Sobolev Spaces and Garding Inequality

Yue Xu

April 27, 2015

A. Some definitions and notation

1. An open connected set $\Omega \subset \mathbf{R}^n$ is called a *domain*. We say that a domain $\Omega^{'} \subset \Omega \subset \mathbf{R}^n$ is a *strictly interior subdomain of* Ω and write $\Omega^{'} \subset\subset \Omega$, if $\overline{\Omega'} \subset \Omega$.

2. x=$(x_1, x_2, \dots, x_n)$$\in$$\mathbf{R}^n$, u=u(x),
α=$(\alpha_1, \alpha_2, \dots, \alpha_n)$$\in$$\mathbf{Z}^n_+$ as a multi-index, $|\alpha|$=$\alpha_1 + \alpha_2 + \cdots + \alpha_n$, then

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

3. $\mathbf{L}^p(\Omega)$, 1≤p≤∞, is the set of all measurable functions u(x) in Ω such that the norm

$$\| u \|_{L^p(\Omega)} = \left(\int_\Omega | u |^p \, dx\right)^{\frac{1}{p}}$$

is finite.

$\mathbf{L}^p_{loc}(\Omega)$, 1≤p≤∞, is the set of all measurable functions u(x) in Ω such that $\int_{\Omega'} | u |^p$ dx<∞ for any bounded strictly interior subdomain $\Omega^{'} \subset\subset \Omega$.

4. $C^k(\Omega)$ is the class of functions in Ω such that u(x) and ∂^αu, $| \alpha | \le$k, are continous in Ω.

$C^\infty_c(\Omega)$ is the class of functions u(x) in Ω such that
a) u(x) is infinitely smooth, which means that ∂^αu is uniformly continous in Ω, $\forall\alpha$.
b) u(x) is compactly supported, supp u is a compact subset of Ω.

B. Weak derivatives

1. **Definition**: Let α be a multi-index. Suppose that u,v$\in \mathbf{L}^1_{loc}(\Omega)$ and

$$\int_\Omega u(x)\,\partial^\alpha \eta(x)\,dx = (-1)^{|\alpha|}\int_\Omega v(x)\,\eta(x)\,dx,\ \forall \eta \in C_c^\infty(\Omega).$$

Then v is called the *weak partial derivative* of u in Ω, and is denoted by D^αu.

2. **Definition**: Suppose that u,v $\in \mathbf{L}^1_{loc}(\Omega)$ and there exists a sequence $\{u_m\} \in C^k(\Omega)$, m$\in \mathbf{N}$, such that

$$\begin{gathered} u_m \to u,\ m \to \infty \\ \partial^\alpha u_m \to v,\ m \to \infty \end{gathered}$$

in $\mathbf{L}^1_{loc}(\Omega)$, here α is a multi-index and $|\alpha|$=k. Then v is called the *weak partial derivative* of u in Ω: D^αu=v.

3. **Example**: Let n=1, Ω=(0,2) and

$$u(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \le x < 2 \end{cases}$$

Define

$$v(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 \le x < 2 \end{cases}$$

Then u'=v in the weak sense. To see this, choose any $\eta \in C_c^\infty(\Omega)$.
We must demonstrate

$$\int_0^2 u\eta' dx = -\int_0^2 v\eta dx$$

But we easily calculate
$\int_0^2 u\eta' dx = \int_0^1 x\eta' dx + \int_1^2 \eta' dx = \eta(1) - \int_0^1 \eta dx + \eta(2) - \eta(1) = -\int_0^1 \eta dx = -\int_0^2 v\eta dx.$

4. **Theorem**: Let $u_m \in \mathbf{L}^1_{loc}(\Omega)$, and $u_m \to u$ in $\mathbf{L}^1_{loc}(\Omega)$ as $m \to \infty$. Suppose that there exists weak derivatives $D^\alpha u_m \in \mathbf{L}^1_{loc}(\Omega)$ and $D^\alpha u_m \to$ v in $\mathbf{L}^1_{loc}(\Omega)$ as $m \to \infty$. Then v=D^αu, i.e. D^αu is closed.

