Scalar Curvature, Connes' Trace Theorem, and Einstein-Hilbert Action for Noncommutative Four Tori

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What is curvature?

Classical geometry: R^i_{jkl}, R_{ij}, R.

Einstein-Hilbert action: $\int_M R\,dvol$.

Einstein field equation $R_{ij} - \frac{R}{2} g_{ij} = 0$.

Chern-Weil theory: $\mathrm{Tr}(e^\Omega)$ $\Omega^i_j = R_{ijkl} dx^k \wedge dx^l$.

Curvature in NCG

Connection-Curvature formalism of Connes in 1981 (NC Chern-Weil theory):

$$\nabla : E \to E \otimes_A \Omega^1 A, \qquad \nabla \in \mathrm{End}_{\mathbb{C}}(E \otimes_A \Omega A)$$

$$\nabla^2 \in \mathrm{End}_{\Omega A}(E \otimes_A \Omega A) = \mathrm{End}_A(E) \otimes_A \Omega A.$$

Any cyclic cocycle $\varphi : A^{\otimes(2n+1)} \to \mathbb{C}$ defines a closed graded trace $\int_\varphi : \Omega A \to \mathbb{C}$. Can define $\int_\varphi \mathrm{Tr}(e^\Omega)$, etc. But won't discuss it here.

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How to define the scalar curvature of a spectral triple (A, H, D)? This is also answered by Connes since late 1980's and is based on ideas of spectral geometry.

Spectral geometry

- (M, g) = closed Riemannian manifold. Laplacian on forms

$$\triangle = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M),$$

has pure point spectrum:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- Fact: Dimension, volume, total scalar curvature, Betti numbers, and hence the Euler characteristic of M are fully determined by the spectrum of Δ (on all p-forms).

Heat trace asymptotics

- Heat equation for functions: $\partial_t + \triangle = 0$
- $k(t, x, y) =$ kernel of $e^{-t\triangle}$. Asymptotic expansion near $t = 0$:

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x, \triangle) + a_1(x, \triangle)t + a_2(x, \Delta)t^2 + \cdots)$$

- $a_i(x, \triangle)$, Seeley-De Witt-Gilkey coefficients.

- Theorem: $a_i(x, \triangle)$ are universal polynomials in the curvature tensor $R = R^1_{jkl}$ and its covariant derivatives:

$$
\begin{aligned}
a_0(x, \triangle) &= 1 \\
a_1(x, \triangle) &= \frac{1}{6} S(x) \qquad \text{scalar curvature} \\
a_2(x, \triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2) \\
a_3(x, \triangle) &= \cdots\cdots
\end{aligned}
$$

Tauberian theory and $a_0 = 1$, implies Weyl's law:

$$N(\lambda) \sim \frac{\text{Vol }(M)}{(4\pi)^{m/2}\Gamma(1+m/2)}\lambda^{m/2} \qquad \lambda \to \infty,$$

where

$$N(\lambda) = \#\{\lambda_i \leq \lambda\}$$

is the eigenvalue counting function.

Meromorphic extension of spectral zeta functions

$$\zeta_{\triangle}(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \qquad \mathrm{Re}(s) > \frac{m}{2}$$

Mellin transform + asymptotic expansion:

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-t} t^{s-1}\, dt \qquad \mathrm{Re}(s) > 0$$

$$\begin{aligned} \zeta_{\triangle}(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} (\mathrm{Trace}(e^{-t\triangle}) - \mathrm{Dim\,Ker}\,\triangle) t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \{ \int_0^{c} \cdots + \int_c^{\infty} \cdots \} \end{aligned}$$

The second term defines an entire function, while the first term has a meromorphic extension to $\mathbb{C}$ with simple poles within the set

$$\frac{m}{2} - j, \qquad j = 0, 1, \cdots$$

Also: 0 is always a regular point.

