Notes on the Quantum Harmonic Oscillator

Brief Discussion of Coherent States, Weyl's Law and the Mehler Kernel

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1 The Harmonic Oscillators

Recall that any object oscillating in one dimension about the point 0 (say, a mass-spring system) obeys Hooke's law $F_x = -k_s x$, where x is the compression of the spring, and k_s is the *spring constant* of the system. By Newton's second law of motion, we then have the equation of the *Classical Harmonic Oscillator*

$$m\frac{\partial^2 x}{\partial t^2} = -k_s x \tag{1}$$

Now, F_x is conservative, so we can recover the potential energy function as follows[1]:

$$-F_x = \frac{\partial V}{\partial x} \implies V(x) = \frac{1}{2}k_s x^2 \tag{2}$$

The solution to (1) is given by

$$x(t) = A\cos\left(\sqrt{\frac{k_s}{m}}\,t + \psi\right) \tag{3}$$

The frequency of oscillation is then

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k_s}{m}} \tag{4}$$

Let the oscillating object be a particle. Now, classical mechanics only provides a good approximation of the system as long as $E >> hf$, where$E$ is the total energy of the particle, f is the classical frequency, and h is the Planck constant.

Assume that this is not the case; we now see the system as a *Quantum Mechanical Oscillator*. As a consequence of the *Heisenberg Uncertainty Principle*

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2} \tag{5}$$

[1]We assume without loss of generality that the constant of integration is 0.

(where $\hat{x}$ represents the position operator, $\hat{p}$ represents the momentum operator, and $\hbar$ is the reduced Planck constant), we lose our grasp of the particle's trajectory, since we now switch our focus from the position of the particle to its *wavefunction* $\psi(x,t)$. The wavefunction of the system gives all possible information about the behaviour of the entire system; it is interpreted as an amplitude, the square of whose modulus $|\psi(x,t)|^2$ gives the probability density function of the particular state being considered.

In quantum mechanics, a *Hamiltonian* $\hat{H}$ is an operator representing the total energy of a system. To create the Hamiltonian for our system, we sum the kinetic and potential energies. Recalling that momentum p is the product of velocity v and mass m ($\hat{p} = m\hat{v}$), then the kinetic energy operator of the system is given by

$$\hat{K} = \frac{1}{2}m\hat{v}^2 = \frac{1}{2}m\left(\frac{\hat{p}}{m}\right)^2 = \frac{1}{2m}\hat{p}^2 \overset{*}{=} \frac{1}{2m}(-i\hbar\nabla)^2 = \frac{\hbar^2}{2m}\nabla^2 \tag{6}$$

The step $* : \hat{p}^2 = -i\hbar\nabla$ follows from the solution $\psi(x,t) = e^{i(kx-\omega t)}$ to the Schrödinger equation for relativistic particles. From (2), and using the relation $k_s = m\omega^2$ (ω represents angular frequency), then $\hat{V} = \frac{1}{2}m\omega^2x^2$, and we get the Hamiltonian

$$\hat{H} = \hat{K} + \hat{V} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2x^2 \tag{7}$$

Using the *time-independent Schrödinger Equation* $\hat{H}\psi = E\psi$, we derive the equation

$$\frac{\partial^2\psi}{\partial x^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}x^2\right)\psi = 0 \tag{8}$$

Here, after making the substitutions

$$\epsilon = \frac{E}{\hbar\omega},\;\; y = \sqrt{\frac{m\omega}{\hbar}}x \implies \frac{\partial^2\psi}{\partial x^2} = \frac{m\omega}{\hbar}\frac{\partial^2\psi}{\partial y^2}$$

we get the equation

$$\frac{\partial^2\psi}{\partial y^2} + (2\epsilon - y^2)\psi = 0 \tag{9}$$

Recognising similarity to the *Gaussian Differential Equation* $\beta'' + (1-y^2)\beta = 0$ with solution $\beta = e^{-\frac{y^2}{2}}$, we use the *ansatz method* to guess that $\psi = \gamma(y)e^{-\frac{y^2}{2}}$. By substituting this guess into (9), we get the differential equation

$$\frac{\partial^2}{\partial y^2}\gamma(y) - 2y\frac{\partial}{\partial y}\gamma(y) + 2\mu\gamma(y) = 0;\;\; \mu = \epsilon - \frac{1}{2}$$

This is called the *Hermite Differential Equation*, and has polynomial solutions described by the relations

$$H_n(x) = (-1)^n e^{x^2}\frac{\partial^n}{\partial x^n}\left(e^{-x^2}\right) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k} \tag{10}$$

