Weyl's Asymptotic Law

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Let Ω be a bounded open subset of $\mathbb{R}^m$ satisfying the *segment property*, i.e., its topological boundary $\partial\Omega$ has a finite open covering $\{O_i\}$ and corresponding nonzero vectors $\{y_i\}$ such that

$$\{x + ty_i \mid 0 < t < 1\} \subset \Omega\,, \qquad \forall x \in \overline{\Omega} \cap O_i\,.$$

We refer to a subset of $\mathbb{R}^m$ with these properties as a *domain*. Denote by $C^\infty(\overline{\Omega})$ the set of smooth functions on Ω all of whose partial derivatives can be extended continuously to $\overline{\Omega}$. Consider the subspace

$$D_D = \{f \in C^\infty(\overline{\Omega}) \mid f|_{\partial\Omega} = 0\}$$

of the Hilbert space $L^2(\Omega)$. Let

$$\Delta = -\mathrm{div} \circ \mathrm{grad} \tag{1}$$

be the Laplace operator on $C^\infty(\Omega)$. If $\partial\Omega$ is smooth, then, by Green's theorem, the restriction of Δ to D_D is symmetric and non-negative, i.e.,

$$\langle \Delta f, g\rangle = \langle \mathrm{grad} f, \mathrm{grad} g\rangle_{L^2} := \int_\Omega (\mathrm{grad} f, \mathrm{grad} g)\, d^m x = \langle f, \Delta g\rangle\,, \tag{2}$$

where $\langle\,.\,,.\rangle$ and $(\,.\,,.)$ denote the inner product on $L^2(\Omega)$ and the Euclidean metric on $\mathbb{R}^m$, respectively. We denote $\langle \mathrm{grad} f, \mathrm{grad} g\rangle_{L^2}$ by $Q(f, g)$.

Consider the Sobolev space

$$H^1(\Omega) := W^{1,2}(\Omega) = \{f \in L^2(\Omega) \,|\, \mathrm{Grad} f \in L^2(\Omega)\}\,,$$

where $\mathrm{Grad} f$ denote the distributional gradient of f. We define the subspace $H_0^1(\Omega)$ of $H^1(\Omega)$ to be the closure of the subspace $C_c^\infty(\Omega) \subset H^1(\Omega)$ under the norm $||\,.\,||_{H^1}$, where $||f||_{H^1}^2 = ||f||_{L^2}^2 + ||\mathrm{Grad} f||_{L^2}^2$. The Dirichlet Laplacian for Ω, denoted by Δ_D^Ω, is defined to be the self-adjoint non-negative Friedrichs extension of $\Delta|_{D_D}$. The domain of Δ_D^Ω is the subspace $H_0^1(\Omega)$ of $L^2(\Omega)$. We have

$$\langle \Delta_D^\Omega f, f \rangle = \overline{Q}(f, f)\,, \qquad \forall f \in H_0^1(\Omega)\,, \tag{3}$$

where $\overline{Q}$ denote the closure of Q.

Consider the subspace $D_N = \{f \in C^\infty(\overline{\Omega}) \mid (v f)|_{\partial\Omega} = 0\}$ of $L^2(\Omega)$, where v denote the outward unit normal vector field on $\partial\Omega$. If $\partial\Omega$ is smooth, then, by Green's theorem, the restriction of Δ to D_N is symmetric and non-negative. The Neumann Laplacian for Ω, denoted by Δ_N^Ω, is defined to be the self-adjoint non-negative Friedrichs extension of $\Delta|_{D_N}$. The domain of Δ_N^Ω is the subspace $H^1(\Omega)$ of $L^2(\Omega)$. For each $f \in H^1(\Omega)$ we have $\langle \Delta_N^\Omega f, f \rangle = \overline{Q}(f, f)$.

The self-adjoint non-negative operators Δ_D^Ω and Δ_N^Ω have compact resolvent [3]. Thus, their (point) spectrum consist of a discrete increasing sequence of non-negative real numbers with no limit point in $\mathbb{R}$, and each eigenvalue has finite multiplicity. In addition, the set of corresponding eigenfunctions is an orthonormal basis for $L^2(\Omega)$.

