On spectral action principle for Robertson-Walker metrics

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Dedicated to Henri Moscovici on the occasion of his birthday

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The spectral action principle of Connes and Chamseddine

Some relevant references:

- A. Connes, *Noncommutative Geometry*, 1994.
- A. Connes, *Gravity coupled with matter and the foundation of noncommutative geometry*, Comm. Math. Phys. 182 (1996) 155-176.
- A. H. Chamseddine, A. Connes, *The spectral action principle,* Comm. Math. Phys. 186 (1997), no. 3, 731–750.
- A. H. Chamseddine, A. Connes, *Spectral action for Robertson-Walker metrics,* J. High Energy Phys. 2012, no. 10, 101.

Classical action

- Classical Lagrangian action

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- Classical equations of motion

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- Quantum expectation values

$$\langle \mathcal{O} \rangle = \int D[\varphi]\, \mathcal{O}(\varphi)\, e^{\frac{i}{\hbar} S}$$

Spectral action of Connes-Chamseddine

- Replace the classical action $S = \int_M \mathcal{L}(\varphi, \partial_\mu \varphi) d^n x$ by the spectral action

$$S = \mathrm{Trace}(f(D/\Lambda)),$$

where D is a Dirac operator, f is a positive even function, and the cutoff Λ is the mass scale.

- *S only depends on the spectrum of D and moments of the cutoff, $f_k = \int_0^\infty f(v) v^{k-1} dv$.*

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- Spectral action makes perfect sense for spectral triples.

Heat Kernel and Seeley-DeWitt Coefficients

- Assume: an asymptotic expansion of the form

$$\mathrm{Trace}\left(e^{-tD^2}\right) \sim \sum a_\alpha t^\alpha \quad (t \to 0) \tag{1}$$

holds.

Theorem (Connes-Chamseddine) Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple fulfilling (1). Then the spectral action can be expanded in powers of the scale $\Lambda \gg 0$ in the form

$$\mathrm{Tr}(f(D/\Lambda)) = \sum_{\beta \in \Pi} f_\beta \Lambda^\beta \int\!\!\!\!\!- |D|^{-\beta} + f(0)\zeta_D(0) + o(1),$$

where the dimension spectrum Π is the set of poles of zeta functions $\zeta_a(s) = \mathrm{Tr}(a|D|^{-s})$, and

$$f_\beta = \int_0^\infty f(v) v^{\beta-1} dv.$$

and $\int\!\!\!\!- \, T = \mathrm{Res}_{s=0} \, \mathrm{Tr}(T|D|^{-s})$.

Gilkey's local formulae

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$$\begin{aligned} a_4 \;=\; & \frac{(4\pi)^{-m/2}}{360}\mathrm{Tr}\big(-12R_{ijij;kk} + 5R_{ijij}R_{klkl} - 2R_{ijik}R_{ljlk} \\ & +2R_{ijkl}R_{ijkl} - 60R_{ijij}E + 180E^2 + 60E_{;kk} + 30\Omega_{ij}\Omega_{ij}\big). \end{aligned}$$

$$
\begin{aligned}
a_6 \quad = \quad & (4\pi)^{-m/2}\,\mathrm{Tr}\Big(\frac{1}{7!}\big(-18R_{ijij;kkll} + 17R_{ijij;k}R_{ulul;k} - 2R_{ijik;l}R_{ujuk;l} \\
& \qquad -4R_{ijik;l}R_{ujul;k} + 9R_{ijku;l}R_{ijku;l} + 28R_{ijij}R_{kuku;ll} \\
& \qquad -8R_{ijijk}R_{ujuk;ll} + 24R_{ijik}R_{ujul;kl} + 12R_{ijkl}R_{ijkl;uu}\big) \\
& + \frac{1}{9\cdot 7!}\big(-35R_{ijij}R_{klkl}R_{pqpq} + 42R_{ijij}R_{klkp}R_{qlqp} \\
& -42R_{ijij}R_{klpq}R_{klpq} + 208R_{ijik}R_{julu}R_{kplp} - 192R_{ijik}R_{uplp}R_{jukl} \\
& +48R_{ijik}R_{julp}R_{kulp} - 44R_{ijku}R_{ijlp}R_{kulp} - 80R_{ijku}R_{ilkp}R_{jlup}\big) \\
& \quad + \frac{1}{360}\big(8\Omega_{ij;k}\Omega_{ij;k} + 2\Omega_{ij;j}\Omega_{ik;k} + 12\Omega_{ij}\Omega_{ij;kk} - 12\Omega_{ij}\Omega_{jk}\Omega_{ki} \\
& \quad -6R_{ijkl}\Omega_{ij}\Omega_{kl} + 4R_{ijik}\Omega_{jl}\Omega_{kl} - 5R_{ijij}\Omega_{kl}\Omega_{kl}\big) \\
& + \frac{1}{360}\big(6E_{;iijj} + 60EE_{;ii} + 30E_{;i}E_{;i} + 60E^3 + 30E\Omega_{ij}\Omega_{ij} \\
& -10R_{ijij}E_{;kk} - 4R_{ijik}E_{;jk} - 12R_{ijij;k}E_{;k} - 30R_{ijij}E^2 \\
& -12R_{ijij;kk}E + 5R_{ijij}R_{klkl}E - 2RijikR_{ijkl}E + 2R_{ijkl}R_{ijkl}E\big)\Big).
\end{aligned}
$$