Scalar curvature

The spectral invariants a_i in the heat asymptotic expansion

$$\mathrm{Trace}(e^{-t\triangle}) \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)$$

are related to residues of spectral zeta function by

$$\mathrm{Res}_{s=\alpha}\zeta_{\triangle}(s) = (4\pi)^{-\frac{m}{2}} \frac{a_{\frac{m}{2}-\alpha}}{\Gamma(\alpha)}, \qquad \alpha = \frac{m}{2} - j > 0$$

Focusing on subleading pole $s = \frac{m}{2} - 1$ and using $a_1 = \frac{1}{6}\int_M S(x) dvol_x$, we obtain a formula for scalar curvature density as follows:

Let $\zeta_f(s) := \mathsf{Tr}\,(f\triangle^{-s}),\ f \in C^\infty(M)$.

$$\mathsf{Res}\,\zeta_f(s)|_{s=\frac{m}{2}-1} = \frac{(4\pi)^{-m/2}}{\Gamma(m/2-1)}\int_M fS(x)dvol_x, \quad m \geq 3$$

$$\zeta_f(s)|_{s=0} = \frac{1}{4\pi}\int_M fS(x)dvol_x - \mathsf{Tr}(fP) \qquad m = 2$$

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$$\log \mathsf{det}(\triangle) = -\zeta'(0),$$ Ray-Singer regularized determinant

Complex structures on A_θ

- Let $\mathcal{H}_0 = L^2(A_\theta)$= GNS completion of A_θ w.r.t. $\mathfrak{t}$.
- Fix $\tau = \tau_1 + i\tau_2, \qquad \tau_2 = \Im(\tau) > 0$, and define
$$\partial := \delta_1 + \tau\delta_2, \qquad \partial^* := \delta_1 + \bar{\tau}\delta_2.$$
- To the conformal structure defined by τ, corresponds a positive Hochschild two cocycle on A_θ^∞ given by
$$\psi(a, b, c) = -\mathfrak{t}(a\partial b\partial^* c).$$
- Connes (book, 1994): Extremals of positive Hochschild cocycles correspond to complex structures.

- Hilbert space of $(1,0)$-forms:

  $\mathcal{H}^{(1,0)} :=$ completion of finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$, w.r.t.

$$\langle a\partial b, a'\partial b'\rangle := \mathfrak{t}((a'\partial b')^* a\partial b).$$

- Flat Dolbeault Laplacian: $\partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$.

Conformal perturbation of the metric, Connes-Tretkoff spectral triple

- Fix $h = h^* \in A_\theta^\infty$. Replace the volume form $\mathfrak{t}$ by $\varphi : A_\theta \to \mathbb{C}$,

$$\varphi(a) := \mathfrak{t}(ae^{-h}), \quad a \in A_\theta.$$

- It is a twisted trace (in fact a KMS state)

$$\varphi(ab) = \varphi(b\Delta(a)), \quad \forall a, b \in A_\theta.$$

where

$$\Delta(x) = e^{-h} x e^{h},$$

is the modular automorphism of a von Neumann factor-has no commutative counterpart.

- Warning: $\triangle$ and Δ are very different operators!

- Hilbert space $\mathcal{H}_\varphi := GNS$ completion of A_θ w.r.t. $\langle , \rangle_\varphi$,

$$\langle a, b\rangle_\varphi := \varphi(b^* a), \quad a, b \in A_\theta$$

- View $\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$. and let

$$\partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi$$

$$\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

$$D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Full perturbed Laplacian:

$$\triangle := D^2 = \begin{pmatrix} \partial_\varphi^* \partial_\varphi & 0 \\ 0 & \partial_\varphi \partial_\varphi^* \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Scalar curvature for A_θ

- The scalar curvature of the curved nc torus $(\mathbb{T}^2_\theta, \tau, k)$ is the unique element $R \in A_\theta^\infty$ satisfying

$$\text{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \text{Trace}\,(aP) = \mathfrak{t}\,(aR), \qquad \forall a \in A_\theta^\infty,$$

  where P is the projection onto the kernel of $\triangle$.

- In practice this is done by finding an asymptotic expansin for the kernel of the operator $e^{-t\triangle}$, using Connes' pseudodifferential calculus for nc tori.

Local expression for the scalar curvature

- Cauchy integral formula:

$$e^{-t\triangle} = \frac{1}{2\pi i}\int_C e^{-t\lambda}(\triangle - \lambda)^{-1}\, d\lambda.$$

- $B_\lambda \approx (\triangle - \lambda)^{-1}$:

$$\sigma(B_\lambda) \sim b_0(\xi,\lambda) + b_1(\xi,\lambda) + b_2(\xi,\lambda) + \cdots,$$

each $b_j(\xi,\lambda)$ is a symbol of order $-2-j$, and

$$\sigma(B_\lambda(\triangle - \lambda)) \sim 1.$$

(Note: λ is considered of order 2.)