Then, for the n^{th} state, the wavefunction is given by $\psi_n = c_n H_n(y) \exp\left(-\frac{y^2}{2}\right)$. To determine the constant c_n, we can use the fact that $|\psi_n(x)|^2$ is a probability density function.

$$c_n^2 \int_{-\infty}^{\infty} H_n^2\left(\sqrt{\frac{m\omega}{\hbar}}\right) e^{-\frac{m\omega}{\hbar}x^2}\, dx = \int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1$$

By making the substitution $z = (\frac{m\omega}{\hbar})^{\frac{1}{2}}$, and using the fact that $\int_{-\infty}^{\infty} H_n^2(u)e^{-u^2}\, du = 2^n n!\sqrt{\pi}$, we can find c_n to complete the wavefunction:

$$\psi_n(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \tag{11}$$

2 Eigenfunctions and Energy States

The Dirac-von Neumann mathematical formulation of quantum mechanics gives the following three axioms satisfied in a separable Hilbert space $\mathcal{H}$ of countably infinite dimension:

- The observables (quantities that can be measured in a corresponding classical mechanical system) of a quantum mechanical system are represented by the set of self-adjoint operators on $\mathcal{H}$.
- The state ω of a quantum mechanical system is described by $\psi_\omega \equiv \{\lambda\psi_\omega : |\lambda| = 1\}$, where ψ_ω represents a unit vector on $\mathcal{H}^2$.
- If the state ω is described by vector $\psi_\omega \in \mathcal{H}$, then the experimental expectation is given by $\langle A \rangle_\omega = \langle \omega, A\omega \rangle$, where $\langle\ ,\ \rangle$ is the inner product on $\mathcal{H}$.

As $\hat{H}$ is a differential operator, it is unbounded, and therefore we will only define $\hat{H}$ on a dense linear subset of the Hilbert space $\mathcal{H}$. Now, we make the following definitions:

Definition 1. An **operator** $\hat{T}$ on $\mathcal{H}$ is a linear map to $\mathcal{H}$ from a linear $\mathcal{H}$-subspace called the **domain** $D(\hat{T})$.

Definition 2. The **graph** $\Gamma(T)$ of an operator $\hat{T}$ is the set of pairs $\left\{\left(\alpha, \hat{T}\alpha\right) \mid \alpha \in D\left(\hat{T}\right)\right\}$. Clearly, $\Gamma(\hat{T}) \subseteq \mathcal{H} \times \mathcal{H}$, and can be viewed as a Hilbert space when equipped with the inner product defined

$$\left\langle \left(\alpha, \hat{T}\alpha\right), \left(\beta, \hat{T}\beta\right) \right\rangle = \left(\alpha + \beta, \hat{T}(\alpha + \beta)\right)$$

Further, $\hat{T}$ is a **closed operator** if $\Gamma(\hat{T})$ is closed in $\mathcal{H}^2$.

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[2] It is possible (and generally the case) that there are elements of the spectrum of an operator that do not correspond to any eigenvalue of the operator; these points of the spectrum are 'meaningless', since they then do not represent any physical state of the system (in any continuous spectrum, for example, or more specifically, the position of a free particle on the line; $|x\rangle$ in $L^2(\mathbb{R})$). To resolve this 'problem', there are formulations of quantum mechanics building on the concept of *Rigged Hilbert Space*, in which there is a one-to-one correspondence between points of the operator spectrum and eigenvalues of eigenfunctions. We can then think of the rigged Hilbert space as an 'extension' of the Hilbert space (Hilbert space $\oplus$ distribution theory).

Definition 3. If $\hat{T}$ and $\hat{S}$ are operators on $\mathcal{H}$, then $\hat{S}$ is called an **extension** of $\hat{T}$ if $\Gamma(\hat{T}) \subset \Gamma(\hat{S})$. Denote this relationship by $T \subset S$; this holds if and only if $D(\hat{T}) \subset D(\hat{S})$ and $S|_{D(\hat{T})} \equiv \hat{T}$.