Henceforth, we denote by ϕ_k^Ω (respectively, ψ_k^Ω) the normalized eigenfunction of Δ_D^Ω (respectively, Δ_N^Ω) with eigenvalue λ_k^Ω (respectively, μ_k^Ω), where k ranges over positive integers, i.e., $k = 1, 2, \cdots$. We arrange the eigenvalues in an increasing sequence

$$0 \le \lambda_1^\Omega \le \lambda_2^\Omega \le \cdots$$
$$0 \le \mu_1^\Omega \le \mu_2^\Omega \le \cdots$$

with each eigenvalue repeated according to its multiplicity. For each $\nu \in \mathbb{R}_{>0}$, let $N(\Delta_D^\Omega, \nu)$ be the number of eigenvalues $\lambda_k^\Omega \le \nu$ counted with multiplicity. Indeed, $N(\Delta_D^\Omega, \nu)$ is the sum of the dimensions of the eigenspaces of Δ_D^Ω corresponding to the eigenvalues $\lambda_k^\Omega \le \nu$. The quantity $N(\Delta_N^\Omega, \nu)$ is defined similarly. In the following, we want to investigate the asymptotic behaviour of $N(\Delta_D^\Omega, \nu)$ as $\nu \to \infty$.

As the first step, we need to study the asymptotic behaviour of $N(\Delta_D^\Omega, \nu)$ and $N(\Delta_N^\Omega, \nu)$ in the special case where, Ω is the open m-dimensional rectangle $\Omega = (0, l_1) \times (0, l_2) \times \cdots \times (0, l_m) \subset \mathbb{R}^m$. In this case, the eigenvalues of Δ_D^Ω and Δ_N^Ω are explicitly given by (see [3], [1]):

$$\lambda^\Omega = \sum_{i=1}^{m} k_i^2 \left(\frac{\pi}{l_i}\right)^2 \tag{4}$$

and

$$\mu^{\Omega} = \sum_{i=1}^{m} (k_i - 1)^2 \left(\frac{\pi}{l_i} \right)^2 , \tag{5}$$

respectively, where each k_i ranges over the positive integers.

Proposition 1 *Let Ω be the open m-dimensional rectangle as above. We have*

$$\lim_{\nu\to\infty} \frac{N(\Delta_D^{\Omega}, \nu)}{\nu^{m/2}} = \lim_{\nu\to\infty} \frac{N(\Delta_N^{\Omega}, \nu)}{\nu^{m/2}} = \frac{\omega_m \, Vol(\Omega)}{(2\pi)^m} , \tag{6}$$

where ω_m is the volume of the unit ball in $\mathbb{R}^m$, and $Vol(\Omega)$ denotes the volume of Ω.

Proof Let Γ be the lattice in $\mathbb{R}^m$ generated by the vectors $\{e_1/l_1, \cdots, e_m/l_m\}$, where $\{e_i\}$ are the standard basis of $\mathbb{R}^m$. Let $\mathbb{B}^m(r) \subset \mathbb{R}^m$ be the closed ball with radius r and centered at the origin. Considering (4), the quantity $N(\Delta_D^{\Omega}, \nu)$ equals the number of the points of Γ in an "octant" of $\mathbb{B}^m(\sqrt{\nu}/\pi)$.