Proposition: The scalar curvature of the spectral triple attached to (A_θ, τ, k) is equal to

$$\frac{1}{2\pi i}\int_{\mathbb{R}^2}\int_C e^{-\lambda} b_2(\xi,\lambda)\, d\lambda\, d\xi,$$

where b_2 is defined as above.

The computations for $k\partial^*\partial k$

- The symbol of $k\partial^*\partial k$ is equal to

$$a_2(\xi) + a_1(\xi) + a_0(\xi)$$

where

$$a_2(\xi) = \xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2,$$

$$a_1(\xi) = 2\xi_1 k\delta_1(k) + 2|\tau|^2 \xi_2 k\delta_2(k) + 2\tau_1 \xi_1 k\delta_2(k) + 2\tau_1 \xi_2 k\delta_1(k),$$

$$a_0(\xi) = k\delta_1^2(k) + |\tau|^2 k\delta_2^2(k) + 2\tau_1 k\delta_1\delta_2(k).$$

- The equation

$$(b_0 + b_1 + b_2 + \cdots)((a_2 + 1) + a_1 + a_0) \sim 1,$$

has a solution with each b_j a symbol of order $-2 - j$.

$$b_0 = (a_2 + 1)^{-1} = (\xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + 1)^{-1},$$

$$b_1 = -(b_0 a_1 b_0 + \partial_1(b_0)\delta_1(a_2)b_0 + \partial_2(b_0)\delta_2(a_2)b_0),$$

$$\begin{aligned} b_2 = & -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_1)b_0 + \partial_2(b_0)\delta_2(a_1)b_0 + \\ & \partial_1(b_1)\delta_1(a_2)b_0 + \partial_2(b_1)\delta_2(a_2)b_0 + (1/2)\partial_{11}(b_0)\delta_1^2(a_2)b_0 + \\ & (1/2)\partial_{22}(b_0)\delta_2^2(a_2)b_0 + \partial_{12}(b_0)\delta_{12}(a_2)b_0) \end{aligned}$$

$$\begin{aligned} = & \; 5\xi_1^2 b_0^2 k^3 \delta_1^2(k) b_0 + 2\xi_1^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k \\ & + \text{about 800 terms.} \end{aligned}$$

Final formula for the scalar curvature (Connes-Moscovici; Fathizadeh-K.)

Theorem: The scalar curvature of (A_θ, τ, k), up to an overall factor of $\frac{-\pi}{\tau_2}$, is equal to

$$R_1(\log \Delta)\Big(\triangle_0(\log k)\Big)+$$

$$R_2(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\delta_1(\log k)^2+|\tau|^2\delta_2(\log k)^2+\tau_1\big\{\delta_1(\log k), \delta_2(\log k)\big\}$$

$$iW(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\tau_2\big[\delta_1(\log k), \delta_2(\log k)\big]\Big)$$

where

$$R_1(x) = -\frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) = (1+\cosh((s+t)/2))\times$$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s+\sinh t - \sinh(s+}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}$$

$$W(s,t) = -\frac{(-s-t+t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}$$

The limiting case

In the commutative case, the above modular curvature reduces to a constant multiple of the formula of Gauss:

$$\frac{1}{\tau_2}\delta_1^2(\log k) + \frac{|\tau|^2}{\tau_2}\delta_2^2(\log k) + 2\frac{\tau_1}{\tau_2}\delta_1\delta_2(\log k).$$

First application: Ray-Singer determinant and conformal anomaly (Connes-Moscovici)

Recall: $\log \mathrm{Det}'(\triangle) = -\zeta'_{\triangle}(0)$, where $\triangle$ is the perturbed Laplacian on $\mathbb{T}^2_\theta$. One has the following *conformal variation formula*. Let $\nabla_i = \log \Delta$ which acts on the i-th factor of products.

Theorem

(analogue of Plyakov's formula) The log-determinant of the perturbed Laplacian $\triangle$ on $\mathbb{T}^2_\theta$ is given by

$$\log \mathit{Det}'(\triangle) = \log \mathit{Det}' \triangle_0 + \log \varphi(1) - \frac{\pi}{12\tau_2}\varphi_0(h\triangle_0 h) -$$

$$\frac{\pi}{4\tau_2}\varphi_0\left(K_2(\nabla_1)(\square_{\Re}(h))\right),$$

Second application: the Gauss-Bonnet theorem for A_θ after Connes and Tretkoff.