Definition 4. Let $\hat{T}$ be a densely defined operator on $\mathcal{H}$. Define $D(\hat{T}^*)$ as the set of all $\gamma \in \mathcal{H}$ for which there exists an $\phi \in \mathcal{H}$ such that $\langle \hat{T}\psi, \gamma \rangle = \langle \psi, \phi \rangle$ for all $\psi \in D(\hat{T})$. Now, define $\phi = \hat{T}^*\gamma$. Then $\hat{T}^*$ is called the **adjoint** of $\hat{T}$. Notice that $\hat{T} \subset \hat{S}$ implies that $\hat{S}^* \subset \hat{T}^*$, and the Riesz Representation Theorem implies that $\left| \left\langle \hat{T}\psi, \gamma \right\rangle \right| \leq C\|\psi\|$ for all $\psi \in D(\hat{T})$.

Definition 5. A densely defined operator $\hat{T}$ on $\mathcal{H}$ is called **Hermitian** if $D(\hat{T}) \subset D(\hat{T}^*)$ and $\hat{T}\gamma = \hat{T}^*\gamma$ for all $\gamma \in D(\hat{T})$.

Definition 6. An operator is called **self-adjoint** if $\hat{T} \equiv \hat{T}^*$. Equivalently, $D(\hat{T}) = D(\hat{T}^*)$ and $\hat{T}$ is Hermitian.

Definition 7. A Hermitian operator $\hat{T}$ is called **essentially self-adjoint** if $\overline{\hat{T}}$ (the closure of $\hat{T}$) is self-adjoint.

A Brief Remark: To avoid the trouble of specifying the domain of unbounded operators, it suffices to specify a *core* of the operator, where the **core** of an operator $\hat{T}$ is the set $D \subset D(\hat{T})$ such that $\overline{\widehat{T|_D}} \equiv \hat{T}$.

According to the Dirac-von Neumann formulation, we must first prove that the Hamiltonian is self-adjoint (or at least essentially self-adjoint). For the proof, we use the following theorem.

Theorem 1. Let $\hat{T}$ be a positive Hermitian unbounded operator on a separable Hilbert space $\mathcal{H}$. If there is an orthonormal basis of $\mathcal{H}$ comprised of eigenfunctions of $\hat{T}$, then $\hat{T}$ is essentially self-adjoint, and $\hat{T}$ is unitarily equivalent to the multiplicity operator

$$\hat{M}x_n = \lambda_n x_n$$

defined on the domain $D = \left\{(x_n) \in \ell^2(\mathbb{N}) \;\middle|\; \sum_{n\in\mathbb{N}} \lambda_n^2 |x_n|^2 < \infty\right\}$, where λ_n is the eigenvalue corresponding to the eigenbasis element $e_n \in D(\hat{T})$.

Proof. Since the multiplication operator is self-adjoint (and we know its spectrum), it suffices to show that $\overline{\hat{T}}$ is unitarily equivalent to $\hat{M}$, with the equivalence given by $\hat{U}x_n = \sum_{n\in\mathbb{N}} x_n e_n$. Now, $D(\hat{T}) \subset D(\hat{T}^*)$; choose a pair $(v, w) \in \Gamma(\hat{T})$. Then, for all $n \in \mathbb{N}$, the hermiticity of $\hat{T}$ allows us to argue that

$$\langle w, e_n \rangle = \langle \hat{T}v, e_n \rangle = \langle v, \hat{T}e_n \rangle = \lambda_n \langle v, e_n \rangle$$

Since $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal eigenbasis, then we have the decomposition

$$w = \sum_{n\in\mathbb{N}} \langle w, e_n \rangle e_n = \sum_{n\in\mathbb{N}} \lambda_n \langle v, e_n \rangle e_n$$

and it follows that

$$\sum_{n\in\mathbb{N}} \lambda_n^2 |\langle v, e_n\rangle|^2 = \|w\|^2 < \infty$$

So $\hat{U}^{-1}D(\hat{T}) \subset D(\hat{M})$, and $\hat{U}\hat{M}\hat{U}^{-1}v = w = \hat{T}v$ when $v \in D(\hat{T})$, so that $\hat{T} \subset \hat{U}\hat{M}\hat{U}^{-1}$. All that remains to be shown that $\Gamma(\hat{T})$ is dense in $\Gamma(\hat{U}\hat{M}\hat{U}^{-1})$. Choose $(v, w) \in \Gamma(\hat{U}\hat{M}\hat{U}^{-1})$. Rewriting v and w in the basis $\{e_n\}_{n\in\mathbb{N}}$, there is

$$v = \sum_{n\in\mathbb{N}} \langle v, e_n\rangle e_n \ , \ \ w = \sum_{n\in\mathbb{N}} \lambda_n \langle v, e_n\rangle e_n$$

