes a_{n-1}.
\end{aligned}$$

A unital algebra map $f : A \to B$ induces a morphism of cyclic modules $f^\natural : A^\natural \to B^\natural$ by $f^\natural(a_0 \otimes \cdots \otimes a_n) = f(a_0) \otimes \cdots \otimes f(a_n)$.

Example 6.4. *We have*

$$Hom_{\Lambda_k}(A^\natural,\ k^\natural) \simeq T(A),$$

where $T(A)$ is the space of traces from $A \to k$. Under this isomorphism a trace τ is sent to the cyclic map $(f_n)_{n\geq 0}$, where

$$f_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \tau(a_0 a_1 \cdots a_n), \quad n \geq 0.$$

Now we can state the following fundamental theorem of Connes [21]:

Theorem 6.2. *For any unital k-algebra A, there is a canonical isomorphism*

$$HC^n(A) \simeq Ext^n_{\Lambda_k}(A^\natural, k^\natural), \qquad \text{for all } n \geq 0.$$

Now the above Example and Theorem, combined together, say that cyclic cohomology is the derived functor of the functor of traces $A \to T(A)$ where the word derived functor is understood to mean as above.

Motivated by the above theorem, one defines the cyclic cohomology and homology of any cyclic module M by

$$HC^n(M) := \mathrm{Ext}^n_{\Lambda_k}(M, k^\natural),$$

and

$$HC_n(M) := \mathrm{Tor}_n^{\Lambda_k}(M, k^\natural),$$

One can use the injective resolution used to prove the above Theorem to show that these Ext and Tor groups can be computed by explicit complexes and bicomplexes, similar to the situation with algebras. For example one has the following first quadrant bicomplex, called the *cyclic bicomplex* of M

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
M_2 & \xleftarrow{1-\lambda} & M_2 & \xleftarrow{N} & M_2 & \xleftarrow{1-\lambda} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
M_1 & \xleftarrow{1-\lambda} & M_1 & \xleftarrow{N} & M_1 & \xleftarrow{1-\lambda} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
M_0 & \xleftarrow{1-\lambda} & M_0 & \xleftarrow{N} & M_0 & \xleftarrow{1-\lambda} & \cdots
\end{array}$$

whose total homology is naturally isomorphic to cyclic homology. Here the operator $\lambda : M_n \to M_n$ is defined by $\lambda = (-1)^n \tau_n$, while

$$b = \sum_{i=0}^{n} (-1)^i \delta_i, \qquad b' = \sum_{i=0}^{n-1} (-1)^i \delta_i,$$

and $N = \sum_{i=0}^{n} \lambda^i$. Using the simplicial and cyclic relations, one can check that $b^2 = b'^2 = 0$, $b(1-\lambda) = (1-\lambda)b'$ and $b'N = Nb'$. These relations amount to saying that the above is a bicomplex.

The (b, B)-bicomplex of a cyclic module is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
M_2 & \xleftarrow{B} & M_1 & \xleftarrow{B} & M_0 \\
\downarrow b & & \downarrow b & & \\
M_1 & \xleftarrow{B} & M_0 & & \\
\downarrow b & & & & \\
M_0 & & & &
\end{array}$$

whose total homology is again isomorphic to the cyclic homology of M (this time we have to assume that k is a field of characteristic 0). Here $B : M_n \to M_{n+1}$ is Connes' boundary operator defined by $B = (1-\lambda)sN$, where $s = (-1)^n \sigma_n$.

A remarkable property of the cyclic category Λ, not shared by the simplicial category, is its *self-duality* in the sense that there is a natural isomorphism of categories $\Lambda \simeq \Lambda^{\text{op}}$. Roughly speaking, Connes' duality functor $\Lambda^{\text{op}} \longrightarrow \Lambda$ acts as the identity on objects of Λ and exchanges face and degeneracy operators while sending the cyclic operator to its inverse. Thus to a cyclic (resp. cocyclic) module one can associate a cocyclic (resp. cyclic) module by applying Connes' duality isomorphism. In the next section we shall see examples of cyclic modules in Hopf cyclic (co)homology that are dual to each other in the above sense.