- Heat trace asymptotic expansion relates geometry to topology, thanks to MacKean-Singer formula:

$$\sum_{p=0}^{m}(-1)^p \mathsf{Tr}\left(e^{-t\Delta_p}\right) = \chi(M) \qquad \forall t > 0$$

- This gives the spectral formulation of the Gauss-Bonnet theorem:

$$\zeta(0) + 1 = \frac{1}{12\pi}\int_\Sigma R\,dv = \frac{1}{6}\chi(\Sigma)$$

Theorem (Connes-Tretkoff; Fathizadeh-K.): Let $\theta \in \mathbb{R}$, $\tau \in \mathbb{C} \setminus \mathbb{R}$, $k \in A_\theta^\infty$ be a positive invertible element. Then

$$\text{Trace}(\triangle^{-s})_{|s=0} + 2 = \mathfrak{t}\,(R) = 0,$$

where $\triangle$ is the Laplacian and R is the scalar curvature of the spectral triple attached to (A_θ, τ, k).

Noncommutative 4-Torus $\mathbb{T}^4_\theta$

$C(\mathbb{T}^4_\theta)$ is the universal C^*-algebra generated by 4 unitaries

$$U_1, U_2, U_3, U_4,$$

satisfying

$$U_k U_\ell = e^{2\pi i \theta_{k\ell}} U_\ell U_k,$$

for a skew symmetric matrix

$$\theta = (\theta_{k\ell}) \in M_4(\mathbb{R}).$$

Complex Structure on $\mathbb{T}^4_\theta$

$$\partial = \partial_1 \oplus \partial_2, \qquad \bar\partial = \bar\partial_1 \oplus \bar\partial_2,$$

$$\partial_1 = \frac{1}{2}(\delta_1 - i\delta_3), \qquad \partial_2 = \frac{1}{2}(\delta_2 - i\delta_4),$$

$$\bar\partial_1 = \frac{1}{2}(\delta_1 + i\delta_3), \qquad \bar\partial_2 = \frac{1}{2}(\delta_2 + i\delta_4).$$

Conformal Perturbations (after Connes-Tretkoff)

Let $h = h^* \in C^\infty(\mathbb{T}^4_\theta)$ and replace the trace φ_0 by

$$\varphi : C(\mathbb{T}^4_\theta) \to \mathbb{C},$$

$$\varphi(a) := \varphi_0(a\, e^{-2h}), \qquad a \in C(\mathbb{T}^4_\theta).$$

φ is a KMS state with the modular group

$$\sigma_t(a) = e^{2ith}\, a\, e^{-2ith}, \qquad a \in C(\mathbb{T}^4_\theta),$$

and the modular automorphism

$$\Delta(a) := \sigma_i(a) = e^{-2h}\, a\, e^{2h}, \qquad a \in C(\mathbb{T}^4_\theta).$$

$$\varphi(a\, b) = \varphi\big(b\, \Delta(a)\big), \qquad a, b \in C(\mathbb{T}^4_\theta).$$

Perturbed Laplacian on $\mathbb{T}^4_\theta$

$$d = \partial \oplus \bar{\partial} : \mathcal{H}_\varphi \to \mathcal{H}_\varphi^{(1,0)} \oplus \mathcal{H}_\varphi^{(0,1)},$$

$$\triangle_\varphi := d^* d.$$

Remark. If $h = 0$ then $\varphi = \varphi_0$ and

$$\triangle_{\varphi_0} = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 = \partial^* \partial$$

(the underlying manifold is Kähler).

Explicit Formula for $\triangle_\varphi$

Lemma. Up to an anti-unitary equivalence $\triangle_\varphi$ is given by

$$e^h\bar{\partial}_1 e^{-h}\partial_1 e^h + e^h\,\partial_1 e^{-h}\bar{\partial}_1\,e^h + e^h\bar{\partial}_2 e^{-h}\,\partial_2 e^h + e^h\partial_2 e^{-h}\bar{\partial}_2 e^h,$$

where ∂_1, ∂_2 are analogues of the Dolbeault operators.