For the Dirac operator $D^2 = \nabla^*\nabla - \frac{1}{4}R$, so

$$
E = \frac{1}{4}R.
$$

Robertson-Walker Metric

Robertson-Walker metric:

$$ds^2 = dt^2 + a^2(t)\left(d\chi^2 + \sin^2(\chi)\left(d\theta^2 + \sin^2(\theta)\, d\varphi^2\right)\right)$$

For $a(t) = \sin(t)$ one obtains the round metric on S^4.

Euler Maclaurin formula and Heat kernel for S^4

Euler Maclaurin formula

$$\sum_{k=a}^{b} g(k) = \int_a^b g(x)dx + \frac{g(a)+g(b)}{2} + \sum_{j=2}^{m} \frac{B_j}{j!}(g^{(j-1)}(b) - g^{(j-1)}(a)) - R_m$$

Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$$

Using the formula, Euler easily computed $\zeta(2)$ up to 15 decimal digits! To compute it directly to 6 decimal digits, you need to add 10^6 terms!

Spectrum of Dirac for round S^4,

	eigenvalues	multiplicity
D	$\pm k$	$\frac{2}{3}(k^3 - k)$
D^2	k^2	$\frac{4}{3}(k^3 - k)$

To find heat kernel coefficients of D^2 we apply the Euler Maclaurin formula for $a = 0$, $b = \infty$ and

$$g(x) = \frac{4}{3}(x^3 - x)f(x) = \frac{4}{3}(x^3 - x)e^{-tx^2}$$

The integral term gives

$$\int_a^b g(x)dx = \frac{4}{3}\int_0^\infty (x^3 - x)e^{-tx^2}dx = \frac{2}{3}(t^{-2} - t^{-1})$$

The term $\frac{g(a)+g(b)}{2}$ is zero since $g(0) = g(\infty) = 0$.
And

$$g^{(2m-1)}(0)/(2m-1)! = (-1)^m \frac{4}{3}\left(\frac{t^{m-2}}{(m-2)!} + \frac{t^{m-1}}{(m-1)!}\right)$$

Putting all these together we get

$$\frac{3}{4}\mathrm{Tr}(e^{-tD^2}) = \frac{1}{2t^2} - \frac{1}{2t} + \frac{11}{120} + \sum_{k=1}^{m}(-1)^k\left(\frac{B_{2k+2}}{2k+2} + \frac{B_{2k+4}}{2k+4}\right)\frac{t^k}{k!} + o(t^m)$$

Euler Maclaurin formula and spectral action for S^4

For general f (with some conditions to control the remainder term), the Euler Maclaurin formula for $g(x) = \frac{4}{3} f(tx^2)(x^3 - x)$ will give the expansion

$$\frac{3}{4}\mathrm{Tr}(f(tD^2)) = \int_0^\infty f(tx^2)(x^3 - x)dx + \frac{11f(0)}{120} - \frac{31f'(0)}{2520}t$$

$$+\frac{41f''(0)}{10080}t^2 - \frac{31f^{(3)}(0)}{15840}t^3 + \frac{10331f^{(4)}(0)}{8648640}t^4 + \ldots + R_m$$

Levi-Civita Connection and the Spin Connection

Now, suppose $\{\theta_\alpha\}$ is an orthonormal basis of vector fields and $\{\theta^\alpha\}$ is its dual basis on forms. For any connection on cotangent bundle ∇, connection one forms ω^α_β is given by

$$\nabla\theta^\alpha = \omega^\alpha_\beta \theta^\beta.$$

If ∇ is a metric connection then

$$\omega^\alpha_\beta = -\omega^\beta_\alpha.$$

Cartan equations gives a recipe to find the torsion and curvature 2-forms using the ω^β_α and exterior derivatives the dual basis.

$$T^\alpha = d\theta^\alpha - \omega^\alpha_\beta \wedge \theta^\beta$$

For torsion free connection we have

$$d\theta^\beta = \omega^\beta_\alpha \wedge \theta^\alpha.$$

Connection one-form for Levi-civita connection

We have the following orthonormal basis for the cotangent space

$$\begin{aligned}
\theta^1 &= dt, \\
\theta^2 &= a(t)\, d\chi, \\
\theta^3 &= a(t)\, \sin\chi\, d\theta, \\
\theta^4 &= a(t)\, \sin\chi\, \sin\theta\, d\varphi.
\end{aligned}$$