6.8 Hopf cyclic cohomology

In their fundamental work on index theory of transversally elliptic operators [39], Connes and Moscovici developed a new cohomology theory for Hopf algebras based on ideas in cyclic cohomology. This theory can be regarded as the right noncommutative analogue of both group and Lie algebra homology, although this was not the original motivation behind it. Instead, the main reason was to obtain a *noncommutative characteristic map*

$$\chi_\tau : HC^*_{(\delta,\sigma)}(H) \longrightarrow HC^*(A), \tag{39}$$

for an action of a Hopf algebra H on an algebra A endowed with an "invariant trace" $\tau : A \to \mathbb{C}$. Here, the pair (δ, σ) consists of a grouplike element $\sigma \in H$ and a character $\delta : H \to \mathbb{C}$ satisfying certain compatibility conditions to be discussed later in this section. While in this section we confine ourselves to Hopf algebras, we refer to the recent surveys [43] and [82] for later developments in the subject inspired by [39].

The characteristic map (39) is induced, on the level of cochains, by a map

$$\chi_\tau : H^{\otimes n} \longrightarrow C^n(A)$$

defined by

$$\chi_\tau(h_1 \otimes \cdots \otimes h_n)(a^0, \cdots, a^n) = \tau(a_0 h_1(a^1) \cdots h_n(a^n)). \tag{40}$$

Maps like this have quite a history in cyclic cohomology, going back to [18]. Notice, for example, that the fundamental cyclic cocycles (31), (32) on the noncommutative 2-torus are of this form, where H is the enveloping algebra of a two-dimensional abelian Lie algebra acting by a pair of commuting derivations δ_1 and δ_2 on $\mathcal{A}_\theta$.

Exercise 6.9. *Let $\delta_1, \cdots, \delta_p$ be a commuting family of derivations on an algebra $\mathcal{A}$ and let $\tau : \mathcal{A} \to \mathbb{C}$ be an invariant trace, i.e. $\tau(\delta_i(a)) = 0$ for all $a \in \mathcal{A}$ and $i = 1, \cdots, p$. Show that*

$$\varphi(a^0, \cdots, a^p) = \sum_{\sigma \in S_n} (-1)^\sigma \tau(a^0 \delta_{\sigma_1}(a^1) \cdots \delta_{\sigma_p}(a^p))$$

is a cyclic p-cocycle on $\mathcal{A}$.

Applied to higher dimensional noncommutative tori, these cocycles give a basis for its periodic cyclic cohomology.

For applications to transverse geometry and number theory [41, 42, 43], it is important to formulate a notion of 'invariant trace' under the presence of a modular pair. Let A be an H-module algebra, δ a character of H, and $\sigma \in H$ a grouplike element. A linear map $\tau : A \to \mathbb{C}$ is called *δ-invariant* if for all $h \in H$ and $a \in A$,

$$\tau(h(a)) = \delta(h)\tau(a).$$

τ is called a σ-*trace* if for all a, b in A,

$$\tau(ab) = \tau(b\sigma(a)).$$

For $a, b \in A$, let

$$\langle a,\, b\rangle := \tau(ab).$$

Then the δ-invariance property of τ is equivalent to the *integration by parts formula*:

$$\langle h(a),\, b\rangle = \langle a,\, \widetilde{S}_\delta(h)(b)\rangle, \tag{41}$$

where the δ-*twisted antipode* $\widetilde{S}_\delta : H \to H$ is defined by $\widetilde{S}_\delta = \delta * S$. That is,

$$\widetilde{S}_\delta(h) = \delta(h^{(1)})S(h^{(2)}).$$

Loosely speaking, this amounts to saying that the formal adjoint of the differential operator h is $\widetilde{S}_\delta(h)$. Following [39, 40], we say $(\delta,\, \sigma)$ is a *modular pair* if $\delta(\sigma) = 1$, and a *modular pair in involution* if

$$\widetilde{S}_\delta^2(h) = \sigma h \sigma^{-1},$$

for all h in H.

Examples 6.2. 1. For any commutative or cocommutative Hopf algebra we have $S^2 = 1$. It follows that $(\varepsilon, 1)$ is a modular pair in involution.

