Why $1 + 2 + 3 + \cdots = -\frac{1}{12}$

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Abstract

This talk is a quick introduction to summability theory and regularization of divergent series. Afer a quick introduction to Abel, Cesaro, and Borel summation techniques, I shall mostly focus on one such theory of summability: zeta function regularization. I will show how to define and compute divergent sums like the one on the title, as well as many others like $1 + 1 + 1 + 1 + \cdots = -1/2$, or $1 + 4 + 9 + 16 + 25 + \cdots = 0$. I shall also discuss the closely related cutoff regularization method. One of Euler's goals in this area of math was to find (nowadays we say to define!) the alternating sum of factorials $1! - 2! + 3! - 4! + 5! - \cdots$. I shall explain this and will give Euler's surprising answer!

Introduction

The title of my talk sounds like an utterly wrong statement! After all, the infinite series $1 + 2 + 3 + 4 + \cdots$ is divergent and in fact diverges to infinity. So shouldn't we just say $1 + 2 + 3 + 4 + \cdots = \infty$? Of course we can. But then with the same limited understanding of summation we shall assign the same value, infinite, to a host of other very different types of series like $1 + 1 + 1 + 1 + \cdots$ or $1 + 4 + 9 + 16 + 25 + \cdots$, etc. The point of my talk is that in doing so we are throwing away a wealth of information hidden in such divergent series. Information that can have practical implications for mathematics and its applications.

This situation is a bit similar to set theory and cardinal numbers. Mathematicians used to think that there are only two types of numbers: finite and infinite. Of course, after Cantor, we know that there is a vast hierarchy of infinities and knowing about these different types of infinities is often very useful.

Enter Euler

One of the first people who realized the importance of divergent series and developed some techniques to sum divergent series was Leonhard Euler. In fact Euler was of the opinion that any series is summable and one should just find the right method of summing it! In the last 250 years many summation techniques have been designed and there is now a vast summability theory: Abel summation, Cezaro summation, Borel summation, zeta function regularization, cutoff regularization, etc.

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- But imagine we want to regularize this infinity and get a finite number. How would you proceed? Somehow we need to extend the class of convergent series to a larger class. There are many possibilities.

Abel summation

- This is in fact due to Euler. To sum a divergent series like $\sum a_n$, define the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Assume $\sum a_n x^n$ is convergent for $|x| < 1$, and assume $\lim_{x \to 1^-} f(x)$ exists. Then we can define

$$\sum_{n=1}^{\infty} a_n := \lim_{x \to 1^-} f(x) \quad \text{(Abel)}.$$

- Example: The Abel sum of $1 - 1 + 1 - 1 + \cdots$ is $\frac{1}{2}$ since $f(x) = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$.

Abelian and Tauberian Theorems

- A Theorem of Abel (this is really due to Abel!) asserts that a convergent sequence $\sum_{n=1}^{\infty} a_n$ is Abel summable and its Abel sum is equal to its standard sum. Thus Abel summation extends the standard summation of series, but the class of Abel summable series is strictly larger than summable series.
- There is a kind of converse to Abel's theorem. Tauber's theorem states that if a series is Abel summable and if $a_n = o(1/n)$, then the series is actualy convergent.

Cesaro summation

- Let $s_k = a_1 + \cdots + a_k$ denote the kth partial sum of the series $\sum_{n=1}^{\infty} a_n$. The series is called Cesaro summable, with Cesaro sum C, if the average value of its partial sums tends to C:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k = C.$$

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$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k = C.$$

- It is easy to see show that a convergent sequence is Cesaro summable and its Cesaro sum is equal to its standard summation. But the converse need not be true:
- Example: the series $1 - 1 + 1 - \cdots$ is clearly not summable, but is Cesaro summable with Cesaro sum $1/2$.

- Example: The series $1 + 2 + 3 + \cdots$ is not Cesaro summable as can be easily verified.

Borel summation

- To sum the divergent series $\sum_{n\geq 1}(-1)^n n!$, Euler used the formula

$$n! = \int_0^\infty e^{-t} t^n \, \frac{dt}{t},$$

and then wrote, quite formally,

$$\sum_{n\geq 0}(-1)^n n! = \sum_{n\geq 1}(-1)^n \int_0^\infty e^{-t} t^n \, \frac{dt}{t}$$

$$= \int_0^\infty e^{-t} \sum_{n\geq 1}(-1)^n t^n \, \frac{dt}{t} = \int_0^\infty \frac{e^{-t}}{1+t} dt$$

Notice that the last integral is convergent!This idea can be extended to a full fledged summability theory, called Borel summation.