The computation by Chamseddin-Connes shows that the connection one-form is given by

$$\omega = \begin{bmatrix} 0 & -\frac{a'(t)}{a(t)}\theta^2 & -\frac{a'(t)}{a(t)}\theta^3 & -\frac{a'(t)}{a(t)}\theta^4 \\ \frac{a'(t)}{a(t)}\theta^2 & 0 & -\frac{\cot(\chi)}{a(t)}\theta^3 & -\frac{\cot(\chi)}{a(t)}\theta^4 \\ \frac{a'(t)}{a(t)}\theta^3 & \frac{\cot(\chi)}{a(t)}\theta^3 & 0 & -\frac{\cot(\theta)}{a(t)\sin(\chi)}\theta^4 \\ \frac{a'(t)}{a(t)}\theta^4 & \frac{\cot(\chi)}{a(t)}\theta^4 & \frac{\cot(\theta)}{a\sin(\chi)}\theta^4 & 0 \end{bmatrix}$$

The Spin Connection

The spin connection is the lift of the Levi-Civita connection defined on T^*M. Now we have the connection one-forms ω, which is a skew symmetric matrix, i.e. $\omega \in \mathfrak{so}(4)$. Using the Lie algebra isomorphism $\mu : \mathfrak{so}(4) \to \mathfrak{spin}(4)$ given by

$$A \mapsto \frac{1}{4} \sum_{\alpha,\beta} \langle A\theta^\alpha, \theta^\beta \rangle c(\theta^\alpha) c(\theta^\beta)$$

Since ω is written in the orthonormal basis θ^α so $\langle \omega\theta^\alpha, \theta^\beta \rangle = \omega^\alpha_\beta$. So the connection one forms for the spinor connection is given by

$$\tilde{\omega} = \frac{1}{2}\omega^1_2\gamma^{12} + \frac{1}{2}\omega^1_3\gamma^{13} + \frac{1}{2}\omega^1_4\gamma^{14} + \frac{1}{2}\omega^2_3\gamma^{23} + + \frac{1}{2}\omega^2_4\gamma^{24} + \frac{1}{2}\omega^3_4\gamma^{34}$$

Connes-Chamseddin Computations

They used Gilkey's local formulae to obtain

$$a_0 = \frac{a(t)^3}{2}$$

$$a_2 = \frac{1}{4}a(t)\left(a(t)a''(t) + a'(t)^2 - 1\right)$$

$$a_4 = \frac{1}{120}(3a^{(4)}(t)a(t)^2 + 3a(t)a''(t)^2 - 5a''(t) + 9a^{(3)}(t)a(t)a'(t) - 4a'(t)^2a''(t))$$

$$a_6 = \frac{1}{5040a(t)^2}(9a^{(6)}(t)a(t)^4 - 21a^{(4)}(t)a(t)^2 - 3a^{(3)}(t)^2a(t)^3 - 56a(t)^2a''(t)^3 + 42a(t)a''(t)^2 + 36a^{(5)}(t)a(t)^3a'(t) + 6a^{(4)}(t)a(t)^3a''(t) - 42a^{(4)}(t)a(t)^2a'(t)^2 + 60a^{(3)}(t)a(t)a'(t)^3 + 21a^{(3)}(t)a(t)a'(t) + 240a(t)a'(t)^2a''(t)^2 - 60a'(t)^4a''(t) - 21a'(t)^2a''(t) - 252a^{(3)}(t)a(t)^2a'(t)a''(t))$$

Connes-Chamseddin Computations

Using Euler-Maclaurin summation and Feynman-Kac formula they computed up to a_{10}:

$$a_8 = -\frac{1}{10080a(t)^4}(-a^{(8)}(t)a(t)^6 + 3a^{(6)}(t)a(t)^4 + 13a^{(4)}(t)^2a(t)^5 - 24a^{(3)}(t)^2a(t)^3 - 114a(t)^3a''(t)^4 + 43a(t)^2a''(t)^3 - 5a^{(7)}(t)a(t)^5a'(t) + 2a^{(6)}(t)a(t)^5a''(t) + 9a^{(6)}(t)a(t)^4a'(t)^2 + 16a^{(3)}(t)a^{(5)}(t)a(t)^5 - 24a^{(5)}(t)a(t)^3a'(t)^3 - 6a^{(5)}(t)a(t)^3a'(t) + 69a^{(4)}(t)a(t)^4a''(t)^2 - 36a^{(4)}(t)a(t)^3a''(t) + 60a^{(4)}(t)a(t)^2a'(t)^4 + 15a^{(4)}(t)a(t)^2a'(t)^2 + 90a^{(3)}(t)^2a(t)^4a''(t) - 216a^{(3)}(t)^2a(t)^3a'(t)^2 - 108a^{(3)}(t)a(t)a'(t)^5 - 27a^{(3)}(t)a(t)a'(t)^3 + 801a(t)^2a'(t)^2a''(t)^3 - 588a(t)a'(t)^4a''(t)^2 - 87a(t)a'(t)^2a''(t)^2 + 108a'(t)^6a''(t) + 27a'(t)^4a''(t) + 78a^{(5)}(t)a(t)^4a'(t)a''(t) + 132a^{(3)}(t)a^{(4)}(t)a(t)^4a'(t) - 312a^{(4)}(t)a(t)^3a'(t)^2a''(t) - 819a^{(3)}(t)a(t)^3a'(t)a''(t)^2 + 768a^{(3)}(t)a(t)^2a'(t)^3a''(t) + 102a^{(3)}(t)a(t)^2a'(t)a''(t))$$