2. The original non-trivial example of a modular pair in involution is the pair $(\delta,\, 1)$ for the Connes-Moscovici Hopf algebra $\mathcal{H}_1$. Let δ denote the unique extension of the modular character

$$\delta : \mathfrak{g}_{\text{aff}} \to \mathbb{R}; \quad \delta(X) = 1, \quad \delta(Y) = 0,$$

to a character $\delta : U(\mathfrak{g}_{\text{aff}}) \to \mathbb{C}$. There is a unique extension of δ to a character, denoted by the same symbol $\delta : \mathcal{H}_1 \to \mathbb{C}$. Indeed the relations $[Y, \delta_n] = n\delta_n$ show that we must have $\delta(\delta_n) = 0$, for $n = 1, 2, \cdots$. One can then check that these relations are compatible with the algebra structure of $\mathcal{H}_1$.

Now the algebra $A_\Gamma = C_0^\infty(F^+(M)) \rtimes \Gamma$ from Section 4.2 admits a δ-invariant trace $\tau : A_\Gamma \to \mathbb{C}$ under its canonical $\mathcal{H}_1$ action. It is given by [39]:

$$\tau(fU_\varphi^*) = \int_{F^+(M)} f(y, y_1)\frac{dy\,dy_1}{y_1^2}, \quad \text{if } \varphi = 1,$$

and $\tau(fU_\varphi^*) = 0$, otherwise.

3. Let $H = A(SL_q(2))$ denote the Hopf algebra of functions on quantum $SL(2)$. As an algebra it is generated by x, u, v, y, subject to the relations

$$ux = qxu, \;\; vx = qxv, \;\; yu = quy, \;\; yv = qvy,$$
$$uv = vu, \;\; xy - q^{-1}uv = yx - quv = 1.$$

The coproduct, counit and antipode of H are defined by

$$\Delta(x) = x \otimes x + u \otimes v, \quad \Delta(u) = x \otimes u + u \otimes y,$$
$$\Delta(v) = v \otimes x + y \otimes v, \quad \Delta(y) = v \otimes u + y \otimes y,$$
$$\epsilon(x) = \epsilon(y) = 1, \quad \epsilon(u) = \epsilon(v) = 0,$$
$$S(x) = y, \;\; S(y) = x, \;\; S(u) = -qu, \;\; S(v) = -q^{-1}v.$$

Define a character $\delta : H \to \mathbb{C}$ by

$$\delta(x) = q, \;\; \delta(u) = 0, \;\; \delta(v) = 0, \;\; \delta(y) = q^{-1}.$$

One checks that $\widetilde{S}_\delta^2 = \mathrm{id}$. This shows that $(\delta, 1)$ is a modular pair in involution for H. This example and its Hopf cyclic cohomology are studied in [80].

More generally, it is shown in [40] that *coribbon Hopf algebras* and compact quantum groups are endowed with canonical modular pairs in involution of the form $(\delta, 1)$ and, dually, ribbon Hopf algebras have canonical modular pairs in involution of the type $(1, \sigma)$.

4. It is shown in [67] that modular pairs in involution are in fact one-dimensional examples of *stable anti-Yetter-Drinfeld modules* over Hopf algebras introduced there. These modules are noncommutative coefficient systems for the general Hopf cyclic cohomology theory developed in [68].

Now let (H, δ, σ) be a Hopf algebra endowed with a modular pair in involution. In [39] Connes and Moscovici attach a cocyclic module $H^\natural_{(\delta,\sigma)}$ to this data as follows. Let

$$H^{\natural,0}_{(\delta,\sigma)} = \mathbb{C}, \quad \text{and} \quad H^{\natural,n}_{(\delta,\sigma)} = H^{\otimes n}, \quad \text{for} \;\; n \geq 1.$$

