A Presentation: Group of Unitary Operators

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1 A Brief Background

Among many other facets of the interesting journey of the body of knowledge we refer to as quantum mechanics has been its interplay with mathematics, or lack there of. This has been a period whereby physics has changed mathematics and correspondingly mathematics has led physics to new directions. It is conjectured that at no point in time this interplay has been this pronounced and interesting.

Part of the folklore of the subject concerns the mathematical physics textbook *Methods of mathematical Physics* put together by Richard Courant from Hilbert's Gottingen University Lectures. The story is told (by mathematicians) that the physicists had dismissed the material as not interesting in the current research areas, until the advent of Schrodinger's Equation. At that point it was realized that the mathematics of the new quantum mechanics was already laid out in it. It was also said that Heisenberg had consulted Hilbert about his matrix mechanics, and Hilbert observed that his own experience with infinite-dimensional matrices had originated from differential equations, an advice which Heisenberg ignored, missing the opportunity to unify the theory as Weyl and Dirac did a few years later. Whatever the basis of anectodes, the mathematics of the theory was at the time unconventional and whereas the physics was new. This forced things to be looked at under new light: For instance, there was spectral theory before the quantum theory, but it was based on quadratic forms rather than the new approach which was based on linear operators.

Prior to the emergence of quantum mechanics as a separate theory, the mathematicians used in physics consisted mainly of partial differential equations, differential geometry, and statistical mechanics. geometric intuition played a strong role and theories of relativity were formulated entirely in terms of geometric concepts.

The phenomenology of quantum physiscs aroese roughly between 1895-1925 and for the next 15 years or so, before the emergence of *quantum theory*, physicists continued to think of the theory within the confines of what is now called **classical mechanics**, and in particular within the same mathematical structures. the situation has changed dramatically and rapidly in the years between 1925-1930 wen working mathematical foundations were found through groundbreaking work of Schrodinger, Werner Heisenberg, Paul

Dirac, John Von Neumann, and Hermann Weyl, and became possible to unify several different approaches. The Mathematical formulation of quantum mechanics is a body of mathematical formalisms which permits a rigorous description of quantum mechanics. It is distinguished from mathematical formalism of theories developed prior to 1900's by use of abstract mathematical structures, such as infinite-dimensional Hilbert Spaces and operators on these spaces. Many of these structures were drawn from functional analysis, a research area that was influenced in part by the needs of quantum mechanics.

In brief, values of physical observables such as energy amd momentum and position were no longer considered as values of functions on phase space, but as eigenvalues - more precisely as spectral values of linear operators in Hilbertspace:

- point spectrum
- absolutre spectrum
- singular continuous spectrum.

1.1 Basic Postulates

At the heart of the description are ideas of **quantum state** and **quantum observable** which are radically different from those used in previous models of physical reality. Physical symmetries act on Hilbert space of quantum states unitarily. The mathematical structure of quantum mechanics is generally described by three basic ingredients:

States of the System States are no longer points in the symplectic phase spacce, but rather one-dimensional subgroups of $\mathbb{H}$, which are rays.here we have a separable complex Hilbert space $\mathbb{H}$ with $\langle \phi \mid \psi. \rangle$ Separability is rather a mathematical convenient hypothesis; with the physical interpretation that countably many observations are sufficient enough to uniquely determine the given state.

Observables Observables are no longer real valued functions. They are densely represented self adjoint operators on the space of states. The observables can be represented by operators on the Hilbert space $\mathcal{H}$, each state is represented by a density matrix. Since the representation

of the algebra $\mathcal{A}$ is irreducible, the density matrix ρ_w associated to the state w is unique. Expectation values are equal to the trace:

$$\forall\, w \in \mathcal{S}(\mathcal{A}) \exists\, \rho_w \;\forall\, a \in \mathcal{A} : w(a) = \text{Tr } \rho_w a.$$

The postulates are necessary and sufficient to imply the standard form of time-evolution: There is a self-adjoint operator H, the Hamiltonian, unique except for an additive constant, generating a unitary group $U_\tau := \exp(-iH\tau/\hbar)$, which represents the time-evolution:

$$\forall\, t, \tau, w, a : w_{t+\tau}(a) = \text{Tr } e^{-iH\tau/\hbar} \rho_w e^{iH\tau/\hbar} a.$$

Dynamics This is also called the Law of Time Evolution. Time evolution is given by a one-parameter group of unitary transformations on $\mathbb{H}$.

1.2 Unitary Operator

- A **unitary space** V is a complex vector space with a distinguished positive definite Hermitian form,

  $$\langle -, - \rangle : V \times V \to \mathbb{C},$$

  which serves as the inner product on V.