$$a_{10} = \frac{1}{665280a(t)^6}(3a^{(10)}(t)a(t)^8 - 222a^{(5)}(t)^2a(t)^7 - 348a^{(4)}(t)a^{(6)}(t)a(t)^7 - 147a^{(3)}(t)a^{(7)}(t)a(t)^7 - 18a''(t)a^{(8)}(t)a(t)^7 + 18a'(t)a^{(9)}(t)a(t)^7 - 482a''(t)a^{(4)}(t)^2a(t)^6 - 331a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 1110a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 1556a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 448a''(t)^2a^{(6)}(t)a(t)^6 - 1074a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 476a'(t)a''(t)a^{(7)}(t)a(t)^6 - 43a'(t)^2a^{(8)}(t)a(t)^6 - 11a^{(8)}(t)a(t)^6 + 8943a'(t)a^{(3)}(t)^3a(t)^5 + 21846a''(t)^2a^{(3)}(t)^2a(t)^5 + 4092a'(t)^2a^{(4)}(t)^2a(t)^5 + 396a^{(4)}(t)^2a(t)^5 + 10560a''(t)^3a^{(4)}(t)a(t)^5 + 39402a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 11352a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + 6336a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 594a^{(3)}(t)a^{(5)}(t)a(t)^5 + 2904a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 264a''(t)a^{(6)}(t)a(t)^5 + 165a'(t)^3a^{(7)}(t)a(t)^5 + 33a'(t)a^{(7)}(t)a(t)^5 - 10338a''(t)^5a(t)^4 - 95919a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - 3729a''(t)a^{(3)}(t)^2a(t)^4 - 117600a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - 68664a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - 2772a''(t)^2a^{(4)}(t)a(t)^4 - 23976a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - 2640a'(t)a^{(3)}(t)a^{(4)}(t)a(t)^4 - 12762a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 1386a'(t)a''(t)a^{(5)}(t)a(t)^4 - 651a'(t)^4a^{(6)}(t)a(t)^4 - 132a'(t)^2a^{(6)}(t)a(t)^4 + 111378a'(t)^2a''(t)^4a(t)^3 + 2354a''(t)^4a(t)^3 + 31344a'(t)^4a^{(3)}(t)^2a(t)^3 + 3729a'(t)^2a^{(3)}(t)^2a(t)^3 + 236706a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 13926a'(t)a''(t)^2a^{(3)}(t)a(t)^3 + 43320a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 5214a'(t)^2a''(t)a^{(4)}(t)a(t)^3 + 2238a'(t)^5a^{(5)}(t)a(t)^3 + 462a'(t)^3a^{(5)}(t)a(t)^3 - 162162a'(t)^4a''(t)^3a(t)^2 - 11880a'(t)^2a''(t)^3a(t)^2 - 103884a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 13332a'(t)^3a''(t)a^{(3)}(t)a(t)^2 - 6138a'(t)^6a^{(4)}(t)a(t)^2 - 1287a'(t)^4a^{(4)}(t)a(t)^2 + 76440a'(t)^6a''(t)^2a(t) + 10428a'(t)^4a''(t)^2a(t) + 11700a'(t)^7a^{(3)}(t)a(t) + 2475a'(t)^5a^{(3)}(t)a(t) - 11700a'(t)^8a''(t) - 2475a'(t)^6a''(t))$$

Connes-Chamseddine conjecture and question about the form of the coefficients:

- Check the agreement between the above formulas for a_8 and a_{10} and the universal formulas.
- Show that the term a_{2n} of the asymptotic expansion of the spectral action for Robertson-Walker metric is of the form $P_n(a, \cdots, a^{(2n)})/a^{2n-4}$ where P_n is a polynomial with rational coefficients and compute P_n.

