Spectral Graph Theory (Summer 2015)
Assignment 1

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1. Define the **Cartesian product** of two graphs. How the eigenvalues of the Markov matrix of the product graph is related to the eigenvalues of each graph? Compute the eigenvalues and multiplicities of the Laplacian $\mathcal{L} = -\Delta$ of the **discrete torus 2-torus** $\mathbb{Z}_m \times \mathbb{Z}_n$. Compute the trace and determinant of $\mathcal{L}$.

2. Compute the eigenvalues and multiplicities of the Laplacian $\mathcal{L} = -\Delta$ for the **complete bipartite graph** $K_{m,n}$.

3. The **Cheeger constant** $h(G)$ of a graph G on n vertices is defined as

$$h(G) = \min_{0<|S|\leq \frac{n}{2}} \frac{|\partial(S)|}{|S|},$$

where the minimum is over all nonempty sets S of at most $n/2$ vertices and $\partial(S)$ is the edge boundary of S, i.e., the set of edges with exactly one endpoint in S. The Cheeger constant is a measure of **"bottle-neckedness"** of a graph. We shall soon prove the Cheeger inequlities

$$h(G)^2/2 \leq \lambda_1 \leq 2h(G)$$

for the first non-zero eigenvalue λ_1. Show that for the hypercube $G = \{0,1\}^d$ we have an equality $\lambda_1 = 2h(G)$. (hint: use our computation of the spectrum of G), and its first eigenvalue

4. Show that for the cycle graph C_n we have $h(C_n) \geq 2/n$ and $\lambda_1 = O(1/n^2)$. What does this say about Cheeger inequalities?

5. Consider the stochastic matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \end{pmatrix}$$

Show that P is ergodic and find its equilibrium state. Find $\lim P^n$ as $n \to \infty$.

6. Consider the Markov matrix

$$P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$$

When is P **ergodic**? When is it **irreducible**? Is it ever non-irreducible? Compute its **equilibrium state(s)**. In the ergodic case, show directly, without appealing to the ergodic theorem, that the limit $\lim_{n\to\infty} P^n$ exists.

7. Show that the **Ehrenfest urn model** is irreducible but not ergodic. Compute its equilibrium state.

8. A stochastic matrix is called **doubly stochastic** if its column sums are all equal to 1: $\sum_i p_{i,j} = 1$ for all j. Find an equilibrium state for a doubly stochastic matrix.

9. Let G be a finite group (need not be abelian) and let $\rho : G \to [0,\, 1]$ be a probability density function on G. Show that

$$p(x, y) = \rho(xy^{-1})$$

defines a doubly stochastic matrix with state space G.