Its face, degeneracy and cyclic operators δ_i, σ_i, and τ_n are defined by

$$\begin{aligned}
\delta_0(h_1 \otimes \cdots \otimes h_n) &= 1 \otimes h_1 \otimes \cdots \otimes h_n \\
\delta_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes \Delta(h_i) \otimes \cdots \otimes h_n \;\; \text{for} \;\; 1 \leq i \leq n \\
\delta_{n+1}(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_n \otimes \sigma \\
\sigma_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes \epsilon(h_{i+1}) \otimes \cdots \otimes h_n \;\; \text{for} \;\; 0 \leq i \leq n \\
\tau_n(h_1 \otimes \cdots \otimes h_n) &= \Delta^{n-1}\widetilde{S}(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma).
\end{aligned}$$

The cyclic cohomology of the cocyclic module $H^{\natural}_{(\delta,\sigma)}$ is called the Hopf cyclic cohomology of the triple (H, δ, σ) and will be denoted by $HC^n_{(\delta,\sigma)}(H)$.

Examples 6.3. 1. For $H = \mathcal{H}_n$, the Connes-Moscovici Hopf algebra, we have [39] (cf. also [102])

$$HP^n_{(\delta,1)}(\mathcal{H}_n) \simeq \bigoplus_{i=n \text{ (mod 2)}} H^i(\mathfrak{a}_n, \mathbb{C})$$

where $\mathfrak{a}_n$ is the Lie algebra of formal vector fields on $\mathbb{R}^n$.

2. For $H = U(\mathfrak{g})$ the enveloping algebra of a Lie algebra $\mathfrak{g}$, we have [39]

$$HP^n_{(\delta,1)}(H) \cong \bigoplus_{i=n \text{ (mod 2)}} H_i(\mathfrak{g}, \mathbb{C}_\delta)$$

3. For $H = \mathbb{C}[G]$ the coordinate ring of a nilpotent affine algebraic group G, we have [39]

$$HP^n_{(\epsilon,1)}(H) \cong \bigoplus_{i=n \text{ (mod 2)}} H^i(\mathfrak{g}, \mathbb{C}),$$

where $\mathfrak{g} = Lie(G)$.

4. If H admits a normalized left Haar integral, then [45]

$$HP^1_{(\delta,\sigma)}(H) = 0, \qquad HP^0_{(\delta,\sigma)}(H) = \mathbb{C}.$$

Recall that a linear map $\int : H \to \mathbb{C}$ is called a normalized left Haar integral if for all $h \in H$, $\int h = \int(h^{(1)})h^{(2)}$ and $\int 1 = 1$. It is known that a Hopf algebra defined over a field admits a normalized left Haar integral if and only if it is cosemisimple [119]. Compact quantum groups and group algebras are known to admit a normalized Haar integral in the above sense. In the latter case $\int : \mathbb{C}G \to k$ sending $g \mapsto 0$ for all $g \neq e$ and $e \mapsto 1$ is a Haar integral. Note that G need not be finite. In this regard, we should also mention that there are interesting examples of finite-dimensional non-cosemisimple Hopf algebras defined as quantum groups at roots of unity. Nothing is known about the cyclic (co)homology of these Hopf algebras.

5. If $H = U_q(\mathrm{sl}_2)$ is the quantum universal algebra of sl_2, we have [45]

$$HP^0_{(\epsilon,\sigma)}(H) = 0, \quad HP^1_{(\epsilon,\sigma)}(H) = \mathbb{C} \oplus \mathbb{C}.$$

6. Let H be a commutative Hopf algebra. The periodic cyclic cohomology of the cocyclic module $\mathcal{H}^{\natural}_{(\epsilon,1)}$ can be computed in terms of the Hochschild homology of the coalgebra H with trivial coefficients.

Proposition 6.1. *([80]) Let H be a commutative Hopf algebra. Its periodic cyclic cohomology in the sense of Connes-Moscovici is given by*

$$HP^n_{(\epsilon,1)}(H) = \bigoplus_{i=n \ (mod\ 2)} H^i(H, \mathbb{C}).$$

For example, if $H = \mathbb{C}[G]$ is the algebra of regular functions on an affine algebraic group G, the coalgebra complex of H is isomorphic to the group cohomology complex of G where instead of regular cochains one uses regular functions $G \times G \times \cdots \times G \to k\mathbb{C}$. Denote this cohomology by $H^i(G, \mathbb{C})$. It follows that

$$HP^n_{(\epsilon,1)}(\mathbb{C}[G]) = \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H^i(G, \mathbb{C}).$$

As is remarked in [80], when the Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ is nilpotent, it follows from Van Est's theorem that $H^i(G, \mathbb{C}) \simeq H^i(\mathfrak{g}, \mathbb{C})$. This gives an alternative proof of Proposition 4 and Remark 5 in [39].