Dirac Operator; spectral analysis via pseudodifferential calculus

$$
\begin{aligned}
D &= \gamma^\alpha \nabla_{\theta_\alpha} = \gamma^\alpha \left(\theta_\alpha + \omega(\theta_\alpha)\right) \\
&= \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{1}{a} \frac{\partial}{\partial \chi} + \gamma^2 \frac{1}{a \sin \chi} \frac{\partial}{\partial \theta} + \gamma^3 \frac{1}{a \sin \chi \sin \theta} \frac{\partial}{\partial \varphi} \\
&+ \frac{3a'}{2a}\gamma^0 + \frac{\cot(\chi)}{a}\gamma^1 + \frac{\cot(\theta)}{2a\sin(\chi)}\gamma^2
\end{aligned}
$$

So the symbol of the Dirac operator would be

$$
\begin{aligned}
\sigma_D(\mathbf{x}, \xi) \quad = i\gamma^0\xi_1 \quad &+ \frac{i}{a}\gamma^1\xi_2 + \frac{i}{a\sin(\chi)}\gamma^2\xi_3 + \frac{i}{a\sin(\chi)\sin(\theta)}\gamma^3\xi_4 \\
&+ \frac{3a'}{2a}\gamma^0 + \frac{\cot(\chi)}{a}\gamma^1 + \frac{\cot(\theta)}{2a\sin(\chi)}\gamma^2
\end{aligned}
$$

Symbol of D^2

Using the symbol multiplication rule one can compute the symbol of the square of the Dirac operator. The symbol of D^2 has following homogeneous parts.

$$p_2 = \xi_1^2 + \frac{1}{a(t)^2}\xi_2^2 + \frac{1}{a(t)^2\sin^2(\chi)}\xi_3^2 + \frac{1}{a(t)^2\sin^2(\theta)\sin^2(\chi)}\xi_4^2,$$

$$\begin{aligned} p_1 = &-\frac{3ia'(t)}{a(t)}\xi_1 - \frac{i}{a(t)^2}\left(\gamma^{12}a'(t) + 2\cot(\chi)\right)\xi_2 \\ &- \frac{i}{a(t)^2}\left(\gamma^{13}\csc(\chi)a'(t) + \cot(\theta)\csc^2(\chi) + \gamma^{23}\cot(\chi)\csc(\chi)\right)\xi_3 \\ &- \frac{i}{a(t)^2}\left(\csc(\theta)\csc(\chi)a'(t)\gamma^{14} + \cot(\theta)\csc(\theta)\csc^2(\chi)\gamma^{34} + \csc(\theta)\cot(\chi)\csc(\chi)\gamma^{24}\right)\xi_4,\end{aligned}$$

$$\begin{aligned} p_0 = &+\frac{1}{8a(t)^2}\Big(-12a(t)a''(t) - 6a'(t)^2 + 3\csc^2(\theta)\csc^2(\chi) - \cot^2(\theta)\csc^2(\chi) \\ &+4i\cot(\theta)\cot(\chi)\csc(\chi) - 4i\cot(\theta)\cot(\chi)\csc(\chi) - 4\cot^2(\chi) + 5\csc^2(\chi) + 4\Big) \\ &- \frac{(\cot(\theta)\csc(\chi)a'(t))}{2a(t)^2}\gamma^{13} - \frac{(\cot(\chi)a'(t))}{a(t)^2}\gamma^{12} - \frac{(\cot(\theta)\cot(\chi)\csc(\chi))}{2a(t)^2}\gamma^{23}\end{aligned}$$

Symbol of the parametrix

We will use the symbol of the right parametrix, $(P-\lambda)\tilde{R}(\lambda) = I$, instead of the left one. So if $\sigma(\tilde{R}(\lambda)) = r_0 + r_1 + r_2 + \cdots$, with r_n which are homogeneous of order $-2-n$ then we find the following recursive formulas for r_n's.

$$r_n = -r_0 \sum_{\substack{|\alpha|+j+2-k=n \\ j<n}} (-i)^{|\alpha|} d_\xi^\alpha p_k \cdot d_x^\alpha r_j/\alpha!,$$

where $r_0 = (p_2-\lambda)^{-1} = (\|\xi\|^2-\lambda)^{-1}$. So the summation, for $n>1$, will only have the following possible summands.

