Riemann Surfaces
Assignment 2

1. Show that the group $PSL_2(\mathbb{Z})$ is generated by transformations $T : z \mapsto z + 1$ and $S : z \mapsto -\frac{1}{z}$. Show that they satisfy

$$S^2 = I, \qquad (ST)^3 = I.$$

Do they generate $SL_2(\mathbb{Z})$? (It can be shown that $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$, is the free product of groups of order 2 and 3.)

2. Find the degree of the map $F : \mathbb{C}P^1 \to \mathbb{C}P^1$, defined by

$$F(z) = \frac{f(z)}{g(z)},$$

where f and g are polynomials of degrees m and n, respectively. Show that any meromorphic function on $\mathbb{C}P^1$ is of the above form.

3. Consider the map $F : \mathbb{C}P^1 \to \mathbb{C}P^1$, defined by

$$F(z) = \frac{(z-1)(z-2)}{z^3}.$$

Find all $z \in \mathbb{C}P^1$ over which F is ramified and find the ramification indices.

4. State and prove a *product formula* for degree of maps between Riemann surfaces.

5. Show that on $\mathbb{C}P^1$ there is *no* holomorphic 1-form except $\omega = 0$.

6. Give an example of a surjective locally homemorphism which is *not* a covering map.

7. Show that $p_* : \pi_1(E, y_0) \to \pi_1(X, x_0)$ is injective where $p : E \to X$ is a covering map.

8. Show that $\int_M \omega$ is independent of the choice of cover and partition of unity that we used to define it.

9. Show that on $\mathbb{C}P^1$ there is *no* holomorphic 1-form except $\omega = 0$.

10. Show that dz is a meromorphic 1-form on $\mathbb{C}P^1$ and it has a pole of order 2 at ∞.

11. Let f be a meromorphic function on $\mathbb{C}P^1$. Shwo that the residues of $\omega = f'(z)dz$ are all zero. What are the residue of $z^{-1}dz$?

12. Show that the *total degree*:= zeros - poles, multiplicities counted, of a meromorphic 1-form on $\mathbb{C}P^1$ is -2.

13. Give an example of a non-zero holomorphic 1-form on $\mathbb{C}/\Lambda$. Show that the space of holomorphic 1-forms on $\mathbb{C}/\Lambda$ is 1-dimensional.

14. Use the Residue Theorem to show that there is no mromorphic function on $\mathbb{C}/\Lambda$ having a single pole of order 1.

15. Let ω be a meromorphic 1-form on a Riemann surface X whose residues are all zero. Show that there is a covering $p : Y \to X$ and a meromorphic function F on Y such that $dF = p^*\omega$.