Connes' Pseudodifferential Calculus (1980)

A smooth map $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$ is a symbol of order $m \in \mathbb{Z}$, if for any $i, j \in \mathbb{Z}^4_{\geq 0}$, there exists a constant c such that

$$||\partial^j \delta^i \big(\rho(\xi)\big)|| \leq c(1 + |\xi|)^{m-|j|},$$

and if there exists a smooth map $k : \mathbb{R}^4 \setminus \{0\} \to C^\infty(\mathbb{T}^4_\theta)$ such that

$$\lim_{\lambda \to \infty} \lambda^{-m} \rho(\lambda \xi) = k(\xi), \qquad \xi \in \mathbb{R}^4 \setminus \{0\}.$$

- Given a symbol $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$, the corresponding ψDO is:

$$P_\rho(a) = (2\pi)^{-4} \int\int e^{-is.\xi}\, \rho(\xi)\, \alpha_s(a)\, ds\, d\xi, \qquad a \in C^\infty(\mathbb{T}^4_\theta).$$

- Differential operators:

$$\rho(\xi) = \sum a_\ell\, \xi^\ell, \quad a_\ell \in C^\infty(\mathbb{T}^4_\theta) \quad \Rightarrow \quad P_\rho = \sum a_\ell\, \delta^\ell.$$

- ΨDO's on $\mathbb{T}^4_\theta$ form an algebra:

$$\sigma(P\,Q) \sim \sum_{\ell \in \mathbb{Z}^4_{\geq 0}} \frac{1}{\ell!}\, \partial^\ell_\xi \rho(\xi)\, \delta^\ell(\rho'(\xi)).$$

• A symbol $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$ of order m is elliptic if $\rho(\xi)$ is invertible for any $\xi \neq 0$, and if there exists a constant c such that

$$||\rho(\xi)^{-1}|| \leq c(1 + |\xi|)^{-m},$$

when $|\xi|$ is sufficiently large.

• Example of an elliptic operator:

$$\triangle_\varphi = e^h \bar{\partial}_1 e^{-h} \partial_1 e^h + e^h\, \partial_1 e^{-h} \bar{\partial}_1\, e^h + e^h \bar{\partial}_2 e^{-h}\, \partial_2 e^h + e^h \partial_2 e^{-h} \bar{\partial}_2 e^h.$$

Symbol of $\triangle_\varphi$

Lemma. The symbol of $\triangle_\varphi$ is equal to

$$a_2(\xi) + a_1(\xi) + a_0(\xi),$$

where

$$a_2(\xi) = e^h \sum_{i=1}^{4} \xi_i^2, \qquad a_1(\xi) = \sum_{i=1}^{4} \delta_i(e^h)\, \xi_i,$$

$$a_0(\xi) = \sum_{i=1}^{4} \big(\delta_i^2(e^h) - \delta_i(e^h)\, e^{-h}\, \delta_i(e^h)\big).$$

Mellin Transform and Asymptotic Expansions

$$\triangle_\varphi^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\triangle_\varphi}\, t^s\, \frac{dt}{t},$$

$$\text{Trace}(a\, e^{-t\triangle_\varphi}) \sim_{t\to 0^+} \; t^{-2} \sum_{n=0}^\infty B_n(a, \triangle_\varphi)\, t^{n/2}.$$

Approximate $e^{-t\triangle_\varphi^2}$ by pseudodifferential operators:

$$e^{-t\triangle_\varphi} = \frac{1}{2\pi i} \int_C e^{-t\lambda}\, (\triangle_\varphi - \lambda)^{-1}\, d\lambda,$$

$$B_\lambda\, (\triangle_\varphi - \lambda) \approx 1,$$

$$\sigma(B_\lambda) = b_0 + b_1 + b_2 + \cdots .$$

Analogue of Weyl's Law for $\mathbb{T}^4_\theta$

Theorem. For the eigenvalue counting function $N(\lambda)$ of the Laplacian $\triangle_\varphi$ on $\mathbb{T}^4_\theta$, we have

$$N(\lambda) \sim \frac{\pi^2 \varphi_0(e^{-2h})}{2} \lambda^2 \qquad (\lambda \to \infty).$$

Corollary. The Dixmier trace of $(1 + \triangle_\varphi)^{-2}$ is given by

$$\mathrm{Tr}_\omega\Big((1 + \triangle_\varphi)^{-2}\Big) = \frac{\pi^2}{2} \varphi_0(e^{-2h}).$$

A Noncommutative Residue for $\mathbb{T}_\theta^4$

Classical symbols: $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}_\theta^4)$

$$\rho(\xi) \sim \sum_{i=0}^{\infty} \rho_{m-i}(\xi) \qquad (\xi \to \infty),$$

$$\rho_{m-i}(t\,\xi) = t^{m-i}\,\rho_{m-i}(\xi), \qquad t > 0, \quad \xi \in \mathbb{R}^4.$$

Theorem. The linear functional

$$\mathrm{Res}(P_\rho) := \int_{\mathbb{S}^3} \varphi_0\big(\rho_{-4}(\xi)\big)\, d\xi$$

is the unique trace on classical pseudodifferential operators on $\mathbb{T}_\theta^4$.

