# Operator Algebra

**Operator algebra** is an algebra of continuous linear operator on a topological vector space with the multiplication is given by the composition of mappings. In particular it is a set of operator with both algebraic and topological closure properties.
Operator algebra is usually used in reference to algebras of bounded operators on a Banach space or even more especially on a separable Hilbert space, endowed with the operator norm topology.

**Observation:** $B(H)$, the set of all bounded linear operators is a non commutative ring.

## Topologies on a Hilbert space $H$

A Hilbert space has two useful topologies, which are defined as follows:
**Strong or norm topology:** Since a Hilbert space has, by definition, an inner product < , >, that inner product induces a norm and that norm induces a metric. So our Hilbert space is a metric space. The strong or norm topology is that metric topology. A subbase is the collection of all sets of the form

$$O(x_0, \varepsilon) = B_\varepsilon(x_0)$$

which is in fact, a base for the metric topology.
In norm topology $\| x_n - x \| \to 0$ as $n \to \infty$ and we say strong convergence.

**Weak topology:** A subbase for the weak topology is the collection of all sets of the form

$$O(x_0, y, \varepsilon) = \{x \in H : |\langle x - x_0, y\rangle| < \varepsilon\}$$

If $\{x_n\}$ is a sequence in $H$ and $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$, then $\{x_n\}$ is weakly convergence.
**Observation:**

**(i)** The weak topology is weaker than the strong topology
**(ii)** Strong convergence implies weak convergence.

**Example:** The sequence $\{x_n(t)\}$ where $x_n(t) = \frac{\sin nt}{\pi}$, $n = 1, 2, 3, \ldots\ldots$ is weakly convergent in $L_2(0, 2\pi)$ but it is not norm convergent in $L_2(0, 2\pi)$.

## Topologies on $B(H)$

The space of bounded linear operators on a Hilbert space $H$ has several topologies.
**Norm topology:** We know $B(H)$ is a normed space and the given norm induces a metric, so $B(H)$ is a metric space. So the norm topology is just defined to be the metric topology.

In the norm topology $T_n \to T$ if and only if $\| T_n - T \| \to 0$

**Strong operator topology (SOT):** A subbase for the strong operator topology is the collection of all sets of the form

$$O(T_0, x, \varepsilon) = \{T \in B(H) : \| (T - T_0)x \| < \in\}$$

We know a base is the collection of all finite intersections of such sets. It follows that a base is the collection of all sets of the form

$$O(T_0, x_1, x_2, ....., x_k, \varepsilon) = \{T \in B(H) : \| (T - T_0)x_i \| < \in \quad i = 1, 2, ......., k\}$$

The corresponding concepts of convergence: $T_n \to T$ strongly if and only if $T_n x \to Tx$ strongly for each *x* in *H* (i.e. $\| T_n x - Tx \| \to 0$ for each *x*)

**Weak operator topology (WOT):** A subbase for the weak operator topology is the collection of all sets of the form

$$O(T_0, x, y, \varepsilon) = \{T \in B(H) : |\langle (T - T_0)x, y \rangle| < \in\}$$

We can also form base from this sets for WOT.
The corresponding concept of convergence: $T_n \to T$ weakly if and only if $T_n x \to Tx$ weakly for each *x* in *H* (i.e. $\langle T_n x, y \rangle \to \langle Tx, y \rangle$ for each *x* and *y*)

**Example:** Consider the bounded operator on $l_2$

(i) Let $T_n$ be defined by $T_n(x_1, x_2, ......) = \left( \frac{1}{n} x_1, \frac{1}{n} x_2, ....... \right)$, then $T_n \to 0$ in the norm topology.

(ii) Let $S_n$ be defined by $S_n(x_1, x_2, ......) = (0, 0, ..........., x_{n+1}, x_{n+2}, ....... )$, where 0 in n places, then $S_n \to 0$ in the strong topology but not in the uniform topology.