Now given (H, δ, σ), a Hopf algebra endowed with a modular pair in involution as above, let $\mathcal{A}$ be an algebra with an H-action and let $\tau : \mathcal{A} \to \mathbb{C}$ be a δ-invariant σ-trace. Then one can check, using in particular the integration by parts formula (41), that the characteristic map (40) is a morphism of cocyclic modules. It follows that we have a well defined map

$$\chi_\tau : HC^n_{(\delta,\sigma)}(H) \to HC^n(\mathcal{A})$$

Example 6.5. Let $\mathcal{A}$ be an n-dimensional smooth noncommutative torus with canonical commutating derivations $\delta_1, \cdots, \delta_n$ defined by $\delta_i(U_j) = \delta_{ij} U_i$. They define an action of $H = U(\mathfrak{g})$ on $\mathcal{A}$ where $\mathfrak{g}$ is the abelian n-dimensional Lie algebra. The canonical trace τ is invariant under this action. The characteristic map χ_τ combined with the antisymmetrization map

$$\bigwedge^k \mathfrak{g} \to U(\mathfrak{g})^{\otimes k}$$

defines a map

$$\bigwedge^k \mathfrak{g} \to HC^k(\mathcal{A}).$$

This is the map of Exercise 6.8.

In the rest of this section we recall a dual cyclic theory for Hopf algebras which was defined and studied in [80]. This theory is needed, for example, when one studies coactions of Hopf algebras and quantum groups on noncommutative spaces. Notice that for compact quantum groups coactions are more natural. Also, as we mentioned before, for cosemisimple Hopf algebras, i.e. Hopf algebras endowed with a normalized Haar integral, the Hopf cyclic cohomology is trivial in positive dimensions, but the dual theory is non-trivial. There is a clear analogy with continuous group cohomology here.

Let (δ,σ) be a modular pair on H such that $\widehat{S}^2 = \mathrm{id}_H$, where $\widehat{S}(h) := h^{(2)}\sigma S(h^{(1)})$. We define a cyclic module $\widetilde{H}_\natural^{(\delta,\sigma)}$ by

$$\widetilde{H}_{\natural,0}^{(\delta,\sigma)} = \mathbb{C}, \qquad \widetilde{H}_{\natural,n}^{(\delta,\sigma)} = H^{\otimes n}, \quad n > 0.$$

Its face, degeneracy, and cyclic operators are defined by

$$\begin{aligned}
\delta_0(h_1 \otimes h_2 \otimes ... \otimes h_n) &= \epsilon(h_1) h_2 \otimes h_3 \otimes ... \otimes h_n \\
\delta_i(h_1 \otimes h_2 \otimes ... \otimes h_n) &= h_1 \otimes h_2 \otimes ... \otimes h_i h_{i+1} \otimes ... \otimes h_n \\
\delta_n(h_1 \otimes h_2 \otimes ... \otimes h_n) &= \delta(h_n) h_1 \otimes h_2 \otimes ... \otimes h_{n-1} \\
\sigma_0(h_1 \otimes h_2 \otimes ... \otimes h_n) &= 1 \otimes h_1 \otimes ... \otimes h_n \\
\sigma_i(h_1 \otimes h_2 \otimes ... \otimes h_n) &= h_1 \otimes h_2 ... \otimes h_i \otimes 1 \otimes h_{i+1} ... \otimes h_n \\
\sigma_n(h_1 \otimes h_2 \otimes ... \otimes h_n) &= h_1 \otimes h_2 \otimes ... \otimes 1, \\
\tau_n(h_1 \otimes h_2 \otimes ... \otimes h_n) &= \delta(h_n^{(2)}) \sigma S(h_1^{(1)} h_2^{(1)} ... h_{n-1}^{(1)} h_n^{(1)}) \otimes h_1^{(2)} \otimes ... \otimes h_{n-1}^{(2)}.
\end{aligned}$$

We denote the cyclic homology of this cyclic module by $\widetilde{HC}_\bullet^{(\delta,\sigma)}(H)$.

