Riemann Surfaces
Assignment 1

1. (Orientation) Let $B(V)$ denote the set of all *ordered basis* of a finite dimensional real vector space V. Two ordered basis are called *equivalent* if the map that sends one to the other has positive determinant.

   a) Show that this is an equivalence relation on $B(V)$ and that there are only *two* equivalence classes. Each equivalence class is called an *orientation* for V. Let $f : V_1 \to V_2$ be an invertible linear map between oriented vector spaces. Define what it means to say f is *orientation preserving.*

   b) Let W be a finite dimensional complex vector spaces and $V = W_{\mathbb{R}}$ its underlying real vector space. Show that V has a canonical orientation. If $f : W_1 \to W_2$ is a $\mathbb{C}$- linear invertible map, show that the induced map between real vector spaces is orientation preserving. How all this is related to the fact that a Riemann surface has a canonical orientation and an (invertible) holomorphic map is orientation preserving?

2. Let X be a compact connected Riemann surface and $F : X \to \mathbb{C}$ be a holomorphic function. Show that F is constant. (Hint: use maximum principle from complex analysis.)

3. A *meromorphic function* on a Riemann surface X is, by definition, a holomorphic map $F : X \to \mathbb{C}P^1$ which is not identically equal to ∞.

   a) Define *zeros* and *poles* and their *orders* for a meromorphic function. (You have to show that your definitions are independent of the choice of holomorphic coordinates). Why the *residue* of a meromorphic function at a pole is not well-defined?

   b) Let $\frac{f(z)}{g(z)}$ be a rational function. Show that it defines a meromorphic function on $\mathbb{C}P^1$. Find its zeros and poles and their orders in terms of linear factorizations of f and g. What is the *degree* of this map ? Let n_i

denote the order of zeros and p_j the orders of poles of a meromorphic function on $\mathbb{C}P^1$. Show that

$$\sum n_i = \sum p_j.$$

c) Let $K(X)$ denote the set of meromorphic functions on X. Show that $K(X)$ is naturally a *field*.

d) Show that any meromorphic function on $\mathbb{C}P^1$ is a rational function and hence $K(\mathbb{C}P^1) = \mathbb{C}(z)$ is the field of rational functions in one variable.

e) Show that the poles and zeros of a meromorphic function on $\mathbb{C}P^1$ can be placed anywhere you wish, provided they are the same in muber.

f) Let $p_1, p_2, \dots p_n$ be a collection of points on $\mathbb{C}P^1$, repetitions permitted, and let L be the space of meromorphic functions with poles of orders at most d_i at p_i. Show that L is a complex vector space of dimension $\sum d_i + 1$.

4. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_{2g+2}$ be real numbers. a) Sketch a graph of real points of the curve

$$y^2 = \prod_{i=1}^{2g+2} (x - \alpha_i)$$

b) By using homogenization, show that by adding two points, one obtains a compact Riemann surface.

5. Prove Euler's formula for homogeneous functions we used in class.

6. Show that the set of points (x, y) is $\mathbb{C}^2$ where $y^2 = sin(x)$ is naturally a Riemann surface.

7. a) Use Liouville's theorem to show that $Aut(\mathbb{C})$ consists of maps $z \to az + b$ for $a \neq 0$.

   b) Show that the automorphisms of the Riemann sphere are given by Möbius maps $PSL(2, \mathbb{C})$.

   c) Use the Schwartz Lemma to identify the stabilizer of 0 is $Aut(D)$ and hence identify $Aut(D)$ and $Aut(H)$.

8. Let $X(R_1, R_2)$ denote the open annular region between concentric circles of radius R_1 and R_2 in the plane. Show that $X(R_1, R_2)$ is (biholomorphically) equivalent to $X(R_1', R_2')$ iff

$$\frac{R_1}{R_2} = \frac{R_1'}{R_2'}.$$

   Conclude that there are uncountabaly many inequivalent Riemann surfaces with the topology of an annular region (= cyclinder).