(iii) Let $W_n$ be defined by $W_n(x_1, x_2, ......) = (0, 0, ......., x_1, x_2, ....... )$, where 0 in n places, then $W_n \to 0$ in the weak operator topology but not in the strong or uniform topologies.

**Example:** Let $(x_n)$ be a dense sequence in the unit ball *B* of *H* and define the metrices

$$d_S(S, T) = \sum 2^{-n} \| (S - T)x_n \| \text{ and } d_W(S, T) = \sum 2^{-n} | \langle (S - T)x_n, x_n \rangle |$$

then $d_S$ induces the strong topology on *B* and $d_W$ the weak topology.

**Observation:**

**(i)** The WOT is weaker than SOT and SOT is weaker than the norm topology.

**(ii)** Norm convergence implies strong convergence and strong convergence implies weak convergence. If we impose additional conditions, then the reverse is also true in the following sense: (i) If $\langle T_n x, y\rangle \to \langle Tx, y\rangle$ uniformly for $\| y\| = 1$, then $\| T_n x - Tx\| \to 0$ and (ii) If $\| T_n x - Tx\| \to 0$ uniformly for $\| x\| = 1$, then $\| T_n - T\| \to 0$

**(iii)** Which of the three topologies (uniform, SOT, WOT) makes the norm (i.e. the function $T \to \| T\|$) continuous?

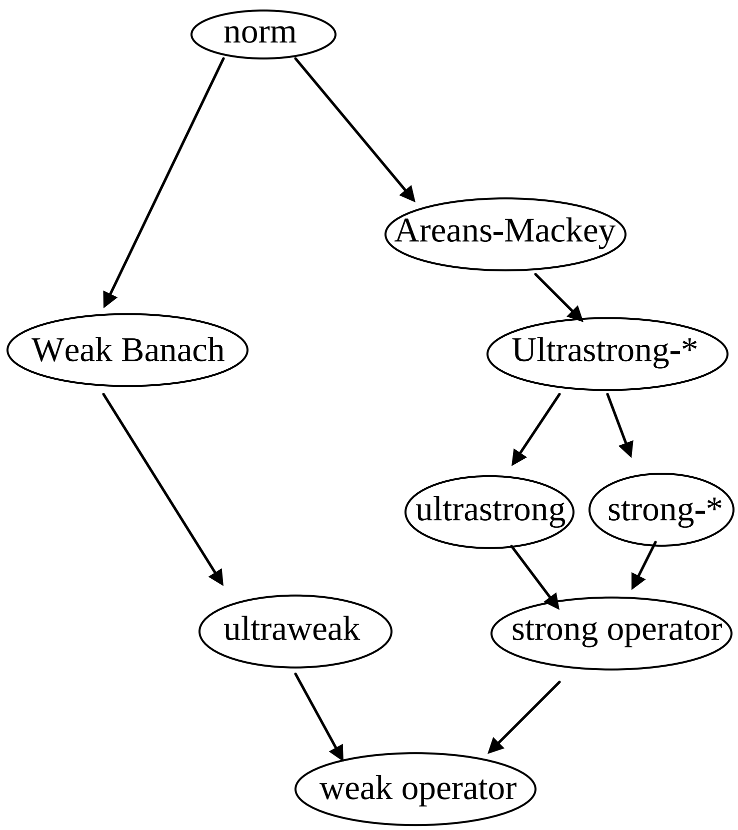
The norm is continuous with respect to the uniform topology and discontinuous with respect to the SOT and WOT

**(iv)** Which of the three topologies (uniform, SOT, WOT) makes the adjoint (i.e. the mapping $T \to T^*$) continuous?

The adjoint is continuous with respect to the uniform and WOT and discontinuous with respect to the SOT.

**(v)** Multiplication is continuous with respect to the norm topology and discontinuous with respect to the SOT and WOT.

There are many topologies that can be defined on $B(H)$. These topologies are all locally convex, which implies that they are defined by a family of semi norms.



The most commonly used topologies are the norm, strong and weak operator topologies. The weak operator topology is useful for compactness arguments. The norm topology is fundamental because it makes $B(H)$ into a Banach space. Strong topology provides the natural language for the generalization of the spectral theorem.