$k=0, \lvert\alpha\rvert=0, j=n-2$	$-r_0 p_0 r_{n-2}$
$k=1, \lvert\alpha\rvert=0, j=n-1$	$-r_0 p_1 r_{n-1}$
$k=1, \lvert\alpha\rvert=0, j=n-2$	$ir_0 \frac{\partial}{\partial\xi_0} p_1 \cdot \frac{\partial}{\partial t} r_{n-2} + ir_0 \frac{\partial}{\partial\xi_1} p_1 \cdot \frac{\partial}{\partial\chi} r_{n-2} + ir_0 \frac{\partial}{\partial\xi_2} p_1 \cdot \frac{\partial}{\partial\theta} r_{n-2}$
$k=2, \lvert\alpha\rvert=1, j=n-1$	$ir_0 \frac{\partial}{\partial\xi_0} p_2 \cdot \frac{\partial}{\partial t} r_{n-1} + ir_0 \frac{\partial}{\partial\xi_1} p_2 \cdot \frac{\partial}{\partial\chi} r_{n-1} + ir_0 \frac{\partial}{\partial\xi_2} p_2 \cdot \frac{\partial}{\partial\theta} r_{n-1}$
$k=2, \lvert\alpha\rvert=2, j=n-2$	$\frac{1}{2} r_0 \frac{\partial^2}{\partial\xi_0^2} p_2 \cdot \frac{\partial^2}{\partial t^2} r_{n-2} + \frac{1}{2} r_0 \frac{\partial^2}{\partial\xi_1^2} p_2 \cdot \frac{\partial^2}{\partial\chi^2} r_{n-2} + \frac{1}{2} r_0 \frac{\partial^2}{\partial\xi_2^2} p_2 \cdot \frac{\partial^2}{\partial\theta^2} r_{n-2}$

By induction, we can see that the each r_n can be written as

$$r_n = \sum_{2j-2-|\alpha|=n} r_{n,j,\alpha} r_0^j \xi^\alpha, \qquad (2)$$

where $n+1 \leq j \leq 2n+1$ (or equivalently, $n \leq |\alpha| \leq 3n$). And the only non-zero parts for $n = 0, 1$ (as the starting point for the recursive relation) are the following terms.

$$\begin{aligned} r_{0,1,\mathbf{0}} &= 1, \\ r_{1,2,\mathbf{e_k}} &= \frac{\partial p_1}{\partial \xi_k}, \\ r_{1,3,2\mathbf{e_l}+\mathbf{e_k}} &= -2ig^{kk}\frac{\partial g^{ll}}{\partial x_k}. \end{aligned}$$

Where $\{\mathbf{e_j}\}$ is the standard orthonormal basis of $\mathbb{R}^4$.

If we plug in 2 into the recursive formula that we have for r_n we find the following recursive formula for the $r_{n,j,\alpha}$.

$$
\begin{aligned}
r_{n,j,\alpha} = & -p_0 r_{n-2,j-1,\alpha} - \sum_k \frac{\partial p_1}{\partial \xi_k} r_{n-1,j-1,\alpha-\mathbf{e_k}} \\
& + i \sum_k \frac{\partial p_1}{\partial \xi_k} \frac{\partial}{\partial x_k} r_{n-2,j-1,\alpha} + i(2-j) \sum_{k,l} \frac{\partial g^{ll}}{\partial x_k} \frac{\partial p_1}{\partial \xi_k} r_{n-2,j-2,\alpha-2\mathbf{e_l}} \\
& + 2i \sum_k g^{kk} \frac{\partial}{\partial x_k} r_{n-1,j-1,\alpha-\mathbf{e_k}} + i(4-2j) \sum_{k,l} g^{kk} \frac{\partial g^{ll}}{\partial x_k} r_{n-1,j-2,\alpha-2e_l-\mathbf{e_k}} \\
& + \sum_k g^{kk} \frac{\partial^2}{\partial x_k^2} r_{n-2,j-1,\alpha} + (4-2j) \sum_{k,l} g^{kk} \frac{\partial g^{ll}}{\partial x_k} \frac{\partial}{\partial x_k} r_{n-2,j-2,\alpha-2\mathbf{e_l}} \\
& + (2-j) \sum_{k,l} g^{kk} \frac{\partial^2 g^{ll}}{\partial x_k^2} r_{n-2,j-2,\alpha-2\mathbf{e_l}} \\
& + (3-j)(2-j) \sum_{k,l,l'} g^{kk} \frac{\partial g^{ll}}{\partial x_k} \frac{\partial g^{l'l'}}{\partial x_k} r_{n-2,j-3,\alpha-2\mathbf{e_l}-\mathbf{2e_{l'}}}
\end{aligned}
$$

Heat Kernel of D^2 in terms of symbols of the parametrix.

Let

$$
\begin{aligned}
e_n &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} r_n(x,\xi,\lambda) d\lambda d\xi \\
&= \frac{1}{2\pi i (2\pi)^4} \sum r_{n,j,\alpha}(x) \int_{\mathbb{R}^4} \xi^\alpha \int_\gamma e^{-t\lambda} r_0^j d\lambda d\xi \\
&= \sum c_\alpha \frac{1}{(j-1)!} r_{n,j,\alpha} a(t)^{\alpha_2+\alpha_3+\alpha_4+3} \sin(\chi)^{\alpha_3+\alpha_4+2} \sin(\theta)^{\alpha_4+1}
\end{aligned}
$$

Where $c_\alpha = \frac{1}{(2\pi)^4} \prod_k \Gamma\left(\frac{\alpha_k+1}{2}\right) \frac{(-1)^{\alpha_k}+1}{2}$.