- A **unitary transformation** is a surjective linear transformation $T : V \to V$ satisfying

  $$\langle u, v \rangle = \langle Tu, Tv \rangle, \quad u, v \in V. \tag{1}$$

  These are isometries of V.

- More generally, a **unitary transformation** is a surjective linear transformation $T : U \longrightarrow V$ between two unitary spaces U, V satisfying

  $$\langle Tv, Tu \rangle_V = \langle v, u \rangle_U, \quad u, v \in U$$

  In this entry will restrict to the case of the first

- A **unitary matrix** is a square complex-valued matrix, A, whose inverse is equal to its conjugate transpose:

  $$A^{-1} = \bar{A}^t.$$

- When V is a Hilbert space, a bounded linear operator $T : V \longrightarrow V$ is said to be a **unitary operator** if its inverse is equal to its adjoint:

$$T^{-1} = T^*$$

In Hilbert spaces unitary transformations correspond precisely to unitary operators.

1.3 Some Examples

1. A standard example of a unitary space is $\mathbb{C}^n$ with inner product

$$\langle u, v \rangle = \sum_{i=1}^{n} u_i \, \overline{v_i}, \quad u, v \in \mathbb{C}^n. \tag{2}$$

2. Unitary transformations and unitary matrices are closely related. On the one hand, a unitary matrix defines a unitary transformation of $\mathbb{C}^n$ relative to the inner product. On the other hand, the representing matrix of a unitary transformation relative to an orthonormal basis is, in fact, a unitary matrix.

3. A unitary transformation is an automorphism. This follows from the fact that a unitary transformation T preserves the inner-product norm:

$$\|Tu\| = \|u\|, \quad u \in V. \tag{3}$$

Hence, if

$$Tu = 0,$$

then by the definition it follows that

$$\|u\| = 0,$$

and hence by the inner-product axioms that

$$u = 0.$$

Thus, the kernel of T is trivial, and therefore it is an automorphism.

4. Moreover, relation can be taken as the definition of a unitary transformation. Indeed, using the polarization identity it is possible to show that if T preserves the norm, then must hold as well.

5. A simple example of a unitary matrix is the change of coordinates matrix between two orthonormal bases. Indeed, let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be two orthonormal bases, and let $A = (A^i_j)$ be the corresponding change of basis matrix defined by

$$v_j = \sum_i A^i_j \, u_i, \quad j = 1, \ldots, n.$$

Substituting the above relation into the defining relations for an orthonormal basis,

$$\begin{aligned} \langle u_i, u_j \rangle &= \delta_{ij}, \\ \langle v_k, v_l \rangle &= \delta_{kl}, \end{aligned}$$

we obtain

$$\sum_{ij} \delta_{ij} A^i_k \overline{A^j_l} = \sum_i A^i_k \overline{A^i_l} = \delta_{kl}.$$

In matrix notation, the above is simply

$$A \bar{A}^t = I,$$

as desired.

6. Unitary transformations form a group under composition. Indeed, if S, T are unitary transformations then ST is also surjective and

$$\langle STu, STv \rangle = \langle Tu, Tv \rangle = \langle u, v \rangle$$

for every $u, v \in V$. Hence ST is also a unitary transformation.

7. Unitary spaces, transformations, matrices and operators are of fundamental importance in quantum mechanics.

8. For a system initially in some state ρ, the final (collapsed) state after a measurement which yields outcome j with probability

$$p_j = Tr(\rho K_j^\dagger K_j)$$

is

$$\rho \to K_j \rho K_j^\dagger / p_j$$

In standard quantum mechanics the continuous (Schroedinger) evolution takes the form

$$\rho \to U\rho U^\dagger$$

where U is a unitary operator. If the Hamiltonian governing this evolution has spectral decomposition

$$H = \sum_{j=1}^{d} \lambda_j |j\rangle\langle j|$$

then this unitary form is given by

$$U = \sum_{j=1}^{d} e^{\lambda_j t} |j\rangle\langle j|.$$

9. The Fourier operator is a unitary operator, i.e. the operator which performs the Fourier transform (with proper normalization). This follows from Parseval's theorem.

10. The spectrum of a unitary operator lies on the unit circle. That is, for any complex number z in the spectrum, one has $|z| = 1$.

    This can be seen as a consequence of the spectral theorem for normal operators. By the theorem, U is unitarily equivalent to multiplication by a Borel-measurable f on L^2, for some finite measure space (X, μ). Now $UU^* = I$ implies $|f(x)|^2 = 1$ in μ a.e. This shows that the essential range of f, therefore the spectrum of U, lies on the unit circle.