The SOT lacks some of the nicer properties that the weak operator topology has, but being stronger, things are some times easier to prove in this topology. It is more natural too, since it is simply the topology of pointwise convergence for an operator.

As an example of this lack of nicer properties,

**Example:** let us mention that the involution map is not continuous in SOT: fix an orthonormal basis $\{e_n : n \in N\}$ of a Hilbert space and consider the unilateral shift S given by

$$S(e_n) = e_{n+1}$$

Then the adjoint $S^*$ is given by

$$S^*(e_n) = e_{n-1}, \quad n \geq 1, \quad S^*(e_0) = 0$$

The sequence $\{S_n\}$ satisfies $\| S_n(x) \| = \| x \|$ for every vector *x*, but $\lim_{n\to\infty}(S^*)_n = 0$ in the SOT topology.

**Observation:** All norms on a finite dimensional vector space are equivalent from a topological point as they induce the same topology although the resulting metric space need not be same. Moreover the strong operator topologies and weak operator topologies coincide on the group $U(H)$ of unitary operators in $B(H)$.

Mention that subsets of a topological vector space are weakly closed means closed with respect to the WOT and strongly closed means closed with respect to SOT. Similarly we can define strong continuity and weak continuity.

**Proposition:** For a functional $\varphi$ on $B(H)$ the following conditions are equivalent:

(i) There are vectors $x_1, x_2, ......, x_n$ and $y_1, y_2, ......, y_n$ in H such that $\varphi(T) = \sum \langle Tx_k, y_k \rangle$ for all T in $B(H)$.

(ii) $\varphi$ is weakly continuous

(iii) $\varphi$ is strongly continuous

**Corollary:** Every strongly closed, convex set in $B(H)$ is weakly closed. In particular, every strongly closed subspace of $B(H)$ is weakly closed.

For any subset $U$ of $B(H)$ we let $U'$ denote the commutant of *U*, i.e.

$$U' = \{T \in B(H) : TS = ST, \forall S \in U\}$$

Here $U'$ is weakly closed and is an algebra. If $U$ is a self-adjoint subset of $B(H)$ (which means if $S \in U$ then $S^* \in U$), then $U'$ is a weakly closed, unital C*-subalgebra of $B(H)$.

Similarly we can also find iterated commutants $(U')'$ and $\left((U')'\right)'$ which we will write $U''$ and $U'''$ respectively.

Note that if $U \subset U_2$ then $U_1{}' \supset U_2{}'$. On the other hand we have $U \subset U''$. It follows that $U''' \subset U' \subset U'''$ for every subset *U*, so that the process of taking commutants stabilizes after at most two steps.

The following double commutant theorem by Von Neumann (1929) is the fundamental result in operator algebra theory.

**Theorem:** For a self-adjoint, unital subalgebra $U$ of $B(H)$ the following conditions are equivalent:

(i) $U = U''$

(ii) *U* is weakly closed

(iii) *U* is strongly closed

**Proof:** $(i) \Rightarrow (ii)$ Since $U'$ is weakly closed, so $U''$ is also weakly closed which is equal to *U*. Hence *U* is weakly closed.

$(ii) \Rightarrow (iii)$ follows by previous corollary.