proof: By definition 1, for $D^\alpha u_m$, we have $\int_\Omega u_m \partial^\alpha \eta\, dx = (-1)^{|\alpha|}\int_\Omega D^\alpha u_m \eta dx$, $\forall \eta \in C_c^\infty(\Omega)$. Let $m \to \infty$, then $\int_\Omega u\partial^\alpha \eta\, dx = (-1)^{|\alpha|}\int_\Omega v\eta dx$, $\forall \eta \in C_c^\infty(\Omega)$. Hence v=$D^\alpha$u.

C. Sobolev Spaces

Fix 1$\leq p \leq \infty$, and let k be a nonnegative integer. We define now certain function spaces, whose members have weak derivatives of various orders lying in various $\mathbf{L}^p$ spaces.

1. **Definition**: The Sobolev space W$^{k,p}(\Omega)$=$\{u \in \mathbf{L}^1_{loc}(\Omega) : D^\alpha u \in \mathbf{L}^p(\Omega)\}$.
If u $\in W^{k,p}$ (Ω), we define its norm to be

$$||u||_{W^{k,p}(\Omega)} := \begin{cases} (\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p dx)^{1/p} & (1 \leq p < \infty) \\ \sum_{|\alpha|\leq k} \text{ess sup}_\Omega |D^\alpha u| & (p = \infty). \end{cases}$$

Remark: W$^{0,p}(\Omega)$=$\mathbf{L}^p(\Omega)$.

2. **Proposition**: W$^{k,p}(\Omega)$ is complete, in other words, W$^{k,p}(\Omega)$ is a Banach space.

proof: Let $\{u_m\}$ be a Cauchy sequence in W$^{k,p}(\Omega)$. It is quivalent to the fact that all sequences $\{D^\alpha u_m\}$ for $|\alpha| \leq$ k are Cauchy sequences in $\mathbf{L}^p(\Omega)$.

Since the space $\mathbf{L}^p(\Omega)$ is complete, there exist u, v$_\alpha$ $\in \mathbf{L}^p(\Omega)$ such that in $\mathbf{L}^p(\Omega)$

$$u_m \to u, D^\alpha u_m \to v_\alpha, \text{ as m} \to \infty$$

By Theorem B.4, $v_\alpha = D^\alpha u$, and therefore D$^\alpha u_m \to D^\alpha u$ in $\mathbf{L}^P(\Omega)$. Hence in W$^{k,p}(\Omega)$,

$$u_m \to u, \text{ as m} \to \infty$$

3. If p=2, the space W$^{k,2}(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{W^{k,2}(\Omega)} = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha u(x) \overline{D^\alpha v(x)} \text{ dx}$$

We denote W$^{k,2}(\Omega)$=H$^k(\Omega)$.

4. **Definition**: We denote by W$^{k,p}_0(\Omega)$ the closure of C$^\infty_c(\Omega)$ in W$^{k,p}(\Omega)$.

Thus u$\in W^{k,p}_0(\Omega)$ if and only if there exist functions u$_m \in C^\infty_c(\Omega)$ such that u$_m \to u$ in W$^{k,p}(\Omega)$.

Similarly, we denote H$^k_0(\Omega) = W^{k,2}_0(\Omega)$.

D. Second-ordered Elliptic Equations

1. **Definitions**

1.1 **Elliptic equations.**

Consider the boundary-value problem

$$\begin{cases} Lu = f & \text{in}\Omega \\ u = 0 & \text{on}\partial\Omega \end{cases} \tag{1}$$

where Ω is an open, bounded subset of $\mathbf{R}^n$ and u: $\overline{\Omega} \to \mathbf{R}$ is unknown, u=u(x). Here f: $\Omega \to \mathbf{R}$ is given, and L denotes a second-order partial differential operator having either the form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{2}$$

or else

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{3}$$

for given coefficient functions a^{ij}, b^i, c (i, j=1,2,...,n).