The linearity requirement in the definition of a unitary operator can be dropped without changing the meaning because it can be derived from linearity and positive-definiteness of the scalar product:

$$\langle \lambda \cdot Ux - U(\lambda \cdot x), \lambda \cdot Ux - U(\lambda \cdot x)\rangle$$

$$\begin{aligned} &= \|\lambda \cdot Ux\|^2 + \|U(\lambda \cdot x)\|^2 - \langle U(\lambda \cdot x), \lambda \cdot Ux\rangle - \langle \lambda \cdot Ux, U(\lambda \cdot x)\rangle \\ &= |\lambda|^2 \cdot \|Ux\|^2 + \|U(\lambda \cdot x)\|^2 - \overline{\lambda} \cdot \langle U(\lambda \cdot x), Ux\rangle - \lambda \cdot \langle Ux, U(\lambda \cdot x)\rangle \\ &= |\lambda|^2 \cdot \|x\|^2 + \|\lambda \cdot x\|^2 - \overline{\lambda} \cdot \langle \lambda \cdot x, x\rangle - \lambda \cdot \langle x, \lambda \cdot x\rangle \end{aligned}$$

Analogously you obtain

$$\langle U(x+y) - (Ux + Uy), U(x+y) - (Ux + Uy)\rangle = 0$$

2 A Few Basic Definitions

2.1 Hamiltonian

In quantum mechanics Hamiltonian H is the operator corresponding to the total energy of the system. Its spectrum is the set of possible outcomes when one measures the total energy of a system. It is of fundamental importance in most formulations of quantum theory because of its close relation to the time-evolution of a system. By analogy with the classical mechanics Hamiltonian is commonly expressed as the sum of operators corresponding to the kinetic potential energies of a system, in the form

$$H = T + V.$$

Although this is not the technical definition of the Hamiltonian in classical mechanics, it is the form it most commonly takes.

The potential operator V typically takes the form of a function $V(r,t)$ of position and time, which simply acts on states as a multiplicative factor. The operator T corresponding to kinetic energy is constructed by analogy with the classical formula $T = \frac{p^2}{2m}$. Schrödinger constructed his momentum operator using the substitution $p \to -i\hbar\nabla$ where ∇ is the operator, i is the unit imaginary number, and $\hbar$ is the reduced Planck constant. Combining this with the potential term yields

$$\mathbf{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)$$

which allows one to apply the Hamiltonian to systems described by a wave function $\Psi(\mathbf{r},t)$. This is the approach commonly taken in introductory treatments of quantum mechanics, using the formalism of Schrödinger's wave mechanics. However, in the more general formalism of Dirac, the Hamiltonian is typically implemented as an operator on Hilbert space in the following way:The eigenvectors of H, denoted $|\ |a\rangle$, provide an orthonormal basis for the Hilbert space. The spectrum of allowed energy levels of the system is given by the set of eigenvalues, denoted E_α, solving the equation:

$$H\,|a\rangle = E_a\,|a\rangle$$

Since H is a Hermitian operator, the energy is always a real number.

2.2 Time evolution - A Unitary Operator

Time evolution is the change of state brought about by the passage of time, applicable to systems with internal state (also called stateful systems). In this formulation, time is not required to be a continuous parameter, but may be discrete or even finite. In classical physics, time evolution of a collection of rigid bodies is governed by the principles of classical mechanics. In their most rudimentary form, these principles express the relationship between forces acting on the bodies and their acceleration given by Newton's laws of motion. These principles can also be equivalently expressed more abstractly by Hamiltonian mechanics or Lagrangian mechanics. In quantum mechanics, the state of any physical system is represented by a vector. Suppose that $|\alpha\rangle$ is such a vector. Time evolution is the process

$$|\alpha\rangle \to e^{-iHt}|\alpha\rangle$$

where H is the Hamiltonian operator. You can think of the state vector as a representation of all properties of the system, in the past, present, and future. The effect of the time evolution operator is then to transform our state vector to the state vector that another observer would use to describe the same system. This would be an observer whose clock shows zero t seconds after ours does.

That point of view is called the Heisenberg picture. (If we're using the Heisenberg picture, we prefer to call it time translation rather than time evolution).