Analogue of Connes' Trace Theorem for $\mathbb{T}_\theta^4$

Theorem. For any classical symbol ρ of order -4 on $\mathbb{T}_\theta^4$, we have

$$P_\rho \in \mathcal{L}^{1,\infty}(\mathcal{H}_0),$$

and

$$\mathrm{Tr}_\omega(P_\rho) = \frac{1}{4}\mathsf{Res}(P_\rho).$$

Remark. Weyl's law is a special case of this theorem: let

$$\rho(\xi) = \frac{1}{(1+|\xi|^2)^2}.$$

Scalar Curvature for $\mathbb{T}^4_\theta$

It is the unique element $R \in C^\infty(\mathbb{T}^4_\theta)$ such that

$$\mathsf{Res}_{s=1}\, \zeta_a(s) = \varphi_0(a\, R), \qquad a \in C^\infty(\mathbb{T}^4_\theta),$$

$$\zeta_a(s) := \mathsf{Trace}(a\, \triangle_\varphi^{-s}), \qquad \Re(s) \gg 0.$$

Connes' Rearrangement Lemma

For any $m = (m_0, m_1, \ldots, m_\ell) \in \mathbb{Z}_{>0}^{\ell+1}$, $\rho_1, \ldots, \rho_\ell \in C^\infty(\mathbb{T}_\theta^4)$:

$$\int_0^\infty \frac{u^{|m|-2}}{(e^h u + 1)^{m_0}} \prod_1^\ell \rho_j \, (e^h u + 1)^{-m_j} \, du$$

$$= e^{-(|m|-1)h} \, F_m(\Delta, \ldots, \Delta)\Big(\prod_1^\ell \rho_j \Big),$$

where

$$F_m(u_1, \ldots, u_\ell) = \int_0^\infty \frac{x^{|m|-2}}{(x+1)^{m_0}} \prod_1^\ell \Big(x \prod_1^j u_k + 1 \Big)^{-m_j} \, dx.$$

Examples of F_m

$$F_{(3,4)}(u) = \frac{60u^3 \log(u) + (u-1)(u(u(3(u-9)u-47)+13)-2)}{6(u-1)^6 u^3}$$

$$F_{(2,2,1)}(u,v) =$$

$$\frac{(v-1)\Big((u-1)(uv-1)(u(u(v-1)+v)-1)-u^2(v-1)(2uv+u-3)\log(uv)\Big)+(u(2v-3)+1)(uv}{(u-1)^3u^2(v-1)^2(uv-1)^2}$$

Identities Relating $\delta_i(e^h)$ and $\delta_i(h)$

$$e^{-h}\,\delta_i(e^h) = g_1(\Delta)\big(\delta_i(h)\big),$$

$$e^{-h}\,\delta_i^2(e^h) = g_1(\Delta)\big(\delta_i^2(h)\big) + 2\,g_2(\Delta_{(1)}, \Delta_{(2)})\big(\delta_i(h)\,\delta_i(h)\big),$$

where

$$g_1(u) = \frac{u-1}{\log u},$$

$$g_2(u,v) = \frac{u(v-1)\log(u) - (u-1)\log(v)}{\log(u)\log(v)(\log(u)+\log(v))}.$$

Final Formula for the Scalar Curvature of $\mathbb{T}_\theta^4$

Theorem.

$$R = e^{-h}\, k(\nabla)\Big(\sum_{i=1}^{4} \delta_i^2(h)\Big) + e^{-h}\, H(\nabla,\nabla)\Big(\sum_{i=1}^{4} \delta_i(h)^2\Big),$$

where

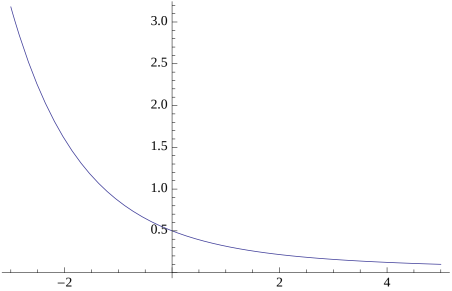
$$\nabla(a) := \frac{1}{2}\log\Delta(a) = [-h, a], \qquad a \in C(\mathbb{T}_\theta^4),$$

$$k(s) = \frac{1-e^{-s}}{2s},$$

$$H(s,t) = -\frac{e^{-s-t}\,((-e^s-3)\,s\,(e^t-1) + (e^s-1)\,(3e^t+1)\,t)}{4\,s\,t\,(s+t)}.$$