We say the partial differential operator is *(uniformly) elliptic* if there exists a constant $\theta > 0$ such that

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{4}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbf{R}^n$.

1.2 **Weak solutions**.

Let us consider first the boundary-value problem (1) when L has the divergence form (2).

The bilinear form B[,] *associated with the divergence form elliptic operator L defined by (2)* is

$$B[u,v] := \int_\Omega \sum_{i,j=1}^{n} a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i}v + cuvdx \tag{5}$$

for $u, v \in H_0^1(\Omega)$.

We say that $u \in H_0^1(\Omega)$ is a *weak solution* of the boundary-value problem (1) if

$$B[u,v] := (f,v) \tag{6}$$

for all $v \in H_0^1(\Omega)$, where (,) denotes the inner product in $\mathbf{L}^2(\Omega)$.

2. **Existance of weak solutions**

We assume for this section H is a real Hilbert space, with norm $\|\|$ and inner product (,). We let $\langle , \rangle$ denote the pairing of H with its dual space.

Lax-Milgram Theorem: Assume that B: H$\times H \to \mathbf{R}$ is a bilinear mapping, for which there exists constants α, β >0, such that

(i) $\beta||u||_H^2 \leq B[u,u]$, for all $u \in$ H

(ii) $|B[u,v]| \leq \alpha||u||_H||v||_H$, for all $u, v \in$ H.

Finally, let f: H$\to$ $\mathbf{R}$ be a bounded linear functional on H. Then there exists a unique element $u \in$ H, such that

$$B[u,v] = \langle f, v\rangle \tag{7}$$

for all $v \in$ H.

proof: 1. For each fixed u$\in$ H, the mapping v$\mapsto$ B[u,v] is a bounded linear functional on H. By the Riesz representation theorem, there exists a unique element $w \in$ H satisfying

$$B[u,v] = (w,v) \tag{8}$$

for all $v \in$ H.

Denote the operator mapping u to w by A, i.e., $w = Au$, and $B[u,v] = (Au,v)$, for all $v \in$H.

2. Using the hypothesis of the theorem, one can show that the operator A is linear, bounded, one to one, and that the range of A, R(A), is closed in H.

3. We demonstrate now

$$R(A) = H \tag{9}$$

For if not, then, since R(A) is closed, there would exist a nonzero element $x \in$H with $x \in R(A)^{\perp}$. But this fact in turn implies the contradiction $\beta||x||^2 \leq B[x,x] = (Ax,x) = 0$.

4. Next, we observe once more from the Riesz representation theorem for f, we have

$$\langle f, v\rangle = (w,v) \text{ for all } v \in \text{H}$$

for some element $w \in$H. We then utilize (8) and (9) to find $u \in$H satisfying A$u = w$. Then

$$B[u,v] = (Au,v) = (w,v) = \langle f, v\rangle$$

and this is (7).

5. Finally, we show there is at most one element $u \in$H verifying (7). For if both $B[u,v] = \langle f,v\rangle$ and $B[u^{'},v] = \langle f,v\rangle$, then $B[u-u^{'},v] = 0$ for all $v \in$H. We set $v = u - u^{'}$ to find $\beta||u-u^{'}||^2 \leq B[u-u^{'}, u-u^{'}] = 0$.

We return now to the specific bilinear form B[,], defined in 1.2(5) by the formula

$$B[u,v] = \int_\Omega \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuvdx$$

for $u, v \in H_0^1(\Omega)$, and try to verify the hypothesis of the Lax-Milgram Theorem.

Theorem: There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$$|B[u,v]| \leq \alpha ||u||_{H_0^1(\Omega)} ||v||_{H_0^1(\Omega)} \tag{10}$$

and

$$\beta ||u||^2_{H_0^1(\Omega)} \leq B[u,u] + \gamma ||u||^2_{L^2(\Omega)} \tag{11}$$

for all $u, v \in H_0^1(\Omega)$.