Another point of view is the Schrödinger picture. Here we think of the state vector as a time-dependent quantity:

$$|\alpha\rangle \to e^{-iHt}|\alpha\rangle$$

We think of this as the state of the system at time t. It's easy to verify that this time dependent state vector satisfies the Schrödinger equation (because the time evolution operator does):

$$i\frac{\partial}{\partial t}|\alpha;t\rangle = H|\alpha;t\rangle$$

Stateful systems often have dual descriptions in terms of states or in terms of observable values. In such systems, time evolution can also refer to the change in observable values. This is particularly relevant in quantum mechanics where the Schrödinger picture and Heisenberg picture are (mostly)

equivalent descriptions of time evolution. In quantum mechanics, the propagation operators are unitary operators on a Hilbert space. The propagators can be expressed as time-ordered exponentials of the integrated Hamiltonian. The Hamiltonian generates the evolution of quantum states. If $|\psi(t)\rangle$ is the state of the system at time t, then

$$H\,|\psi(t)\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle$$

This equation is known as the Schrödinger equation. (It takes the same form as the Hamilton-Jacobi equation, which is one of the reasons H is also called the Hamiltonian.) Given the state at some initial time ($t = 0$), we can integrate it to obtain the state at any subsequent time. In particular, if H is independent of time, then

$$|\psi(t)\rangle = \exp\left(-\frac{\mathrm{i}Ht}{\hbar}\right)|\psi(0)\rangle$$

.The exponential operator on the right hand side of the Schrödinger equation is usually defined by the corresponding a power series in H. One might notice that taking polynomials of unbounded and not everywhere defined operators may not make mathematical sense, much less power series. Rigorously, to take functions of unbounded operators, a functional calculus is required. In the case of the exponential function, the continuous, or just the holomorphic functional calculus suffices. We note again, however, that for common calculations the physicists' formulation is quite sufficient.By the homomorphism¡/a¿ property of the functional calculus, the operator

$$U = \exp\left(-\frac{\mathrm{i}Ht}{\hbar}\right)$$

is a unitary operator. It is the time evolution operator, or propagator, of a closed quantum system. If the Hamiltonian is time-independent, U(t) form a one parameter unitary group; this gives rise to the physical principle of detailed balance.

2.3 One-parameter group

Let G be a Lie Group. A *one-parameter subgroup* of G is a group homomorphism

$$\phi\colon \mathbb{R} \to G$$

that is also a differentiable map at the same time. We view $\mathbb{R}$ additively and G multiplicatively, so that $\phi(r+s) = \phi(r)\phi(s)$.

Examples.

1. If $G = \mathrm{GL}(n,k)$, where $k = \mathbb{R}$ or $\mathbb{C}$, then any one-parameter subgroup has the form
$$\phi(t) = e^{tA},$$
where $A = \frac{d\phi}{dt}(0)$ is an $n \times n$ matrix over k. The matrix A is just a tangent vector to the Lie group $\mathrm{GL}(n,k)$. This property establishes the fact that there is a one-to-one correspondence between one-parameter subgroups and tangent vectors of $\mathrm{GL}(n,k)$. The same relationship holds for a general Lie group. The one-to-one correspondence between tangent vectors at the identity (the Lie algebra) and one-parameter subgroups is established via the exponential map instead of the matrix exponential.

2. If $G = \mathrm{O}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$, the orthogonal group over R, then any one-parameter subgroup has the same form as in the example above, except that A is skew-symmetric: $A^{\mathrm{T}} = -A$.

3. If $G = \mathrm{SL}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$, the special linear group over R, then any one-parameter subgroup has the same form as in the example above, except that $\mathrm{tr}(A) = 0$, where tr is the trace operator.

4. If $G = \mathrm{U}(n) = \mathrm{O}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$, the unitary group over C, then any one-parameter subgroup has the same form as in the example above, except that A is skew-Hermitian: $A = -A^* = -\overline{A}^{\mathrm{T}}$ and $\mathrm{tr}(A) = 0$.

3 Stone's theorem

In mathematics, Stone's theorem on one-parameter unitary groups is a basic theorem of functional analysis which establishes a one-to-one correspondence between self-adjoint operators on a Hilbert space H and one-parameter families of unitary operators
$$\{U_t\}_{t\in\mathbb{R}}$$
which are strongly continuous, that is
$$\lim_{t\to t_0} U_t\xi = U_{t_0}\xi \quad \forall t_0 \in \mathbb{R}, \xi \in H$$

and are homomorphisms

$$U_{t+s} = U_t U_s.$$

Such one-parameter families are ordinarily referred to as strongly continuous one-parameter unitary groups. The theorem is named after Marshall Stone who formulated and proved this theorem in 1932.