Define the following partial sums:

$$v_k = \sum_{1\le n\le k} \langle v, e_n\rangle e_n \ , \ \ w_k = \hat{T}v_n = \sum_{1\le n\le k} \langle v, e_n\rangle e_n \ , \ \ k \in \mathbb{N}$$

These sums are well-defined since $e_n \in D(\hat{T})$, $\hat{T}$ is a linear operator, and the sums are finite. In the product space norm, we have

$$\lim_{n\to\infty} \|(v, w) - (v_k, w_k)\|^2 \le \lim_{n\to\infty} \|v - v_k\|^2 + \lim_{n\to\infty} \|w - w_k\|^2 = 0$$

□

Let $\mathcal{H}$ by $L^2(\mathbb{R})$, and define the inner product on $\mathcal{H}$ as $\langle \alpha, \beta\rangle = \displaystyle\int_{-\infty}^{\infty} \alpha^* \beta \, dx$. Now, it is clear that the Hamiltonian $\hat{H}$ (7) is a linear unbounded operator. Also, by twice integrating by parts, we know that $\hat{H}$ is Hermitian (as is is the sum of the kinetic and potential energy operators, both Hermitian), where $D(\hat{H})$ is the set of all square-integrable, second-differentiable, normalisable functions that vanish at infinity.

Definition 8. A function (or vector) ψ is **normalised** if $\|\psi\| = 1$. A function is **normalisable** if it differs from a normalised function (or vector) by a constant multiple.

Also, the expectation of all states should be nonnegative, so (in line with the Dirac-von Neumann formulation), we specify that $\langle \psi, \hat{H}\psi\rangle \ge 0$.Now, we complete our investigation of the Hamiltonian operator by proving that the wavefunctions give an orthonormal basis for $\mathcal{H}$.

Proposition 1: *The wavefunctions* ψ_n (11) are orthogonal.

Proof. We are required to show that $\langle \psi_i, \psi_k\rangle = 0$ whenever $i \ne k$. Since the wavefunction is given by

$$\psi_n(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

the only part dependent on n is the Hermite polynomial term $H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$. Therefore it is sufficient to demonstrate that the Hermite polynomials are pairwise orthogonal (thereby justifying the statement preceding (11)).

Using the definition given in (10), we use the weight function e^{-x^2} to proceed as:

$$\langle H_i, H_k \rangle = \int_{-\infty}^{\infty} H_i(x) H_k(x) e^{-x^2}\, dx = (-1)^k \int_{-\infty}^{\infty} H_i(x) \frac{\partial^k}{\partial x^k}\left(e^{-x^2}\right) dx \overset{\text{integration}}{\underset{\text{by parts}}{=\cdots=}} 0 \quad \forall i \neq k$$

$$\langle H_k, H_k \rangle = \int_{-\infty}^{\infty} H_k(x) H_k(x) e^{-x^2}\, dx = (-1)^k \int_{-\infty}^{\infty} H_k(x) \frac{\partial^k}{\partial x^k}\left(e^{-x^2}\right) dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} \frac{\partial^k}{\partial x^k} H_k(x)\, dx = \cdots = 2^k k! \int_{-\infty}^{\infty} e^{-x^2}\, dx = 2^k k! \sqrt{\pi}$$

So $\langle H_i(x), H_k(x) \rangle = 2^k k! \sqrt{\pi} \delta_{i,k}$, and so the wavefunctions are also pairwise orthogonal. □

Proposition 2: *The wavefunctions are normalised.*

Proof.

$$\|\psi_n\|^2 = \int_{-\infty}^{\infty} \psi^* \psi\, dx = \sqrt{\frac{m\omega}{\pi\hbar}} \frac{1}{2^n n!} \int_{-\infty}^{\infty} \left(e^{-\frac{m^2\omega^2}{2\hbar}x^2}\right)^2 H_n^2\left(\sqrt{\frac{m\omega}{\hbar}}x\right) dx$$

Making the substitution $k = \sqrt{\frac{m\omega}{\hbar}}$, then we have $\int_{-\infty}^{\infty} H_n^2\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\left(e^{-\frac{m^2\omega^2}{2\hbar}x^2}\right)^2 dx$

$$= \int_{-\infty}^{\infty} H_n^2(kx)\, e^{-k^2x^2} dx$$

$$I_n = \int_{-\infty}^{\infty} e^{k^2x^2} \left(\frac{\partial^n}{\partial x^n}\left(e^{-k^2x^2}\right)\right)^2 dx$$