Denote by $\mathcal{N}(r)$ the number of the points of Γ in $\mathbb{B}^m(r)$. Let Σ be the fundamental domain for the action of Γ on $\mathbb{R}^m$. Denote by $\mathcal{P}(r)$ the number of copies of Σ contained in $\mathbb{B}^m(r)$. Let $d = \sqrt{\sum_{i=1}^m 1/l_i^2}$ be the diameter of Σ. We have

$$\mathcal{P}(r) \le \mathcal{N}(r) \le \mathcal{P}(r+d) . \tag{7}$$

Thus, using

$$\omega_m (r-d)^m \le \mathcal{P}(r) \mathrm{Vol}(\Sigma) \le \omega_m r^m , \tag{8}$$

we get

$$\frac{\omega_m (r-d)^m}{\mathrm{Vol}(\Sigma)} \le \mathcal{N}(r) \le \frac{\omega_m (r+d)^m}{\mathrm{Vol}(\Sigma)} , \tag{9}$$

which implies

$$\lim_{r\to\infty} \frac{\mathcal{N}(r)}{r^m} = \frac{\omega_m}{\mathrm{Vol}(\Sigma)} = \omega_m \mathrm{Vol}(\Omega) . \tag{10}$$

So we have $N(\Delta_D^{\Omega}, \nu) = \mathcal{N}(\sqrt{\nu}/\pi)/2^m + \epsilon$, where the "error" term ϵ corresponds to the points of the $(m-1)$-dimensional lattices resulted from intersection of Γ with each coordinate hyperplane in $\mathbb{R}^m$. Since ϵ is of order $\nu^{(m-1)/2}$, using (10), we get (6) for $N(\Delta_D^{\Omega}, \nu)$. A similar argument yields to (6) for $N(\Delta_N^{\Omega}, \nu)$.

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The next step is to show the *domain monotonicity* of the eigenvalues of Δ_D and Δ_N, and also, find a relation between the corresponding eigenvalues of Δ_D^{Ω} and Δ_N^{Ω} on the same domain Ω.

We recall the max-min principle for the eigenvalues of self-adjoint positive operators. Consider a domain Ω in $\mathbb{R}^m$. Let $E(f) := \overline{Q}(f,f)/||f||_{L^2}^2$ be the Rayleigh quotient. Denote by $\langle g_1, \cdots, g_n \rangle$ the span of the the functions $g_i \in L^2(\Omega),\ i = 1, \cdots, n$. According to the max-min principle, we have

$$\lambda_k^\Omega \leq E(f)\,, \qquad \forall f \in \langle \phi_1^\Omega, \cdots, \phi_{k-1}^\Omega \rangle^\perp \cap H_0^1(\Omega)\,, \tag{11}$$

$$\mu_k^\Omega \leq E(f)\,, \qquad \forall f \in \langle \psi_1^\Omega, \cdots, \psi_{k-1}^\Omega \rangle^\perp \cap H^1(\Omega)\,. \tag{12}$$

Lemma 2 *Consider the domains $\tilde{\Omega}$ and Ω, where $\tilde{\Omega} \subset \Omega \subset \mathbb{R}^m$. We have*

$$\lambda_k^\Omega \leq \lambda_k^{\tilde{\Omega}} \tag{13}$$

for each $k = 1, 2, \cdots$.

Proof Note that each $f \in L^2(\tilde{\Omega})$ can be viewed as as an element of $L^2(\Omega)$ by setting it equal to zero in $\Omega \backslash \tilde{\Omega}$. Consider the linear map

$$\begin{aligned} T : \langle \phi_1^{\tilde{\Omega}}, \cdots, \phi_k^{\tilde{\Omega}} \rangle &\to \langle \phi_1^\Omega, \cdots, \phi_{k-1}^\Omega \rangle^* \\ \phi_i^{\tilde{\Omega}} &\mapsto \langle\, .\,,\, \phi_i^{\tilde{\Omega}} \rangle, \qquad i = 1, \cdots, k \end{aligned}$$

Since $\mathrm{Ker}(T) \neq \emptyset$, there exist a function $f_0 = \sum_{i=1}^k a_i \phi_i^{\tilde{\Omega}},\ a_i \in \mathbb{R}$ such that $f_0 \in \langle \phi_1^\Omega, \cdots, \phi_{k-1}^\Omega \rangle^\perp \cap H_0^1(\Omega)$. Considering (11), we have

$$\lambda_k^\Omega \leq E(f_0) = \frac{\langle \Delta_D^{\tilde{\Omega}} f_0\,, f_0 \rangle}{||f_0||_{L^2}^2} = \frac{\sum_{i=1}^k \lambda_i^{\tilde{\Omega}}\, a_i^2}{\sum_{i=1}^k a_i^2} \leq \lambda_k^{\tilde{\Omega}}\,. \tag{14}$$

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Lemma 3 *Let Ω be a domain in $\mathbb{R}^m$. We have*

$$\mu_k^\Omega \leq \lambda_k^\Omega \tag{15}$$

for each $k = 1, 2, \cdots$.