Let U be a strongly continuous 1-parameter unitary group, then there exists a unique self-adjoint operator A such that

$$U_t := e^{itA} \quad t \in \mathbb{R}.$$

Conversely, let A be a self-adjoint operator on a Hilbert space H. Then

$$U_t := e^{itA} \quad t \in \mathbb{R}$$

is a strongly continuous one-parameter family of unitary operators.The infinitesimal generator of U_{tt} is the operator iA. This mapping is a bijective correspondence. A will be a bounded operator the operator-valued function $t \mapsto U_t$ continuous.

Example: The family of translation operators

$$[T_t\psi](x) = \psi(x + t)$$

is a one-parameter unitary group of unitary operators; the infinitesimal generator of this family is an extension of the differential operator

$$\frac{d}{dx} = i\frac{1}{i}\frac{d}{dx}$$

defined on the space of complex-valued continuously differentiable functions of compact support on R. Thus

$$T_t = e^{t\, d/dx}.$$

In other words, motion on the line is generated by the momentum operator.

4 An important one: Stone von Neumann

One might ask 'why bother?'...Historically this result was significant because it was a key step in proving that Heisenberg's matrix mechanics, which represents quantum mechanical observables and dynamics in terms of infinite

matrices, is unitarily equivalent to Schrödinger's wave mechanical formulation. Stone's theorem was extended to Stone-von Neumann. In mathematics and in theoretical physics, the Stone–von Neumann theorem is any one of a number of different formulations of the uniqueness of the canonical commutation relations between position and momentum operators. The name is for Marshall Stone and John von Neumann. In quantum mechanics, physical observables are represented mathematically by linear operators on Hilbert spaces. For a single particle moving on the real line $\mathbb{R}$, there are two important observables: position and momentum. In the quantum-mechanical description of such a particle, the position operator Q and momentum operator P are respectively given by

$$[Q\psi](x) = x\psi(x)$$

$$[P\psi](x) = \frac{\hbar}{i}\psi'(x)$$

on the domain V of infinitely differentiable functions of compact support on $\mathbb{R}$. We assume $\hbar$ is a fixed non-zero real number — in quantum theory $\hbar$ is (up to a factor of 2p) Planck's const ant, which is not dimensionless; it takes a small numerical value in terms of units in the m acroscopic world. The operators P, Q satisfy the commutation relation

$$QP - PQ = -\frac{\hbar}{i}\mathbf{1}.$$

Already in his classic volume, Hermann Weyl observed that this commutation law was impossible for linear operators P, Q acting on finite dimensional spaces (as is clear by applying the trace of a matrix, unless $\hbar$ vanishes. A little analysis shows that in fact any two self-adjoint operators satisfying the above commutation relation cannot be both bounded.

One would like to classify representations of the canonical commutation relation by two self-adjoint operators acting on separable Hilbert spaces, up to unitary equivalence. By Stone's theorem, there is a one-to-one correspondence between self-adjoint operators and (strongly continuous) one parameter unitary groups. Let Q and P be two self-adjoint operators satisfying the canonical commutation relation, and e^{itQ} and e^{isP} be the corresponding unitary groups given by functional calculus. A formal computation with power series shows that

$$\mathrm{e}^{itQ}\,\mathrm{e}^{isP} - \mathrm{e}^{ist}\,\mathrm{e}^{isP}\,\mathrm{e}^{itQ} = 0.$$

Conversely, given two one parameter unitary groups $U(t)$ and $V(s)$ satisfying the relation

$$U(t)V(s) = \mathrm{e}^{ist}\, V(s)U(t) \quad \forall s,t, \qquad (*)$$

formally differentiating at 0 shows that the two infinitesmal generators satisfy the canonical commutation relation. These formal calculations can be made rigorous. Therefore there is a one-to-one correspondence between representations of the canonical commutation relation and two one parameter unitary groups ¡$U(t)$ and ¡$V(s)$ satisfying (*). This formulation of the canonical commutation relations for one parameter unitary groups is called the Weyl form of the CCR.The problem now thus becomes classifying two jointly irreducible one parameter unitary groups $U(t)$ and $V(s)$ satisfying the Weyl relation on separable Hilbert spaces. The answer is the content of the ¡b¿Stone-von Neumann theorem: all such pairs of one parameter unitary groups are unitarily equivalent. In other words, for any two such $U(t)$ and $V(s)$ acting jointly irreducibly on a Hilbert space H, there is a unitary operator$W : L^2(\mathbb{R}) \to H$so that

$$W^*U(t)W = \mathrm{e}^{isQ} \quad \text{and} \quad W^*V(t)W = \mathrm{e}^{isP},$$

where P and Q are the position and momentum operators from above

The question was:

Can we classify two jointly irreducible one-parameter unitary groups $U(t)$ and $V(s)$ satisfying Weyl relation on Hilbert spaces?