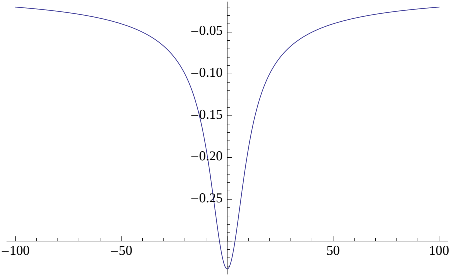
The One Variable Function for $\mathbb{T}^4_\theta$

$$k(s) = \frac{1}{2} - \frac{s}{4} + \frac{s^2}{12} - \frac{s^3}{48} + \frac{s^4}{240} - \frac{s^5}{1440} + O\left(s^6\right).$$



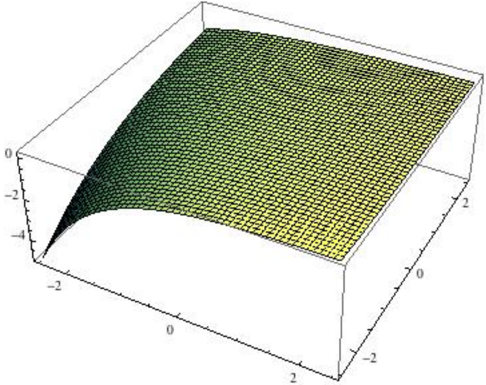
The One Variable Function for $\mathbb{T}_\theta^2$

$$R_1(x) = \frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)}.$$

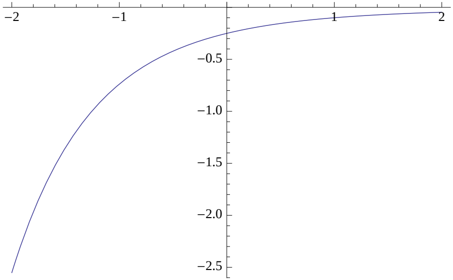


The Two Variable Function for $\mathbb{T}^4_\theta$

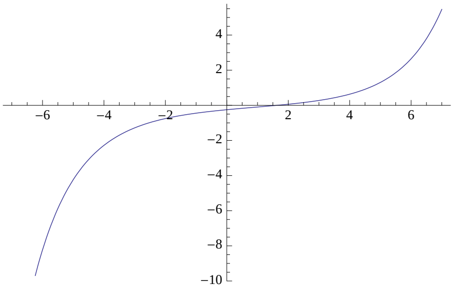
$$H(s,t) = \left(-\frac{1}{4}+\frac{t}{24}+O\left(t^3\right)\right)+s\left(\frac{5}{24}-\frac{t}{16}+\frac{t^2}{80}+O\left(t^3\right)\right)$$
$$+s^2\left(-\frac{1}{12}+\frac{7t}{240}-\frac{t^2}{144}+O\left(t^3\right)\right)+O\left(s^3\right).$$



$$
\begin{aligned}
H(s,s) &= -\frac{e^{-2s}\left(e^{s}-1\right)^{2}}{4s^{2}} \\
&= -\frac{1}{4}+\frac{s}{4}-\frac{7s^{2}}{48}+\frac{s^{3}}{16}-\frac{31s^{4}}{1440}+\frac{s^{5}}{160}+O\left(s^{6}\right).
\end{aligned}
$$



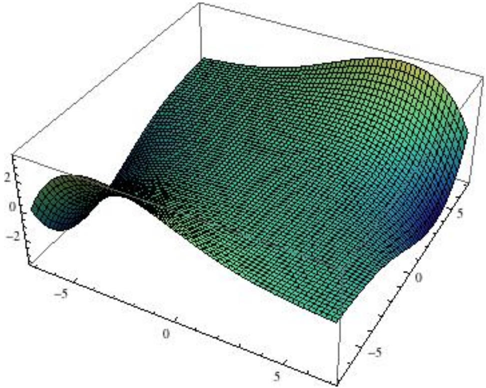
$$G(s) := H(s,-s) = \frac{-4s-3e^{-s}+e^{s}+2}{4s^2}$$
$$= -\frac{1}{4}+\frac{s}{6}-\frac{s^2}{48}+\frac{s^3}{120}-\frac{s^4}{1440}+\frac{s^5}{5040}+O\left(s^6\right).$$