Remark 9. *It is not difficult to check that $(\delta \circ S^{-1}, \sigma^{-1})$, is a modular pair in involution if and only if (δ,σ) is a modular pair with $\widehat{S}^2 = id_H$. In other words (δ,σ) is a modular pair in involution in the sense of Connes and Moscovici [39] if and only if $(\delta \circ S, \sigma^{-1})$ is a modular pair in involution in the sense of [80].*

Now let A be an H-comodule algebra. A linear map, $\tau : A \to \mathbb{C}$ is called a δ-trace if

$$\tau(ab) = \tau(b^{(0)}a)\delta(b^{(1)}) \qquad \forall a, b \in A.$$

It is called σ-invariant if for all $a \in A$,

$$\tau(a^{(0)})a^{(1)} = \tau(a)\sigma.$$

Now consider the map $\chi_\tau : A_\natural \to \widetilde{H}_\natural^{(\delta,\sigma)}$ defined by

$$\chi_\tau(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \tau(a_0 a_1^{(0)} \cdots a_n^{(0)}) a_1^{(1)} \otimes a_2^{(1)} \otimes \cdots \otimes a_n^{(1)}.$$

It is proved in [80] that χ_τ is a morphism of cyclic modules. This looks rather uninspiring, but once dualized we obtain a characteristic map for coactions

$$\chi_\tau^* : \widetilde{HC}^n_{(\delta,\sigma)}(H) \longrightarrow HC^n(A),$$

which can be useful as will be shown in Example (6.7) below.

Next we state a theorem which computes the Hopf cyclic homology of cocommutative Hopf algebras in terms of the Hochschild homology of the underlying algebra:

Theorem 6.3. *([80]) If H is a cocommutative Hopf algebra, then*

$$\widetilde{HC}_n^{(\delta,1)}(H) = \bigoplus_{i\geq 0} H_{n-2i}(H, \mathbb{C}_\delta),$$

where $\mathbb{C}_\delta$ is the one-dimensional module defined by δ.

Example 6.6. One knows that for any Lie algebra $\mathfrak{g}$,

$$H_n(U(\mathfrak{g}), \mathbb{C}_\delta) \simeq H_n^{Lie}(\mathfrak{g}, \mathbb{C}_\delta).$$

So by Theorem 6.3 we have

$$\widetilde{HC}_n^{(\delta,1)}(U(\mathfrak{g})) \simeq \bigoplus_{i\geq 0} H_i^{Lie}(\mathfrak{g}, \mathbb{C}_\delta).$$

Example 6.7. Let $H = \mathbb{C}\Gamma$ be the group algebra of a discrete group Γ. Then from Theorem 6.3 we have

$$\widetilde{HC}_n^{(\epsilon,1)}(\mathbb{C}\Gamma) \simeq \bigoplus_{i\geq 0} H_{n-2i}(\Gamma, \mathbb{C}),$$
$$\text{and } \widetilde{HP}_n^{(\epsilon,1)}(\mathbb{C}\Gamma) \simeq \bigoplus_{i=n \pmod 2} H_i(\Gamma, \mathbb{C}).$$

Now any Hopf algebra H is a comodule algebra over itself via the coproduct map $H \longrightarrow H \otimes H$. The map $\tau : \mathbb{C}\Gamma \to \mathbb{C}$ defined by

$$\tau(g) = \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases}$$

is a δ-invariant σ-trace for $\delta = \epsilon$, $\sigma = 1$. The dual characteristic map

$$\chi_\tau^* : \widetilde{HC}^n_{(\epsilon,1)}(\mathbb{C}\Gamma) \to HC^n(\mathbb{C}\Gamma)$$