The answer was the content of Stone-von Neumann theorem. We list the theorem and two consequences, giving the proof of only one.

Theorem: Let $\mathcal{A}$ be a C^*-algebra with a unique irreducible representation in a separable Hilbert space $\mathcal{H}$. Let $\{T_\tau | \tau \in \mathbf{R}\}$ be a one-parameter group of affine invertible mappings $\mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A})$, such that $\forall\, w \in \mathcal{S}(\mathcal{A})$, $\forall\, a \in \mathcal{A}$ the function $\tau \mapsto (T_\tau w)(a)$ is measurable, then $\tau \mapsto T_\tau w$ is strongly continuous.

Theorem: Given a one parameter group of mappings of the states of $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space,

$$T_\tau : w \mapsto \mathcal{T}_\tau w, \qquad T_0 = \text{identity}, \qquad \forall\, \{\sigma, \tau\} \subset \mathbf{R} : T_\sigma T_\tau = T_{\sigma+\tau},$$

where each mapping can be represented by a unitary transformation,

$$\forall\, \tau \in \mathbf{R}\ \exists U_\tau = U^*_{-\tau} = U^{-1}_{-\tau} \quad \forall\, a \subset \mathcal{B}(\mathcal{H}) : (T_\tau w)(a) = w(U_{-\tau} a U_\tau),$$

and which acts weakly continuous on pure states applied to finite rank operators:

for a of finite rank, w pure, $\tau \mapsto (T_\tau w)(a)$ is a continuous function,

then the unitary operators U_τ can be chosen in such a way that they form a strongly continuous group,

$U_0 = \mathbf{1}, \quad \forall\, \{\sigma, \tau\} \subset \mathbf{R} : U_\sigma U_\tau = U_{\sigma+\tau}, \quad \forall\, \psi \in \mathcal{H} : \tau \mapsto \|U_\tau \psi\|$ is a continuous function.

Proof:

(a) We choose some reference vector $|\psi\rangle \in \mathcal{H}$, fixed throughout the proof. Since for any operator $a = |\psi\rangle\langle\varphi|$ and all pure states represented by the vectors $|\psi + i^n\eta\rangle$ the expectation values of $U_{-\tau} a U_\tau$ are continuous, we know that

$$\frac{1}{4}\sum_{n=0}^{3} i^{-n}\langle\psi + i^n\eta|U_{-\tau} a U_\tau|\psi + i^n\eta\rangle = \langle\psi|U_{-\tau}|\psi\rangle\langle\varphi|U_\tau|\eta\rangle$$

is a continuous function. If we have constructed a group U_τ such that $\tau \mapsto \langle\psi|U_\tau|\psi\rangle$ is continuous (that will be achieved in (b) to (g)), then we know that there is some interval around $\tau = 0$, where $\langle\psi|U_\tau|\psi\rangle \neq 0$, and therefore also $\langle\varphi|U_\tau|\eta\rangle$ is continuous. This property holds for all vectors $|\varphi\rangle$, $|\eta\rangle$, extends to all $\tau \in \mathbf{R}$ by the group property and implies the strong continuity.

(b) Setting $|\varphi\rangle = |\eta\rangle = |\psi\rangle$ in the formula in (a), we see that $|\langle\psi|U_\tau|\psi\rangle|^2$ is a continuous function and is independent of the chosen phases for U_τ. For some ε, which shall be smaller than 2^{-10}, there exists a $\tau_0 > 0$, such that

$$\forall\, \{\sigma, \tau\} \subset [-\tau_0, \tau_0] : ||\langle\psi|U_\sigma|\psi\rangle| - |\langle\psi|U_\tau|\psi\rangle|| \leq \varepsilon < 1/2^{10}.$$

(c) U_{τ_0} considered as fixed, there is a phase $\alpha \in [-\pi, \pi]$, such that for

$$V_0 := e^{-i\alpha} U_{\tau_0}$$

the expectation value $\langle\psi|V_0|\psi\rangle$ is real and positive. Two of the representatives for $T_{\tau_0/2}$ are square roots of V_0. We choose the one with

non-negative real part of the expectation value and denote it V_1. In this way we go on and construct a family $\{V_n\}$:

$$V_{n+1}^2 = V_n, \qquad \text{Re } \langle\psi|V_n|\psi\rangle \geq 0.$$

Since $w(V_n^{-1}aV_n) = (T_\tau w)a$ for $\tau = 2^{-n}\tau_0$, we have constructed a group of unitaries U_τ, representing T_τ for all time differences $\tau = b\tau_0$, where b is a binary digit with finite length:

