Smooth Manifolds
Topics discussed per day

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This document is meant to record the topics discussed in my course, Smooth Manifolds, per each class. It can also be useful to draw your attention to important points discussed in each class.

• Monday, October 5: The differential $f_* : T_mM \to T_{f(m)}N$ of a smooth map $f : M \to N$, its matrix in a coordinate basis (the Jacobian matrix); the pullback $f^* : T^*_{f(m)}N \to T^*_mM$; tangnet bundle TM as a manifold: its topology and smooth atlas; cotangent bundle T^*M; Here are a few questions to think about: show that the tangent space of S^n at a point m is canonically isomorphic to the hyperplane orthogonal to the vector m; use this to show that

$$TS^n \simeq \{(m, v);\ m \in S^n,\, v \in \mathbb{R}^{n+1},\, \langle m, v\rangle = 0\} \subset S^n \times \mathbb{R}^{n+1}$$

the tangent bundle of $TS^1 \simeq S^1 \times \mathbb{R}$, why? *parallelizable manifolds* are defined as those manifolds M for which $TM \simeq M \times \mathbb{R}^n$. S^2 is not parallelizable (think about the 'hairy ball' theorem!), but S^3 is; Lie groups are parallelizable but there are parallelizable manifolds that are not Lie groups (example?) immersion, submanifold, imbeddding, diffeomorphisms, examples are left to the next class.

• Thursday, October 1: Basis $\frac{\partial}{\partial x_i}$ for T_mM, change of basis formula, smooth vector fields; implicitly used a lemma: there is a 1-1 correspondence between smooth vector fields on M and derivations of $C^\infty(M)$; Lie bracket $[X, Y]$ of vector fields, Lie algebra structure; the cotangent space T^*_mM and its basis

$dx^i, i = 1, \cdots, n$; the differential of a smooth map $f : M \to N$.

• Tuesday, September 29: Proved a nice formula in $\mathbb{R}^n$:

$$f(x) - f(0) = \sum_{i=1}^{n} a_i x_i + \sum_{i,j=1}^{n} x_i x_j h_{ij}(x).$$

(proof: start with a one variable formula $g(1) - g(0) = \int_0^1 g'(x)dx$ and continue by integration by parts-there is a trick involved though!) It shows right away that $F_0/F_0^2 \simeq \mathbb{R}^n$ and is in particular n-dimensional; observed that the algebra $\tilde{F}_m(M)$ and its filtration is independent of the manifold M and just depends on n: any local chart gives an algebra isomorphism with $\tilde{F}_0(\mathbb{R}^n)$; so to prove something about $\tilde{F}_m$, suffices to prove it for $\tilde{F}_0(\mathbb{R}^n)$.

• Monday, September 28: the germ of a function around a point $m \in M$, the algebra of germs of functions $\tilde{F}_m$ and its canonical adic filtration

$$\tilde{F}_m \supset F_m \supset F_m^2 \supset F_m^3 \supset \cdots$$

tangent vector at a point $m \in M$ as a derivation on $\tilde{F}_m$ and the tangent space $T_m M$; the canonical isomorphism

$$T_m M \simeq (F_m/F_m^2)^*$$

and the formula $\dim_{\mathbb{R}} T_m M = \dim M$.

• Thursday September 24: cartesian product of manifolds, example: tori; an atlas for the real projective n-space $\mathbb{RP}^n$, comment on orientability of $\mathbb{RP}^n$ for n odd.

• Monday, September 14: First lecture: definition of a smooth manifold (coordinate chart, atlas, maximal atlas); every atlas is contained in a unique maximal atlas; examples: open subsets of $\mathbb{R}^n$, open subsets of smooth manifolds, an atlas for the n-sphere.