Remark: (11) is called ***Garding Inequality***.

proof: 1. We readily check

$$\begin{aligned} |B[u,v]| &\leq \sum_{i,j=1}^n ||a^{ij}||_{L^\infty} \int_\Omega |Du||Dv|dx \\ &+ \sum_{i=1}^n ||b^i||_{L^\infty} \int_\Omega |Du||v|dx + ||c||_{L^\infty} \int_\Omega |u||v|dx \\ &\leq \alpha ||u||_{H_0^1(\Omega)} ||v||_{H_0^1(\Omega)}, \end{aligned}$$

for some appropriate constant α.

2. In view of the ellipticity condition (4) we have

$$\begin{aligned} \theta \int_\Omega |Du|^2 dx &\leq \int_\Omega \sum_{i,j=1}^n a^{ij} u_{x_i x_j} dx \\ &= B[u,u] - \int_\Omega \sum_{i=1}^n b^i(x) u_{x_i} u + cu^2 dx \\ &\leq B[u,u] + \sum_{i=1}^n ||b^i||_{L^\infty} \int_\Omega |Du||u|dx + ||c||_{L^\infty} \int_\Omega u^2 dx \end{aligned}$$

3. Now from Cauchy's Inequality with ϵ, we observe

$$\int_\Omega |Du||u|dx \leq \epsilon \int_\Omega |Du|^2 dx + \frac{1}{4\epsilon} \int_\Omega u^2 dx$$

for $\epsilon > 0$.

We insert this estimate into the inequation in step 2 and then choose ϵ >0 so small that

$$\epsilon \sum_{i=1}^{n} ||b^i||_{L^\infty} < \frac{\theta}{2}.$$

Thus

$$\frac{\theta}{2} \int_\Omega |Du|^2 dx \leq B[u,u] + C \int_\Omega u^2 dx,$$

where C=$\frac{1}{4\epsilon} \sum_{i=1}^{n} ||b^i||_{L^\infty} + ||c||_{L^\infty}$.

In addition we recall from Poincare's inequality that

$$||u||_{L^2(\Omega)} \leq C^{'} ||Du||_{L^2(\Omega)}.$$

It easily follows that

$$\beta ||u||^2_{H^1_0(\Omega)} \leq B[u,u] + \gamma ||u||^2_{L^2(\Omega)}$$

where $\beta = \frac{\theta}{2(C'^2+1)}, \gamma = C$.

First Existence Theorem for weak solutions: There is a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function f$\in L^2(\Omega)$, there exists a unique weak solution $u \in H^1_0(\Omega)$ of the boundary-value problem

$$\begin{cases} Lu + \mu u = f & \text{in}\Omega \\ u = 0 & \text{on}\partial\Omega \end{cases} \quad (12)$$

proof: 1. Take γ from the former Theorem, let $\mu \geq \gamma$, and define then the bilinear form

$$B_\mu[u,v] := B[u,v] + \mu(u,v)$$

for $u, v \in H^1_0(\Omega)$, which corresponds as in 1.1 L$_\mu u := Lu + \mu u$. As before (,) means the inner product in L$^2(\Omega)$. Then $B_\mu[,]$ satisfies the hypothesis of the Lax-Milgram Theorem.

2. Now fix $f \in L^2(\Omega)$ and set $\langle f, v \rangle := (f, v)_{L^2(\Omega)}$. This is a bounded linear functional on L$^2(\Omega)$, and thus on $H^1_0(\Omega)$.

We apply the Lax-Milgram Theorem to find a unique function $u \in H^1_0$ satisfying

$$B_\mu[u,v] = \langle f, v \rangle$$

for all $v \in H^1_0(\Omega)$, u is consequently the unique weak solution of (12).

References

[1] Lawrence C. Evans, Partial Differential Equations, American Methematical Society, 1997: 251-325.

[2] Adams, Robert A. (1975), Sobolev Spaces, Boston, MA: Academic Press, ISBN 978-0-12-044150-1.