Integrating by parts with $u = e^{k^2x^2} \frac{\partial^n}{\partial x^n}\left(e^{-k^2x^2}\right)$ and $dv = \frac{\partial^n}{\partial x^n}\left(e^{-k^2x^2}\right)$

$$= -\int_{-\infty}^{\infty} \left(e^{k^2x^2} \frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{-k^2x^2}\right) + 2^2 e^{k^2x^2} \frac{\partial^n}{\partial x^n}\left(e^{k^2x^2}\right)\right) \frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{k^2x^2}\right) dx$$

$$= -\int_{-\infty}^{\infty} e^{k^2x^2} \left(\frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{-k^2x^2}\right)\right)^2 dx + 2k^2 \int_{-\infty}^{\infty} x e^{-k^2x^2} \frac{\partial^n}{\partial x^n}\left(e^{-k^2x^2}\right) \frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{-k^2x^2}\right) dx$$

Integrating by parts again with $dv = \left(1 + 2k^2x^2e^{2x^2}\right) \frac{\partial^n}{\partial x^n}\left(e^{-k^2x^2}\right) + x e^{k^2x^2} \frac{\partial^{n-1}}{\partial x^{n-1}}\left(e^{-k^2x^2}\right)$ and $u = e^{k^2x^2} \frac{\partial^n}{\partial x^n}\left(e^{-k^2x^2}\right)$

$$= \cdots$$
$$= 2nI_{n-1}$$

Solving the recurrence relation $I_n = 2nI_{n-1}$, we get that $I_0 = \int_{\infty}^{\infty} e^{k^2x^2}\, dx = \sqrt{\frac{\pi}{k}}$, which implies that $I_n = 2^n n! \sqrt{\frac{\pi}{k}}$. Finally,

$$\|\psi_n\|^2 = \frac{1}{2^n n!} \sqrt{\frac{m\omega}{\pi\hbar}}\, I_n = \frac{1}{2^n n!} \sqrt{\frac{m\omega}{\pi\hbar}}\, 2^n n! \sqrt{\frac{\pi}{\frac{m\omega}{\hbar}}} = 1$$

$\square$

Proposition 3: $\{\psi_n\}_{n\in\mathbb{N}}$ *is a basis for* $\mathcal{H}$.

Proof. Consider the space $A = \text{Span}\{\psi_n : n \in \mathbb{N}\}$. Notice that each ψ_* is of the form $P(x)e^{-kx^2}$, with $P(x) \in \mathbb{Z}[x]$. Also, we have the (multiplicative) time evolution operator $\hat{U}(t)$ with the following properties:

- $\hat{U} \equiv \text{id}$ when $t = t_0$, where t_0 is the initial time
- $\hat{U}$ preserves the normalisation of states. This implies that $\hat{U}$ is a unitary operator.
- $\hat{U}$ satisfies the composition property $\hat{U}(\beta)\hat{U}(\alpha) = \hat{U}(\alpha+\beta)$

Since the Schrödinger equation must be satisfied, then $i\hbar\frac{\partial}{\partial t}\hat{U} = \hat{H}\hat{U}$. Therefore, for the time-independent Hamiltonian, the operator takes the form $\hat{U} = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)$. $\overline{A}$ is a basis for $\mathcal{H}$ if and only if the state of the system is expressible as a linear combination of basic states for all times $t \in \mathbb{R}$. Equivalently, it suffices to show the following:

1. *Every state falls in the closure of the span of the wavefunctions.*

   Since the Hermite polynomials obey the relation $H_{n+1}(x) = 2xH_n(x) - \frac{\partial}{\partial x}H_n(x)$, every state of the system is given some $p(x)\hat{U}(t)\psi_0$, where $p(x)$ is a polynomial and $t \in \mathbb{R}$. Also, by considering the Taylor expansion of the exponential function, it is clear that there always exists a polynomial $p(x)$ so that any sequence of states $\{\psi_n\}_n$ converges in $\overline{A}$. Then, the proposition holds.

2. *The orthogonal complement of* $\overline{A}$ *is the zero space.*

   Consider a state $\psi(x) \in \overline{A}$. Then for every $f \in \overline{A}$,$\langle f, \psi\rangle = 0$. Then, $\displaystyle\int_{-\infty}^{\infty} \psi e^{-ikx}\,dx = 0$; this represents the Fourier transform of f. Since the Fourier transform is a unitary operator, then $\|f\| = 0$, and so $f = 0$. Therefore, $\overline{A}^{\perp} = \{0\}$.