Now let $e_{n,j,\alpha} = r_{n,j,\alpha} \frac{a(t)^{\alpha_2+\alpha_3+\alpha_4+3} \sin(\chi)^{\alpha_3+\alpha_4+2} \sin(\theta)^{\alpha_4+1}}{(j-1)!}$ and we want to compute $\sum c_\alpha e_{n,j,\alpha}$ To get the heat coefficient $a_n = \int_{S_a^3} e_n d\chi d\theta d\phi$.

$$\begin{aligned} a_n &= \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} e_n d\chi d\theta d\phi \\ &= 2\pi \sum c_\alpha \int_0^{\pi} \int_0^{\pi} e_{n,j,\alpha}(t) d\chi d\theta \end{aligned}$$

new term a_{12}

$$
\begin{aligned}
a_{12} = {} & \frac{1}{17297280a(t)^8}\Big(3a^{(12)}(t)a(t)^{10} - 1057a^{(6)}(t)^2a(t)^9 - 1747a^{(5)}(t)a^{(7)}(t)a(t)^9 - 970a^{(4)}(t)a^{(8)}(t)a(t)^9 - \\
& 317a^{(3)}(t)a^{(9)}(t)a(t)^9 - 34a''(t)a^{(10)}(t)a(t)^9 + 21a'(t)a^{(11)}(t)a(t)^9 + 5001a^{(4)}(t)^3a(t)^8 + 2419a''(t)a^{(5)}(t)^2a(t)^8 + \\
& 19174a^{(3)}(t)a^{(4)}(t)a^{(5)}(t)a(t)^8 + 4086a^{(3)}(t)^2a^{(6)}(t)a(t)^8 + 2970a''(t)a^{(4)}(t)a^{(6)}(t)a(t)^8 - \\
& 5520a'(t)a^{(5)}(t)a^{(6)}(t)a(t)^8 - 511a''(t)a^{(3)}(t)a^{(7)}(t)a(t)^8 - 4175a'(t)a^{(4)}(t)a^{(7)}(t)a(t)^8 - \\
& 745a''(t)^2a^{(8)}(t)a(t)^8 - 2289a'(t)a^{(3)}(t)a^{(8)}(t)a(t)^8 - 828a'(t)a''(t)a^{(9)}(t)a(t)^8 - 62a'(t)^2a^{(10)}(t)a(t)^8 - \\
& 13a^{(10)}(t)a(t)^8 + 45480a^{(3)}(t)^4a(t)^7 + 152962a''(t)^2a^{(4)}(t)^2a(t)^7 + 203971a'(t)a^{(3)}(t)a^{(4)}(t)^2a(t)^7 + \\
& 21369a'(t)^2a^{(5)}(t)^2a(t)^7 + 1885a^{(5)}(t)^2a(t)^7 + 410230a''(t)a^{(3)}(t)^2a^{(4)}(t)a(t)^7 + 163832a'(t)a^{(3)}(t)^2a^{(5)}(t)a(t)^7 + \\
& 250584a''(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^7 + 244006a'(t)a''(t)a^{(4)}(t)a^{(5)}(t)a(t)^7 + 42440a''(t)^3a^{(6)}(t)a(t)^7 + \\
& 163390a'(t)a''(t)a^{(3)}(t)a^{(6)}(t)a(t)^7 + 35550a'(t)^2a^{(4)}(t)a^{(6)}(t)a(t)^7 + 3094a^{(4)}(t)a^{(6)}(t)a(t)^7 + \\
& 34351a'(t)a''(t)^2a^{(7)}(t)a(t)^7 + 19733a'(t)^2a^{(3)}(t)a^{(7)}(t)a(t)^7 + 1625a^{(3)}(t)a^{(7)}(t)a(t)^7 + \\
& 6784a'(t)^2a''(t)a^{(8)}(t)a(t)^7 + 520a''(t)a^{(8)}(t)a(t)^7 + 308a'(t)^3a^{(9)}(t)a(t)^7 + 52a'(t)a^{(9)}(t)a(t)^7 - \\
& 2056720a'(t)a''(t)a^{(3)}(t)^3a(t)^6 - 1790580a''(t)^3a^{(3)}(t)^2a(t)^6 - 900272a'(t)^2a''(t)a^{(4)}(t)^2a(t)^6 - \\
& 31889a''(t)a^{(4)}(t)^2a(t)^6 - 643407a''(t)^4a^{(4)}(t)a(t)^6 - 1251548a'(t)^2a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - \\
& 43758a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 4452042a'(t)a''(t)^2a^{(3)}(t)a^{(4)}(t)a(t)^6 - 836214a'(t)a''(t)^3a^{(5)}(t)a(t)^6 - \\
& 1400104a'(t)^2a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 48620a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 181966a'(t)^3a^{(4)}(t)a^{(5)}(t)a(t)^6 - \\
& 18018a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 319996a'(t)^2a''(t)^2a^{(6)}(t)a(t)^6 - 11011a''(t)^2a^{(6)}(t)a(t)^6 - \\
& 115062a'(t)^3a^{(3)}(t)a^{(6)}(t)a(t)^6 - 11154a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 42764a'(t)^3a''(t)a^{(7)}(t)a(t)^6 - \\
& 4004a'(t)a''(t)a^{(7)}(t)a(t)^6 - 1649a'(t)^4a^{(8)}(t)a(t)^6 - 286a'(t)^2a^{(8)}(t)a(t)^6 + 460769a''(t)^6a(t)^5 + \\
& 1661518a'(t)^3a^{(3)}(t)^3a(t)^5 + 83486a'(t)a^{(3)}(t)^3a(t)^5 + 13383328a'(t)^2a''(t)^2a^{(3)}(t)^2a(t)^5 + \\
& 222092a''(t)^2a^{(3)}(t)^2a(t)^5 + 342883a'(t)^4a^{(4)}(t)^2a(t)^5 + 36218a'(t)^2a^{(4)}(t)^2a(t)^5 + \\
& 7922361a'(t)a''(t)^4a^{(3)}(t)a(t)^5 + 6367314a'(t)^2a''(t)^3a^{(4)}(t)a(t)^5 + 109330a''(t)^3a^{(4)}(t)a(t)^5 +
\end{aligned}
$$