Zeta function regularization

- To regularize a divergent series like $\sum_1^\infty a_n$, we replace it by the function
$$Z(s) = \sum_{n=1}^{\infty} a_n^{-s},$$
and let
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- For this to make sense: $Z(s)$ must be convergent (hence holomorphic) for $Re(s)$ large enough and it must have an analytic continuation to $s = -1$.

- Let us apply this to the series $1 + 2 + 3 + 4 + \cdots$.

- In this case $Z(s) = \zeta(s)$ is the Riemann zeta function, originally defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad \Re(s) > 1.$$

It is convergent (and holomorphic) only in $\Re(s) > 1$. But it has an analytic continuation to $\mathbb{C} \setminus \{1\}$, with a simple pole at $s = 1$. So the regularization

$$1 + 2 + 3 + \cdots = \zeta(-1)$$

is well defined and finite. How to compute $\zeta(-1)$? Let us first understand the idea of analytic continuation.

What is analytic continuation and how it is done in practice?

- Our original formula for $\zeta(s)$

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is divergent in the left half plane $\Re(s) \leq 1$, but $\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$. How is this possible?

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  is divergent in the left half plane $\Re(s) \leq 1$, but $\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$. How is this possible?

- Standard method to find an analytic continuation of an analytic function $f(z)$: Find a different formula for $f(z)$ which is manifestly defined and holomorpic on a larger domain.

Analytic continuation of $\zeta(s)$

- A simple example: $f(z) = \sum_{n=0}^{\infty} z^n$ is only convergent and defined for $|z| < 1$, but a different formula for it, $f(z) = \frac{1}{1-z}$, is clearly analytic in $\mathbb{C} \setminus \{1\}$.

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- We apply the same idea to Riemann zeta function. Is there a different formula for $\zeta(s)$ that is manifestly analytic in a larger domain? Yes, and in fact there are many formulas and all of them are rather hard to find. In his 1859 magnificient paper, Riemann gave at least two other formulas for $\zeta(s)$ that leads to its analytic continuation. Here we give yet another formula that is based on Euler-Maclaurin summation formula. We need to know about Bernoulli numbers first.

Enter Bernoulli numbers

- Bernoulli numbers $B_m, m = 0, 1, 2, \cdots$ are defined by their generating function:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$$

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- Easy to see that

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_{2n+1} = 0, \quad n = 1, 2, 3,$$

These numbers are ubiquitous: they appear in analysis, geometry, topology, and numerical analysis.

Euler-Maclaurin Summation

- This formula turns summation into integration and vice-versa, with a remainder term that can be effectively computed/estimated:

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x)\,dx + \sum_{k=1}^{n} \frac{B_k}{k!}\left(f^{(k-1)}(b) - f^{(k-1)}(a)\right) + R_n.$$

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- A heuristic proof: look for a function g s.t.

$$g(x+1) - g(x) = f(x)$$

Then

$$f(a)+f(a+1)+\cdots+f(b-1) = g(a+1)-g(a)+\cdots+g(b)-g(b-1)$$

$$= g(b) - g(a)$$

Euler-Maclaurin Summation

- How to find this g? Let $D = \frac{d}{dx}$. Taylor's formula gives:

$$f(x) = g(x+1) - g(x) = (\sum \frac{D^n}{n!})g(x) = (e^D - 1)g(x)$$

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$$g(x) = \frac{D}{e^D - 1}h(x), \qquad Dh(x) = f(x), \quad h(x) = \int_a^x f(t)dt.$$

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- Solution (Bernoulli numbers appear!)

$$g(x) = (\sum_{n=0}^{\infty} B_n \frac{D^n}{n!})h(x).$$

Notice that $g(b) - g(a) = h(b) - h(a) = \int_a^b f(x)dx$.

Bernoulli polynomials

Bernoulli polynomials $B_k(x), k = 0, 1, 2 \cdots$ are defined recursively:

$$B_0(x) = 1$$

$$B_n'(x) = nB_{n-1}(x) \quad \text{and} \quad \int_0^1 B_n(x)\,dx = 0 \text{ for } n \geq 1$$

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Here are the first few

$$\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x - 1/2 \\ B_2(x) &= x^2 - x + 1/6 \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \end{aligned}$$

Bernoulli polynomials

- Alternatively, they can be defined by their generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

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Remainder term for EM summation

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- Example: for $n = 1$, we get

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x)\,dx - \frac{1}{2}\left(f(b) - f(a)\right)) + \int_a^b f'(x)(x - [x])dx$$

Proof: Integration by parts!