Proof Since $H_0^1(\Omega) \subset H^1(\Omega)$, there exist a function $f_0 = \sum_{i=1}^k a_i \phi_i^\Omega,\ a_i \in \mathbb{R}$ such that $f_0 \in \langle \psi_1^\Omega, \cdots, \psi_{k-1}^\Omega \rangle^\perp \cap H^1(\Omega)$. Considering (12), we have

$$\mu_k^\Omega \leq E(f_0) = \frac{\langle \Delta_D^\Omega f_0\,, f_0 \rangle}{||f_0||_{L^2}^2} = \frac{\sum_{i=1}^k \lambda_i^\Omega\, a_i^2}{\sum_{i=1}^k a_i^2} \leq \lambda_k^\Omega\,. \tag{16}$$

■

Consider the disjoint domains Ω_1, Ω_2 in the domain $\Omega \subset \mathbb{R}^m$ such that $\Omega = \text{int}\overline{(\Omega_1 \cup \Omega_2)}$, and $\Omega \backslash \Omega_1 \cup \Omega_2$ has Lebesgue measure zero. (Intuitively, one gets $\Omega_1 \cup \Omega_2$ by eliminating a "surface" $(\Omega \backslash \Omega_1 \cup \Omega_2) \subset \text{int}\Omega$ from Ω). Note that, since $H^1(\Omega_1 \cup \Omega_2) = H^1(\Omega_1) \oplus H^1(\Omega_2)$, we have

$$\begin{aligned} \Delta_D^{\Omega_1 \cup \Omega_2} &= \Delta_D^{\Omega_1} \oplus \Delta_D^{\Omega_2} \\ \Delta_N^{\Omega_1 \cup \Omega_2} &= \Delta_N^{\Omega_1} \oplus \Delta_N^{\Omega_2} \,. \end{aligned} \tag{17}$$

Lemma 4 *Let $\Omega_1, \Omega_2 \subset \Omega$ be as above. We have*

$$\mu_k^{\Omega_1 \cup \Omega_2} \leq \mu_k^{\Omega} \,, \tag{18}$$

$$\lambda_k^{\Omega} \leq \lambda_k^{\Omega_1 \cup \Omega_2} \tag{19}$$

for each $k = 1, 2, \cdots$.

Proof If $f \in H^1(\Omega)$, its restriction to $\Omega_1 \cup \Omega_2$ is in $H^1(\Omega_1) \oplus H^1(\Omega_2)$. So there exist a function $f_0 = \sum_{i=1}^k a_i \psi_i^{\Omega}$, $a_i \in \mathbb{R}$ such that $f_0 \in \langle \psi_1^{\Omega_1 \cup \Omega_2}, \cdots, \psi_{k-1}^{\Omega_1 \cup \Omega_2} \rangle^{\perp} \cap H^1(\Omega_1 \cup \Omega_2)$. In addition, since $\Omega \backslash \Omega_1 \cup \Omega_2$ has Lebesgue measure zero, we have

$$\int_{\Omega_1 \cup \Omega_2} |\text{Grad} f_0|^2 d^m x = \int_{\Omega} |\text{Grad} f_0|^2 d^m x \,. \tag{20}$$

Thus, considering (12), we get

$$\begin{aligned} \mu_k^{\Omega_1 \cup \Omega_2} \, ||f_0||_{L^2}^2 \leq \int_{\Omega_1 \cup \Omega_2} |\text{Grad} f_0|^2 d^m x &= \int_{\Omega} |\text{Grad} f_0|^2 d^m x \\ &= \langle \Delta_N^{\Omega} f_0 \,, f_0 \rangle \\ &= \sum_{i=1}^k \mu_i^{\Omega} \, a_i^2 \leq \mu_k^{\Omega} \, ||f_0||_{L^2}^2 \,. \end{aligned} \tag{21}$$

The inequality (19) is a special case of (13).

