Smooth Manifolds
Problem set 1

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1. Show that the stereographic projections from the north and south pole defines a smooth atlas for the n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1}; ||x|| = 1\}.$$

2. i) Show that the atlas $\{(U_i, \varphi_i)\}, i = 1, 2, \cdots, n+1$ defined in class is a smooth atlas for the n-dimensional projective space $\mathbb{RP}^n$.
ii) An atlas is called *oriented* if for all i, j the change of coordinates $\varphi_i \circ \varphi_j^{-1}$ have positive Jacobian. Show that if n is odd the above atlas is oriented, and if n is even it is not oriented.

3. i) Consider the vector fields in the plane

$$X = (xy)\frac{\partial}{\partial x} + (x^2 + y)\frac{\partial}{\partial y}, \quad Y = (x + y)\frac{\partial}{\partial x} + (xy)\frac{\partial}{\partial y}.$$

Compute their bracket $[X, Y]$.
ii) Prove that if $(U, x_1, \cdots, x_n)$ is a local coordinate system on M, then

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

4. i) Let θ be an *irrational* real number. Define a map $\varphi : \mathbb{R} \to S^1 \times S^1$ by

$$\varphi(t) = (e^{2\pi it}, e^{2\pi i\theta t}).$$

Show that $(\mathbb{R}, \varphi)$ is a *dense* submanifold of $S^1 \times S^1$.
ii) Generalize the above result. Let $\theta_1, \cdots, \theta_n$ be real numbers which are *linearly independent* over $\mathbb{Q}$. Define a map $\varphi : \mathbb{R} \to S^1 \times \cdots S^1$ by

$$\varphi(t) = (e^{2\pi i\theta_1 t}, \cdots, e^{2\pi i\theta_n t}).$$

Show that $(\mathbb{R}, \varphi)$ is a *dense* submanifold of the n-torus $S^1 \times \cdots \times S^1$.

5. Generalize the formula we had in class

$$\varphi(x) - \varphi(0) = \sum_{i=1}^{n} a_i x_i + \sum_{i,j=1}^{n} x_i x_j h_{ij}(x),$$

by showing that

$$\varphi(x) - \varphi(0) = \sum_{i=1}^{n} a_i x_i + \sum_{i,j=1}^{n} b_{ij} x_i x_j + \sum_{i,j,k=1}^{n} x_i x_j x_k q_{ijk}(x).$$

Generalize further by going to higher order terms.