This is sufficient to show that $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$. $\square$

By the theorem, we have that $\hat{H}$ is essentially self-adjoint, and can therefore be extended to a self-adjoint operator $\overline{\hat{H}}$. Hence, $\hat{H}$ is an observable.

We can factorise the Hamiltonian (7) as

$$\hat{H} = \frac{1}{2m}\left(-\hbar\frac{\partial}{\partial x} + m\omega x\right)\left(\hbar\frac{\partial}{\partial x} + m\omega x\right) - \frac{\hbar\omega}{2}$$

Similarly, we can reverse the order of the factorisation to get

$$\hat{H} = \frac{1}{2m}\left(\hbar\frac{\partial}{\partial x} + m\omega x\right)\left(-\hbar\frac{\partial}{\partial x} + m\omega x\right) + \frac{\hbar\omega}{2}$$

Then, the commutator of these is $\hbar\omega$. Normalising (to get commutator 1), then we can define the following operators in the factorisation of $\hat{H}$:

$$\hat{A}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\hat{p} \ , \ \hat{A}^{-} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p} \tag{12}$$

Then $[\hat{A}^{+}, \hat{A}^{-}] = 1$, so that

$$\hat{H} = \hbar\omega\left(A^{\pm}A^{\mp} \pm \frac{1}{2}\right) \tag{13}$$

Recall the Schrödinger equation $\hat{H}\psi = E\psi$. Suppose that ψ is a solution of this equation for some energy E, and let $\hat{A}^{-}$ act on ψ. Then

$$\begin{aligned} \hat{H}\hat{A}^{-}\psi &= \left[\hbar\omega\left(\hat{A}^{-}\hat{A}^{+} - \frac{1}{2}\right)\right]\hat{A}^{-}\psi \ = \ \hat{A}^{-}\left[\hbar\omega\left(\hat{A}^{+}\hat{A}^{-} - \frac{1}{2}\right)\right]\psi \\ &= \hat{A}^{-}\left[\hbar\omega\left(\hat{A}^{+}\hat{A}^{-} + \frac{1}{2}\right) - \hbar\omega\right]\psi \ = \ \hat{A}^{-}\left(\hat{H} - \hbar\omega\right)\psi \\ &= (E - \hbar\omega)\,\hat{A}^{-}\psi \end{aligned} \tag{14}$$

Similarly, $\hat{H}\hat{A}^{+}\psi = (E + \hbar\omega)\hat{A}^{+}\psi$. So, we realise two things:

- The action $\hat{A}^{-}$ reduces the energy of the state by $\hbar$, and $\hat{A}^{+}$ increases the energy, so the operators $\hat{A}^{-}$ and $\hat{A}^{+}$ are known respectively as *lowering* and *raising* operators, and collectively as *ladder operators.*

- $\hat{A}^{\pm}\psi$ is an eigenfunction of the Schrödinger equation, with corresponding eigenvalue $E \pm \hbar\omega$

By repeatedly applying the lowering operator, then it is clear that $E \pm n\hbar\omega$ is the eigenvalue corresponding to $(\hat{A}^{\pm})^{n}$. Then, we will eventually arrive at a state with negative energy, which is unobservable (and meaningless). So, we want the ground state ψ_0 of the system satisfy $\hat{A}^{-}\psi_0 = 0$. Then

$$E_0\psi_0 = \hat{H}\psi_0 = \hbar\omega\left(\hat{A}^{+}\hat{A}^{-} - \frac{1}{2}\right)\psi_0 = \frac{1}{2}\hbar\omega\psi_0$$

Finally, the energy of the the n^{th} state is given by $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$; these are the eigenvalues of the system, corresponding to the the wavefunction (11).

Similarly, notice that every state of the system can derived from repeated application of the raising operator to ψ_0, so we have that $|n\rangle = \dfrac{\hat{A}^{n}}{\sqrt{n!}} = |0\rangle$. Then the ladder operators have the following representations as square matrices in the $|n\rangle$ basis.

$$\hat{A}^{+} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \hat{A}^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

3 Coherent States

A coherent state is a right eigenstate of the annihilation (lowering) operator $\hat{A}^-$. These states also obey the equations of the Classical Harmonic Oscillator, and minimise the uncertainty between the position and momentum operators $\hat{x}$ and $\hat{p}$.

4 The Mehler Kernel

5 The Weyl Law