combined with the inclusion $H^n(\Gamma, \mathbb{C}) \hookrightarrow \widetilde{HC}^n_{(\epsilon,1)}(\mathbb{C}\Gamma)$ gives us a map

$$H^n(\Gamma, \mathbb{C}) \to HC^n(\mathbb{C}\Gamma)$$

from group cohomology to cyclic cohomology. The image of a normalized group n-cocycle $\varphi(g_1, \cdots, g_n)$ under this map is the cyclic n-cocycle $\hat{\varphi}$ defined by

$$\hat{\varphi}(g_0, \cdots, g_n) = \begin{cases} \varphi(g_1, \cdots, g_n) & g_0 g_1 \cdots g_n = e \\ 0 & g_0 g_1 \cdots g_n \neq e \end{cases}$$

Thus the characteristic map for Hopf cyclic homology reduces to a well known map in noncommutative geometry [24]. This should be compared with Example 6.5.

It would be very interesting to compute the Hopf cyclic homology $\widetilde{HC}_n$ of quantum groups. We cite one of the very few results known in this direction. Let $H = A(SL_q(2, \mathbb{C}))$ be the Hopf algebra of quantum SL_2. As an algebra it is generated by a, b, c, d, with relations

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd,$$

$$bc = cb, \quad ad - q^{-1}bc = da - qbc = 1.$$

The coproduct, counit and antipode of H are defined by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes c$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d$$

$$\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0,$$

$$S(a) = d, \quad S(d) = a, \quad S(b) = -qb, \quad S(c) = -q^{-1}c.$$

We define a modular pair (σ, δ) by

$$\delta(a) = q, \quad \delta(b) = 0, \quad \delta(c) = 0, \quad \delta(d) = q^{-1},$$

$\sigma = 1$. Then we have $\widetilde{S}^2_{(1,\delta)} = \mathrm{id}$.

Theorem 6.4. *([80]) For q not a root of unity, one has*

$$\widetilde{HC}_1(A(SL_q(2, \mathbb{C}))) = \mathbb{C} \oplus \mathbb{C}, \qquad \widetilde{HC}_n(A(SL_q(2, \mathbb{C}))) = 0, \quad n \neq 1,$$

and

$$\widetilde{HP}_0(A(SL_q(2, \mathbb{C}))) = \widetilde{HP}_1(A(SL_q(2, \mathbb{C}))) = 0.$$

In [67, 68], following the lead of [1, 80, 81], Hajac-Khalkhali-Rangipour-Sommerhäuser define a full fledged Hopf cyclic cohomology theory for algebras or coalgebras endowed with actions or coactions of a Hopf algebra. This extends the pioneering work of Connes and Moscovici in two different directions. It allows coefficients for the theory and instead of Hopf algebras one now works with algebras or coalgebras with a Hopf action. It turns out that the periodicity condition $\tau_n^{n+1} = \mathrm{id}$ for the cyclic operator puts very stringent conditions on the type of coefficients that are allowable and the correct class of Hopf modules turned to be the class of *stable anti-Yetter-Drinfeld modules* over a Hopf algebra. It also sheds light on Connes-Moscovici's modular pairs in involution by interpreting them as one-dimensional stable anti-Yetter-Drinfeld modules.

The category of anti-Yetter-Drinfeld modules over a Hopf algebra H is a twisting, or variant rather, of the category of Yetter-Drinfeld H-modules. Notice that this latter category is widely studied primarily because of its connections with quantum group theory and with invariants of knots and low-dimensional topology [124]. Technically it is obtained from the latter by replacing the antipode S by S^{-1} although this connection is hardly illuminating. We refer to [82] and references therein for a survey of anti-Yetter-Drinfeld modules and Hopf cyclic cohomology.

Remark 10. *The picture that is emerging is quite intriguing and seems to be at the crossroads of three different areas: von Neumann algebras, quantum groups and low-dimensional topology, and cyclic cohomology. We have Connes-Moscovici's modular pairs in involution which was suggested by type III factors and non-unimodular Lie groups and turned out to be examples of anti-Yetter-Drinfeld modules. As we saw above the latter category is of fundamental importance in Hopf cyclic cohomology. One obviously needs to understand these connections much better.*

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