$$b = b_I + \sum_{n=1}^{N} \delta_n 2^{-n}, \qquad b_I \in \mathbf{Z}, \qquad \delta_n \in \{0,1\},$$

$$U_{b\tau_0} := e^{ib\alpha} V_N^{(2^N b)}.$$

(d) Next we have to show that the continuity of the absolute value of $\langle\psi|U_\tau|\psi\rangle$ implies a spectral concentration of $|\psi\rangle$. We represent each V_N as the exponential of a quasi-Hamiltonian H_N (here we assume units, where $\tau_0/\hbar = 1$):

$$V_N = \exp(-i2^{-N}H_N), \qquad -2^N\pi < H_N \leq 2^N\pi, \qquad \text{for } n \leq N : V_n = \exp(-i2^{-n}H_N).$$

We conclude from (b) and the definition of V_0 that $\langle\psi|V_0|\psi\rangle \geq 1-\varepsilon$, and, using the spectral representation of H_N,

$$\langle\psi|f(H_N)|\psi\rangle = \int f(E)d\mu_N(E),$$

that

$$1-\varepsilon \leq \langle\psi|V_0|\psi\rangle = \langle\psi|\cos H_N|\psi\rangle \leq \sum_{n,|n|\leq 2^{N-1}} a_n + (1 - \sum_n a_n)\cos\vartheta,$$

where

$$a_n := \int_{2\pi n - \vartheta}^{2\pi n + \vartheta} d\mu_N(E).$$

So $\qquad 1 \geq \sum_n a_n \geq 1 - \varepsilon/(1-\cos\vartheta) > 1 - \sqrt{\varepsilon} \qquad$ for $\vartheta = 2\varepsilon^{1/4}$.

(e) We specify the spectral concentration further, using $\varepsilon \geq 1 - \langle\psi|V_N^{-\nu}|\psi\rangle\langle\psi|V_N^{\nu}|\psi\rangle$ for all $\nu \leq 2^N - 1$:

$$\begin{aligned}
1-\varepsilon &\leq \frac{1}{2^N}\sum_{\nu=0}^{2^N-1}\langle\psi|\exp(i\nu 2^{-N}H_N)|\psi\rangle\langle\psi|\exp(-i\nu 2^{-N}|\psi\rangle = \\
&= \frac{1}{2^N}\sum_{\nu}\iint \exp(i\nu 2^{-N}(E-E')d\mu_N(E)d\mu_N(E') = \\
&= \iint \frac{1}{2^N}\sum_{\nu}\cos(\nu 2^{-N}(E-E'))d\mu_N(E)d\mu_N(E') = \\
&= \iint \cos\left((1-2^{-N})\frac{E-E'}{2}\right)\frac{\sin\frac{E-E'}{2}}{2^N\sin 2^{-N}\frac{E-E'}{2}}d\mu_N(E)d\mu_N(E') \leq \\
&\leq \sum_n a_n^2 + \frac{\vartheta}{\pi}\sum_{n\neq n'}a_n a_n' + \varepsilon \leq a_{\max}^2 + (1-a_{\max})^2 + \varepsilon^{1/4} + \varepsilon.
\end{aligned}$$

The last two inequalities are consequences of (d) and the convexity of the function $a \mapsto a^2$. Consequently, for the maximum of the concentration weights a_n:

$$\begin{aligned}
(1-a_{\max})^2 &\leq \varepsilon^{1/4} \\
a_{\max} &\geq 1 - \varepsilon^{1/8} > 3/4.
\end{aligned}$$

With this inequality, the definition of the V_n, and their representations as functions of H_N, we conclude that $a_{\max} = a_0$.

(f) We consider the phases:

$$\forall\, n < 2^N : |\mathrm{Im}\ \langle\psi|V_N^n|\psi\rangle| \leq 2\varepsilon^{1/4} + \varepsilon^{1/8} \leq 2\varepsilon^{1/8}.$$

Combined with the inequality for the absolute value in (b):

$$\|\psi - V_N^n\psi\|^2 \leq (\varepsilon^2 + 4\varepsilon^{1/4})^{1/2} \leq 5\varepsilon^{1/4}.$$

N was finite, but arbitrary, so

$$|b| \leq 1 \Longrightarrow \|\psi - U_{b\tau_0}\psi\| \leq |b\alpha| + 3\varepsilon^{1/8}.$$

