Noncommutative Geometry for Poets

Masoud Khalkhali
Mathematics Department, University of Western Ontario
London, ON, Canada

So here I am, in the middle way, having had twenty years-
Twenty years largely wasted, the years of *l'entre deux guerres*
Trying to use words, and every attempt
Is a wholly new start, and a different kind of failure
Because one has only learnt to get the better of words
For the thing one no longer has to say, or the way in which
One is no longer disposed to say it. And so each venture
Is a new beginning, a raid on the inarticulate,
With shabby equipment always deteriorating
In the general mess of imprecision of feeling,
Undisciplined squads of emotion. And what there is to conquer
By strength and submission, has already been discovered
Once or twice, or several times, by men whom one cannot hope
To emulate-but there is no competition-
There is only the fight to recover what has been lost
And found and lost again and again: and now, under conditions
That seem unpropitious. But perhaps neither gain nor loss.
For us, there is only the trying. The rest is not our business.
T.S. Eliot, *Four Quartets*

Classical spaces

Most areas of mathematics and its applications deal with the notion of *space* either as a primary object of study, or as an organizational tool to codify information in a suggestive manner, amenable to geometric intuition. For example in topology or differential geometry one studies a multi dimensional space from a topological or metric stand point, e.g. counting its holes of a given dimension, or measuring its curvature and so on. Similarly in general relativity and high energy physics spacetime together with classical or quantum fields defined over it is the primary object of study. On the other hand linear spaces of functions, spaces of sections of a vector bundle, and in general nonlinear spaces of functions between two manifolds are devices that help us to gain information about the domain and target spaces. Similarly when a computer scientist talks about a hypercube he or she just uses the language of n-dimensional cartesian geometry to codify bits of information.

We have come a long way, starting with Euclid's simple, but far reaching, ideas of space and geometry in dimensions 1, 2 and 3, to accept modern notions of n-dimensional or even infinite dimensional spaces, non-Euclidean and Riemannian spaces and so forth as a natural playground for our geometric intuition .

We take these generalized notions of space for granted and, even worse, often forget that earlier generations of mathematicians had to surmount great psychological, social, and- not the least!, mathematical barriers to introduce these notions and to work with them. Motivations to do this came, as usual, from within mathematics as well its applications, e.g. to physics. Great minds of the 19th century mathematics, Gauss, Riemann, Poincare, Klein, just to name a few, were behind such drastic change of attitude and perspective with respect to the notion of space. On a technical level Cantor's discovery of set theory and formalist approach of Hilbert and, much later, Bourbaki played an important role and provided the necessary tools to make this paradigm shift possible.

And then one should not forget the impetus from physics. Starting with Gallileo and Newton, if not Aristotle!, physicists and natural philosophers have often marvelled on the nature of space and time. After all physical reality takes place in space and time. The revolution brought in by the special theory of relativity of Einstein, its ultimate geometrization by Minkowski, and the general theory of relativity puts the idea of space in the center of large areas of fundamental physics.

Nowadays we refer to these types of spaces as *classical spaces*: a set of points endowed with some extra structure, perhaps a topology or a smooth or metric structure, or a measure and so on. But this is only half of the story. There is an alternative algebraic way to describe a classical space which is very relevant to our story and pointed the way to the future development of the concept of space.

A fundamental duality

An important and old theme in mathematics is the *algebra-geometry correspondence*. This is an old idea, as old as mathematics itself, but it is the case that each generation of mathematicians find new incarnations of this principle and, *to preserve the principle*, push the boundaries by discovering new concepts and notions on either side of the equation *algebra = geometry*.

On a physiological level this is perhaps related to a division in human brain: one computes and manipulates symbols with the left hemispheric side of the brain and one visualizes things with the right brain. computations evolve in time and has a temporal character while visualization is instant and immediate. It was for good reason that Hamilton, one of the creators of modern algebraic methods, called his abstract approach to algebra, e.g. to complex numbers and quaternions, the *science of pure time*.

In modern terms, the algebra $\leftrightarrow$ geometry correspondence, in its simplest form, means that the information about a (classical) space can be encoded in the (commutative) algebra of functions on that space. The words classical and commutative are meant to represent the same idea here. Great duality theorems of mathematics, like the Gelfand-Naimark theorem, Hilbert's Nullstellensatz, and similar theorems, teach us that one can look at a classical space from two distinct, but equivalent, points of view: algebraic or geometric. Thus, for example, one learns that the information about a, say, compact Hausdorff space, is totally encoded in the algebra of continuous complex valued functions on that space (the theorem of Gelfand and Naimark). Algebras that appear this way are commutative C^*-algebras. This is a remarkable theorem since it tells us that any natural construction that involves compact spaces and continuous maps between has a purely algebraic reformulation and vice-versa any statement about commutative C^*-algebras and C^*-algebraic maps between them has a purely topological meaning.

There is however, and this is a very important point, a vast array of noncommutative C^*-algebras that naturally appear, say, in harmonic analysis (as completions of group algebras of noncommutative groups), in differential geometry (as noncommutative algebras attached to foliations), in quantum mechanics (properly formulated using bounded instead of unbounded operators), and in general as algebras of bounded operators on Hilbert space. Noncommutative geometry extends the above mentioned duality between commutative algebras and spaces by dealing with a not necessarily commutative algebra, say a C^*-algebra, as the algebra of 'functions on a noncommutative space'.

Alain Connes' vision of space

Once again we are at the threshold of a paradigm shift in our geometric intuition. The new notion of space, called a *noncommutative space*, and the mathematics that goes with it, called *noncommutative geometry* is the brainchild of one man, Alain Connes, who is responsible for all the major results in this theory.