Example: for $n = 2$, we get

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x)\,dx - \frac{1}{2}\left(f(b) - f(a)\right) + \frac{1}{12}\left(f'(b) - f'(a)\right)$$

$$-\frac{1}{2}\int_a^b f''(x)\bar{B}_2(x)dx$$

Faulhaber-Bernoulli formulae

- As a first application we get a formula for power sums. Let $f(x) = x^p$. Then the remainder $R_n = 0$ for $n > p$ and we get an exact formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} (-1)^j \binom{p+1}{j} B_j n^{p+1-j}.$$

$$= \frac{1}{p+1} n^{p+1} + \frac{1}{2} n^p + \frac{p}{2} B_2 n^{p-1} + \cdots .$$

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- Example:

$$1^5 + 2^5 + 3^5 + \cdots + n^5 = \frac{2n^6 + 6n^5 + 5n^4 - n^2}{12}$$

A new formula for $\zeta(s)$

- As a second application of Euler-Maclaurin summation we obtain a new formula for $\zeta(s)$ that is manifestly extendible to larger domains. Let $f(x) = x^{-s}$. We get, for $s \neq 1$,

$$\sum_{m=1}^{N} \frac{1}{m^s} = \frac{1 - N^{1-s}}{s-1} + \frac{1 + N^{-s}}{2}$$

$$+ \sum_{k=2}^{n} B_k s(s+1) \cdots (s+k-2)(1 - N^{-s-k+1})/k! + R_n$$

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- Let $N \to \infty$ with $Re(s) > 1$ fixed. We get

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=2}^{n} B_k s(s+1) \cdots (s+k-2)/k!$$

$$-\frac{1}{n!} s(s+1) \cdots (s+n-1) \int_1^{\infty} \bar{B}_n(x) x^{-s-n} dx$$

- Example: for $n = 1$, we get

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty (x - [x] - 1/2)x^{-s-1}dx.$$

The integral is convergent if $Re(s) > 0$. This already extends zeta to the larger domain $Re(s) > 0$ (with a simple pole at s=1).

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- In general, since $\bar{B}_n(x)$ is periodic, it is bounded on $[1, \infty]$ and hence $\int_1^{\infty} \bar{B}_n(x)x^{-s-n}dx$ is convergent if $Re(s) > 1 - n$.

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- In general, since $\bar{B}_n(x)$ is periodic, it is bounded on $[1, \infty]$ and hence $\int_1^\infty \bar{B}_n(x)x^{-s-n}dx$ is convergent if $Re(s) > 1 - n$.

- Thus the new expression for $\zeta(s)$ shows that it can be analytically continued to the larger domain $Re(s) > 1 - n$. By choosing larger and larger values of n we see that $\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$.

Special zeta values

The formula that we obtained before,

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty (x - [x] - 1/2)x^{-s-1}dx,$$

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shows that

$$\zeta(0) = -\frac{1}{2}.$$

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shows that

$$\zeta(-1) = -\frac{1}{12}$$

And in general we get

$$\zeta(-m) = -\frac{B_{m+1}}{m+1}, \qquad m \geq 0$$

Special zeta values

These are usually written in a mystifying way:

$$
\begin{aligned}
1+1+1+\cdots &= -\frac{1}{2} \\
1+2+3+\cdots &= -\frac{1}{12} \\
1^2+2^2+3^2+\cdots &= 0 \\
1^3+2^3+3^3+\cdots &= \frac{1}{120} \\
&\cdots\cdots \\
1^m+2^m+3^m+\cdots &= -\frac{B_{m+1}}{m+1}
\end{aligned}
$$

These formulas were obtained by Euler in 18th century. His interpretation of the sums were different though.

Cutoff Regularization

- Here is another method to regularize divergent sums.
- Fix a smooth rapidly decreasing function $f : [0, \infty) \to \mathbb{R}$, with $f(0) = 1$. f is called a cutoff or regulator. Replace $\sum_{n=0}^{\infty} a_n$ by

$$S(\Lambda) = \sum_{n=0}^{\infty} a_n f(\frac{n}{\Lambda}), \qquad \Lambda > 0.$$

  Very often it happens that $S(\Lambda)$ is convergent, and has an asymptotic expansion near ∞

$$S(\Lambda) = \sum_{k=-N}^{\infty} C_k \Lambda^k, \qquad \Lambda \to \infty.$$

Cutoff summation

- In good cases, C_0 is independent of the choice of the cutoff function f. We then define the regularized sum by

$$a_0 + a_1 + a_2 + \cdots = C_0.$$

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- To compute the constant C_0, we can try using Euler-Maclaurin summation formula, as we explain next.