By unitary equivalence

$$|b_1 - b_2| \leq 1 \Longrightarrow \|U_{b_1\tau_0}\psi - U_{b_2\tau_0}\psi\| \leq |(b_1-b_2)\alpha| + 3\varepsilon^{1/8},$$

and, since $|\alpha| \leq \pi$,

$$\|U_{b_1\tau_0}\psi - U_{b_2\tau_0}\psi\| \leq 4\varepsilon^{1/8} \qquad \text{if } |b_1\tau_0 - b_2\tau_0| \leq \varepsilon\tau_0(\varepsilon).$$

(g) Now for any smaller ε we can do the same argument starting with some $\tau_0(\varepsilon) = b\tau_0$ instead of τ_0. Because of the inequalities already proven for the first chosen ε, the $U_{b\tau_0}$ will not change in the construction (c). So we have proven that $\tau \mapsto \langle\psi|U_\tau|\psi\rangle$ is equicontinuous for $\{\tau = b\tau_0\}$ (which is dense in $\mathbf{R}$), and can thus be extended to a continuous function for all real τ.

5 Heisenberg Group

The commutation relations for P, Q look very similar to the commutation relations that define the of general Heisenberg group H_n for n a positive integer. This is the Lie group of $(n+2)\times(n+2)$ square matrices of the form

$$\mathrm{M}(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1_n & b \\ 0 & 0 & 1 \end{pmatrix}$$

In fact, using the Heisenberg group, we can formulate a far-reaching generalization of the Stone von Neumann theorem. Note that the center of H_n consists of matrices $M[0,0,c]$. **Theorem** For each non-zero real number h there is an irreducible representation U_h acting on the Hilbert space

$$[U_h(\mathrm{M}(a,b,c))]\psi(x) = e^{i(b\cdot x+hc)}\psi(x+ha).$$

All these representations are unitarily inequivalent and any irreducible representation which is not trivial on the center of H_n is unitarily equivalent to exactly one of these.Note that U_h is a unitary operator because it is the composition of two operators which are easily seen to be unitary: the translation to the $left$ by h and multiplication by a function of absolute value 1. To show $U_h$is multiplicative is a straightforward calculation.

6 Relating to Fourier Transform

For any non-zero h, the mapping

$$\alpha_h : \mathrm{M}(a,b,c) \to \mathrm{M}(-h^{-1}b, ha, c-ab)$$

is an automorphism of H_n which is the identity on the center of H_n. In particular, the representations $U_h\alpha$ and $U_h\alpha$ a are unitarily equivalent. This

means that there is a unitary operator W on $L^2(\mathbb{R}^n)$ such that for any g in H^n

$$WU_h(g)W^* = U_h\alpha(g).$$

Moreover, by irreducibility of the representations U_h, it follows that up to a scalar, such an operator W is unique (Schur's lemma)

6.0.1 Theorem: Unitary again

The operator W is, up to a scalar multiple, the Fourier transform on $L^2(\mathbb{R}^n)$.This means that (ignoring the factor of $(2p)^{n/2}$ in the definition of the Fourier transform)

$$\int_{\mathbb{R}^n} e^{-ix\cdot p} e^{i(b\cdot x+hc)}\psi(x+ha)\ dx = e^{i(ha\cdot p+h(c-b\cdot a))}\int_{\mathbb{R}^n} e^{-iy\cdot(p-b)}\psi(y)\ dy.$$

The previous theorem can actually be used to prove the unitary nature of the Fourier transform, also known as the Plancherel theorem. Moreover, note that

$$(\alpha_h)^2\,\mathrm{M}(a,b,c) = \mathrm{M}(-a,-b,c).$$

The operator W_1 such that

$$W_1U_hW_1^* = U_h\alpha^2(g)$$

$$[W_1\psi](x) = \psi(-x).$$

From this fact the Fourier inversion formula easily follows.

References

[Bis] Bishop, E.: Foundations of Constructive Analysis. McGraw-Hill, N.Y. (1967)

[Coh63] Cohen, P.J.: The Independence of the Continuum Hypothesis. Proc. Nat. Acad. Sci. US *50*, 1143–48 (1963); ibid. *51*, 105–110 (1964)

[Coh71] Cohen, P.J.: Comments on the Foundation of Set Theory. [Sco] 9–11

[FVe] Fannes, M., Verbeure, A.: On the Time Evolution Automorphisms of the CCR-Algebra for Quantum Mechanics. Commun. Math. Phys. *35*, 257–264 (1974)

[GRW] Ghirardi, G.C., Rimini, A., Weber, T.: Unified dynamics for microscopic and macroscopic systems. Phys. Rev. *D34*, 470–479 (1986)

[Gil] Giles, R.: Foundations for Quantum Mechanics. [Hoo], 277-322