The First Two Variable Function for $\mathbb{T}^2_\theta$

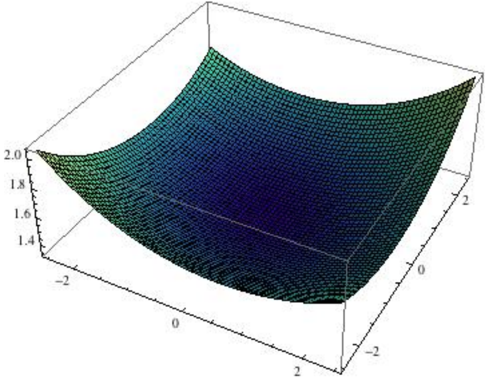
$R_2(s,t) =$

$$-\frac{(1+\cosh((s+t)/2))(-t(s+t)\cosh s+s(s+t)\cosh t-(s-t)(s+t+\sinh s+\sinh t-\sinh(s+t)}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}$$



The Second Two Variable Function for $\mathbb{T}_\theta^2$

$$W(s,t) = \frac{(-s - t + t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$



Commutative Case $\theta = 0 \in M_4(\mathbb{R})$

We have

$$k(0) = 1/2, \qquad H(0,0) = -1/4.$$

Therefore, in the commutative case $\theta = 0$, since $\nabla = 0$, the formula for the scalar curvature of $\mathbb{T}_\theta^4$ reduces to

$$R = \frac{\pi^2}{2} \sum_{i=1}^{4} \left(\delta_i^2(h) - \frac{1}{2}\delta_i(h)^2 \right).$$

This, up to a normalization factor, is the scalar curvature of the ordinary 4-torus equipped with the metric

$$ds^2 = e^{-h} \left(dx_1^2 + dx_2^2 + dx_3^3 + dx_4^2\right),$$

where $h \in C^\infty(\mathbb{T}^4, \mathbb{R})$.

Motivation for the Computations

In the 2-dimensional case:

P. B. Cohen, A. Connes, *Conformal geometry of the irrational rotation algebra*, MPI preprint 1992-93

$$\zeta_h(0) + 1 = \varphi\big(f(\Delta)(\delta_1(e^{h/2}))\,\delta_1(e^{h/2})\big) + \varphi\big(f(\Delta)(\delta_2(e^{h/2}))\,\delta_2(e^{h/2})\big).$$

Two theories were developed: the spectral action principle (Chamseddine-Connes) and twisted spectral triples (Connes-Moscovici);

A. Connes, P. Tretkoff, *The Gauss-Bonnet Theorem for the Non- commutative Two Torus*, 2009

$$\zeta_h(0) + 1 = 0 \qquad\qquad (\tau = i).$$

This created the need to investigate the Gauss-Bonnet for general conformal structures (Khalkhali-F) and stimulated the computation of scalar curvature for $\mathbb{T}^2_\theta$ (Connes-Moscovici; Khalkhali-F).

Einstein-Hilbert Action for $\mathbb{T}^4_\theta$

Theorem. We have the local expression (up to a factor of π^2):

$$\varphi_0(R) = \frac{1}{2}\sum_{i=1}^{4}\varphi_0\Big(e^{-h}\delta_i^2(h)\Big) + \sum_{i=1}^{4}\varphi_0\Big(G(\nabla)(e^{-h}\delta_i(h))\,\delta_i(h)\Big).$$

Extrema of the Einstein-Hilbert Action

Theorem. For any Weyl factor $e^{-h} \in C^\infty(\mathbb{T}_\theta^4)$, we have:

$$\varphi_0(R) \leq 0,$$

and the equality happens if and only if h is a constant.

Proof.

$$\varphi_0(R) = \sum_{i=1}^{4} \varphi_0\big(e^{-h} T(\nabla)(\delta_i(h))\, \delta_i(h)\big),$$

where

$$T(s) = \frac{1}{2}\frac{e^{-s}-1}{-s} + G(s) = \frac{-2s + e^s - e^{-s}(2s+3) + 2}{4s^2}.$$

$$T(s) = \frac{1}{4} - \frac{s}{12} + \frac{s^2}{16} - \frac{s^3}{80} + \frac{s^4}{288} - \frac{s^5}{2016} + O\left(s^6\right).$$