■

Definition A standard 2^{-n} cube in $\mathbb{R}^m$ is a cube of the form

$$\left[\frac{b_1}{2^n}, \frac{b_1 + 1}{2^n}\right) \times \cdots \times \left[\frac{b_m}{2^n}, \frac{b_m + 1}{2^n}\right)$$

with $b_1, \cdots, b_m$ integers. Given a Lebesgue measurable set $\Omega \subset \mathbb{R}^m$, denote by Ω_n^- (respectively, Ω_n^+) those standard 2^{-n} cubes contained in Ω (respectively, those standard 2^{-n} cubes that intersect Ω). Let $W_n^{\pm}(\Omega) = \text{Vol}(\Omega_n^{\pm})$. The set Ω is called a *contented (Jordan measurable)* set if we have

$$\lim_{n \to \infty} W_n^-(\Omega) = \lim_{n \to \infty} W_n^+(\Omega) = \text{Vol}(\Omega) \,. \tag{22}$$

Theorem 5 *Let Ω be a contented domain in $\mathbb{R}^m$. We have*

$$\lim_{\nu\to\infty} \frac{N(\Delta_D^\Omega, \nu)}{\nu^{m/2}} = \frac{\omega_m\, Vol\,(\Omega)}{(2\pi)^m}\,. \tag{23}$$

Proof Let $\{C_{n,j}^\pm\}$ be the interiors of the cubes in the definition of $\Omega_n^\pm$ so that $\bar{\Omega}_n^\pm = \bigcup_j \bar{C}_{n,j}^\pm$. By (13) and (19) we have

$$\lambda_k^\Omega \le \lambda_k^{\Omega_n^-} \le \lambda_k^{\cup_j C_{n,j}^-}\,, \tag{24}$$

which implies

$$N(\Delta_D^\Omega, \nu) \ge N(\Delta_D^{\Omega_n^-}, \nu) \ge N(\oplus_j \Delta_D^{C_{n,j}^-}, \nu) = \sum_j N(\Delta_D^{C_{n,j}^-}, \nu)\,. \tag{25}$$

Thus, by (6), we have

$$\begin{aligned}\liminf_{\nu\to\infty} \frac{N(\Delta_D^\Omega, \nu)}{\nu^{m/2}} &\ge \sum_j \lim_{\nu\to\infty} \frac{N(\Delta_D^{C_{n,j}^-}, \nu)}{\nu^{m/2}} = \sum_j \frac{\omega_m \mathrm{Vol}\,(C_{n,j}^-)}{(2\pi)^m} \\ &= \frac{\omega_m\, W_n^-(\Omega)}{(2\pi)^m}\,. \end{aligned}\tag{26}$$

On the other hand, by (15) and (18), we have

$$\lambda_k^\Omega \ge \lambda_k^{\Omega_n^+} \ge \mu_k^{\Omega_n^+} \ge \mu_k^{\cup_j C_{n,j}^+}\,, \tag{27}$$

which implies

$$N(\Delta_D^\Omega, \nu) \le \sum_j N(\Delta_N^{C_{n,j}^+}, \nu)\,. \tag{28}$$

Therefore, by (6), we have

$$\limsup_{\nu\to\infty} \frac{N(\Delta_D^\Omega, \nu)}{\nu^{m/2}} \le \sum_j \lim_{\nu\to\infty} \frac{N(\Delta_N^{C_{n,j}^+}, \nu)}{\nu^{m/2}} = \frac{\omega_m\, W_n^+(\Omega)}{(2\pi)^m}\,. \tag{29}$$

Since Ω is contented, if we let $n \to \infty$, then, using (26) and (29), we get (23). ■

References

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