$$+7065862a'(t)^3a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 360386a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 1918386a'(t)^3a''(t)^2a^{(5)}(t)a(t)^5 + 98592a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + 524802a'(t)^4a^{(3)}(t)a^{(5)}(t)a(t)^5 + 55146a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 226014a'(t)^4a''(t)a^{(6)}(t)a(t)^5 + 23712a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 8283a'(t)^5a^{(7)}(t)a(t)^5 + 1482a'(t)^3a^{(7)}(t)a(t)^5 - 7346958a'(t)^2a''(t)^5a(t)^4 - 72761a''(t)^5a(t)^4 - 11745252a'(t)^4a''(t)a^{(3)}(t)^2a(t)^4 - 725712a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - 27707028a'(t)^3a''(t)^3a^{(3)}(t)a(t)^4 - 819520a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - 8247105a'(t)^4a''(t)^2a^{(4)}(t)a(t)^4 - 520260a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - 1848228a'(t)^5a^{(3)}(t)a^{(4)}(t)a(t)^4 - 205296a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - 973482a'(t)^5a''(t)a^{(5)}(t)a(t)^4 - 110136a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 36723a'(t)^6a^{(6)}(t)a(t)^4 - 6747a'(t)^4a^{(6)}(t)a(t)^4 + 17816751a'(t)^4a''(t)^4a(t)^3 + 721058a'(t)^2a''(t)^4a(t)^3 + 2352624a'(t)^6a^{(3)}(t)^2a(t)^3 + 274170a'(t)^4a^{(3)}(t)^2a(t)^3 + 24583191a'(t)^5a''(t)^2a^{(3)}(t)a(t)^3 + 1771146a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 3256248a'(t)^6a''(t)a^{(4)}(t)a(t)^3 + 389376a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 135300a'(t)^7a^{(5)}(t)a(t)^3 + 25350a'(t)^5a^{(5)}(t)a(t)^3 - 15430357a'(t)^6a''(t)^3a(t)^2 - 1252745a'(t)^4a''(t)^3a(t)^2 - 7747848a'(t)^7a''(t)a^{(3)}(t)a(t)^2 - 967590a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 385200a'(t)^8a^{(4)}(t)a(t)^2 - 73125a'(t)^6a^{(4)}(t)a(t)^2 + 5645124a'(t)^8a''(t)^2a(t) + 741195a'(t)^6a''(t)^2a(t) + 749700a'(t)^9a^{(3)}(t)a(t) + 143325a'(t)^7a^{(3)}(t)a(t) - 749700a'(t)^{10}a''(t) - 143325a'(t)^8a''(t))\Big)$$

Check on Sphere S^4

For $a(t) = sin(t)$ we have

$$a_{12}(sphere) = \frac{10331 \sin^3(t)}{8648640}.$$

Hence

$$\int_0^\pi a_{12}(spher)dt = \frac{4}{3}\frac{10331}{8648640} = \frac{10331}{6486480}.$$

Which agrees with the direct computation done in Connes-Chamseddine.

	Seeley-DeWitt coefficients	Connes-Chamseddin method	Pseudo-diff method	Test case S^4
a_0	✓	✓	✓	✓
a_2	✓	✓	✓	✓
a_4	✓	✓	✓	✓
a_6	✓	✓	✓	✓
a_8	X	✓	✓	✓
a_{10}	X	✓	✓	✓
a_{12}	X	X	✓	✓