$(iii) \Rightarrow (i)$ We already know $U \subset U''$, to prove the converse let $T \in U''$. We want to show that *T* lies in the strong closure of *U*. A neighborhood base of zero in the strong topology is given by the system of all sets of the form $\{S \in B(H) : \| Sx_i \| < \varepsilon \ i = 1, 2, ....., n\}$ where $x_1, x_2, ........, x_n$ are arbitrary vectors in *H*. and $\varepsilon > 0$. So it suffices to show that for given $x_1, x_2, ........, x_n \in H$ and $\varepsilon > 0$ there is a $S \in U$ with $\| Tx_i - Sx_i \| < \varepsilon$ for $i = 1, 2, ....., n$. For this let $B(H)$ act diagonally on $H^n$. The commutant of *U* in $B(H^n)$ is the algebra of all $n \times n$ matrices with entries in $U'$, and the bicommutant of $U$ in $B(H^n)$ is the algebra $U''I$, where $I = I_n$ denotes the $n \times n$ unit matrix. Consider the vector $x = (x_1, x_2, ........, x_n)^t$ in $H^n$. The closure of $Ux$ in $H^n$ is a closed, *U*-stable subspace of $H^n$. As *U* is a *-algebra, the orthogonal complement $(Ux)^\perp$ is *U*-stable as well; therefore the orthogonal projection *P* onto the closure of $Ux$ is in the commutant of *U* in $B(H^n)$. It follows that $T \in U''I$ commutes with *P* and leaves $\overline{Ux}$ stable. One concludes $Tx \in \overline{Ux}$, and so there is, to given $\varepsilon > 0$, an element *S* of *U* such that $\| Tx - Sx \| < \varepsilon$, which implies the desired $\| Tx_i - Sx_i \| < \varepsilon$ for $i = 1, 2, ....., n$.

An algebra satisfying the conditions of above is called a Von Neumann algebra These algebras appear quite naturally in many connections. From the preceding we see that if $D$ is any self-adjoint subset of $B(H)$, then $D'$ is a Von Neumann algebra.

We can also define Von Neumann algebra by following ways:
Let $H$ be a Hilbert space. A *- subalgebra $M$ of $B(H)$, which contains the unit 1 of $B(H)$ and is closed with respect to the weak topology, is called a Von Neumann algebra.

Since weak-closedness implies norm-closedness, every Von Neumann algebra is a C*-algebra. So Von Neumann algebras are a special class of C*-algebras. Those C*-algebras that appear in topology / geometry are separable C*- algebras in most cases. However Von Neumann algebras are not **separable** with respect to norm topology unless they are finite dimensional. We illustrated it by an example.

**Example:** The algebra $B(H)$ is a VonNeumann algebra. Let $\{e_n; n \in N\}$ be an orthonormal system of $H$. Choose a sequence $\alpha = (\alpha_n)$ so that $\alpha_n \in \{0, 1\}$. Define $T_\alpha \in B(H)$ by

$$T_\alpha \phi = \sum_n \alpha_n \langle \phi, e_n \rangle e_n$$

Then $\{T_\alpha\}_\alpha$ is an uncountable set, and $\| T_\alpha - T_\beta \| = 1 \quad (\alpha \neq \beta)$. Therefore $\{T_\alpha\}$ cannot be approximated by a countable set.

**Example:** The ring $L^\infty(R)$ of essentially bounded measurable functions on the real line is a commutative Von Neumann algebra.

**Example:** Let $M_1 \subset B(H_1)$ and $M_2 \subset B(H_2)$ be Von Neumann algebras. On the direct sum $H_1 \oplus H_2$ of Hilbert spaces, the space

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in B(H_1 \oplus H_2); a \in M_1, b \in M_2 \right\}$$

is a Von Neumann algebra. This Von Neumann algebra is called the direct sum of $M_1$ and $M_2$, and it is denoted $M_1 \oplus M_2$

A commutative, self-adjoint algebra $U$ of operators in $B(H)$ is **maximal commutative** if it is not contained properly in any larger commutative *-subalgebra of $B(H)$. Since if $T \in U'$ and $T = T^*$, then the algebra generated by $U$ and $T$ will be commutative and self-adjoint. Thus just as the condition $U \subset U'$ characterizes the commutative *-subalegbras of $B(H)$, the condition $U = U'$ characterizes the maximal commutative algebras in the class of *-subalegbras of

$B(H)$. In particular each maximal commutative algebra is weakly closed and contains $I$, and is therefore a Von Neumann algebra.

Examples of operator algebras which are not self-adjoint include: nest algebras, many commutative subspace lattice algebras, many limit algebras.

Nest algebras are a class of operator algebras that generalize the upper triangular matrix algebras to a Hilbert space context.

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