Cutoff and Euler-Maclaurin

- In our original Euler-Maclaurin formula, if we let $a = 0, b = \infty, n = \infty$, for a rapid decay function g we get an asymptotic expansion

$$\sum_{k=0}^{\infty} g(k) \sim \int_0^{\infty} g(x)\, dx - \sum_{k=1}^{\infty} \frac{B_k}{k!} g^{(k-1)}(0).$$

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- For $g(x) = xf(\frac{x}{\Lambda})$, we get an asymptotic expansion

$$\sum_{k=0}^{\infty} kf(\frac{k}{\Lambda}) \sim \Lambda^2 \int_0^{\infty} xf(x)\,dx - \frac{B_2}{2} + O(\frac{1}{\Lambda}).$$

Cutoff Regularization

- For $g(x) = x^2 f(\frac{x}{\Lambda})$, we get an asymptotic expansion

$$\sum_{k=0}^{\infty} k^2 f(\frac{k}{\Lambda}) \sim \Lambda^3 \int_0^{\infty} x f(x)\, dx - \frac{B_3}{3} + O(\frac{1}{\Lambda}).$$

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- In genral, for $g(x) = x^m f(\frac{x}{\Lambda})$, we get an asymptotic expansion

$$\sum_{k=0}^{\infty} k^m f(\frac{k}{\Lambda}) \sim \Lambda^{m+1} \int_0^{\infty} xf(x)\, dx - \frac{B_{m+1}}{m+1} + O(\frac{1}{\Lambda}).$$

- Thus according to our cutoff regularization scheme, the constant term should give the regularized sum:

$$1^m + 2^m + 3^m + \cdots = -\frac{B_{m+1}}{m+1}$$

which coincides with our zeta function regularization. This is not accidental.

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- As a direct check, we evaluate, with cutoff $f(x) = e^{-x}$,

$$S(\Lambda) = \sum_{n=1}^{\infty} n e^{-\frac{n}{\Lambda}} = \frac{a}{(1-a)^2}, \qquad a = e^{-\frac{1}{\Lambda}},$$

which has the expansion

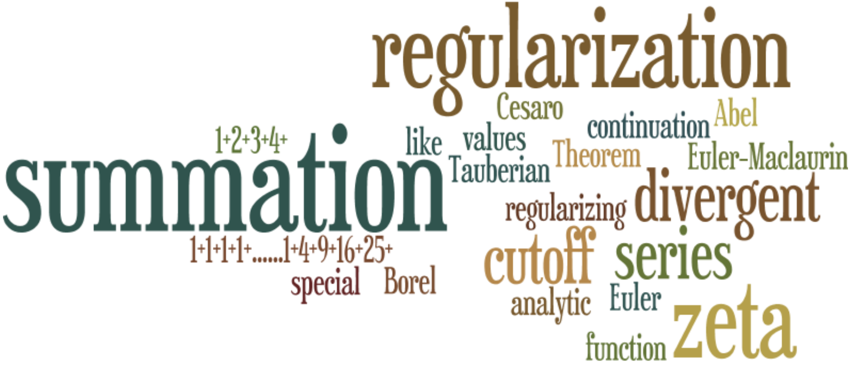
$$S(\Lambda) = \Lambda^2 - \frac{1}{12} + \frac{1}{\Lambda^2}\frac{1}{240} + O(\frac{1}{\Lambda^4}).$$

- This matches perfectly with what we got from Euler-Maclaurin summation formula.

- As a good excercise, one should check by a direct calculation as above that for the cutoff $f(x) = e^{-x}$ one obtains the same result for $\sum_{k=1}^{\infty} k^m$ as we found using Euler-Maclaurin summation formula.

Summary

- We sketched several approaches to regularizing divergent series like $1 + 2 + 3 + \cdots$: Abel summation, Cesaro summation, Borel summation, zeta function regularization, and cutoff regularization. The zeta regularization needed more sophisticated tools like analytic continuation, and computing special zeta values which was done using Euler-Maclaurin summation formula.



regularization
Cesaro
continuation
Abel
1+2+3+4+
like
values
summation
Tauberian
Theorem
Euler-Maclaurin
regularizing
divergent
1+1+1+1+......1+4+9+16+25+
cutoff
series
special
Borel
analytic
Euler
zeta
function