From a measure theoretic perspective, von Neumann algebras provide the noncommutative analogue of Borel measure theory, while C^*-algebras furnish one with the right notion of a noncommutative locally compact space. Noncommutative analogues of smooth manifolds in full generality have yet to be discovered but at the moment we know what should be at least a noncommutative spin Riemannian manifold with its natural Dirac operator. This is captured by the concept of a *spectral triple*. In fact a major idea in the theory is to first formulate classical notions and theorems in spectral and Hilbertian terms, and then pass to the noncommutative case. For example Weyl's law on the asymptotic behavior of eigenvalues of Laplacian of a compact manifold allows one to define the dimension and volume of a manifold in noncommutative terms and to generalize it to noncommutative geometry.

The passage from a classical space to a noncommutative space is remarkably similar to what was done in quantum mechanics by Heisenberg in 1925. From a mathematical point of view, transition from classical mechanics to quantum mechanics amounts to passing from the commutative algebra of classical observables to the noncommutative algebra of quantum mechanical observables. Recall that in classical mechanics an observable (e.g. energy, position, momentum, etc.) is a function on a manifold called the phase space of the system. Immediately after Heisenbergs work, ensuing papers by Dirac and Born-Heisenberg-Jordan , made it clear that a quantum mechanical observable is a (selfadjoint) operator on a Hilbert space called the state space of the system. Thus the commutative algebra of functions on a space is replaced by the noncommutative algebra of operators on a Hilbert space. A little more than fifty years after these developments, Alain Connes realized that a similar procedure can in fact be applied, with great benefit, to areas of mathematics where the classical notions of space looses its applicability and pertinence and can be replaced by a new idea of space, represented by a noncommutative algebra.

It is extremely important to realize that this is not a game of generalization for the sake of generalization. Finding the right concepts and theories is very hard here and totally new phenomena appears that have no classical counterparts. In fact what makes the whole project of noncommutative geometry a viable and extremely important enterprize are the following three fundamental points, already emphasized in Connes' book, "Noncommutative Geometry":

• There is a vast repertoire of noncommutative spaces and there are very general methods to construct them. For example, consider a *bad quotient* of a nice and smooth space by an equivalence relation. Typically the quotient space is not even Hausdorff and has very bad singularities so that it is beyond the reach of classical geometry and topology. Orbit spaces of group actions and the space of leaves of a foliation are examples of this situation. In algebraic topology one replaces such bad quotients by homotopy quotients, by using the general idea of a classifying space. This is however not good enough and not general enough as the classifying space is only a homotopy construction and does see any of the smooth structure. A key observation of Connes is that in all these situations one can attach a nice noncommutative space, a C^* or von Neumann algebra that captures most of the information hidden in these quotients. The general construction starts by first replacing the equivalence relation by a groupoid and then considering the groupoid algebra, a generalization of the group algebra.

• The possibility of extending many of the tools of classical geometry and topology that are used to probe classical spaces to this noncommutative realm. Of all the topological invariants of spaces, topological K-theory of Atiyah and Hirzebruch and it most important theorem, Bott periodicity theorem, has the most natural and straightforward extension to the noncommutative world. Using the duality theorem of Serre-Swan on the correspondence between the geometric concept of a vector bundle and algebraic notion of a finitely generated projective module, one easily extends the topological K-functor to the class of noncommutative Banach algebras. With this extension in fact the proof of Bott periodicity becomes easier! It is much harder to find the right noncommutative analogue of de Rham cohomology and Chern-Weil theory. This was achieved by Connes in 1981 and the resulting theory is called *cyclic cohomology*. Another big result of recent years is the local index formula of Connes and Moscovici. This result is a vast extension of the classical Atiyah-Singer index theorem to the noncommutative setup.

• Applications. Even if we wanted to restrict ourselves just to classical spaces, methods of noncommutative geometry would still be very relevant and useful. For example the most natural and general proofs of the Novikov conjecture on the homotopy invariance of higher signatures of non-simply connected manifolds use the machinery of noncommutative geometry. The relevant noncommutative space here is the (completion of the) group ring of the fundamental group of the manifold. For another example, we mention a recent proof of the gap labelling conjecture about the spectrum of a Schrodinger operator associated to a quasicrystal that makes use of Connes' index theorem for foliations. We also mention the geometrization of the standard Salam-Weinberg-Glashow model of elementary particles over a noncommutative spacetime as its basic ingredient due to Connes and Connes-Lott. This is simply impossible over a classical space.

Moving to more recent applications, we mention the approach by Connes to Riemann hypothesis, the spectral realization of zeros of zeta from the noncommutative space of Q-lattices (joint with Marcolli), as well as his (joint with Kreimer) work on the mathematical underpinnings of renormalization in quantum field theory as a Riemann-Hilbert Correspondence. These results tie up experimentally tested areas of high energy physics with number theory and algebraic geometry. In fact the ensuing work (joint with Marcolli) revealed a beautiful motivic Galois group, conjectured and named *cosmic Galois group* by Cartier, hidden in quantum field theory. These results have brought noncommutative geometry much closer to central areas of modern number theory, algebraic geometry and high energy physics and will be the subject of intensive studies in coming years.

But that is not all and in fact by now, just 25 years after the creation of these ideas, one can say examples abound! With the new freedom afforded by allowing noncommutativity as a viable option the reader can move on to his or her own favorite mathematics or physics subject and find more examples and applications!