Representation Theory of Finite Groups

Assignment 1
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1. (**Pontryagin Duality**) Let G be a finite abelian group and $g \neq e$ an element of G. Show that there is character χ such that $\chi(g) \neq 1$. Conclude that the duality map $c : G \to \hat{\hat{G}}$ is an injective homomorphism (and hence is surjective since $|G| = |\hat{G}| = |\hat{\hat{G}}|$). This finishes the proof of Pontryagin duality for finite abelian groups. (Hint: Use the structure theorem of finite abelian groups to reduce this to cyclic groups).

2. Let ω be a primitive n-th root of unity (e.g. $\omega = e^{2\pi i/n}$; in general there are $\varphi(n)$ primitive$n$-th roots, where φ is *Euler's phi function*). Show, by a direct computation, that the characters $\chi_p(m) = \omega^{mp}$, satisfy the orthogonality relations $< \chi_p, \chi_q >= n\delta_{pq}$. This gives an alternative proof of the orthogonality relations for the cyclic groups. Deduce the general orthogonality theorem from this.

3. Let $H \subset G$ be a subgroup of a finite abelian group. Show that for any character $\chi \in \hat{G}$

$$\sum_{h \in H} \chi(h) = \begin{cases} |H| & \chi = 1 \,\text{on H} \\ 0 & \text{otherwise} \end{cases}$$

(Hint: Use the orthogonality relation in H).

4. (**Characters for infinite groups**). Let G be an abelian group (need not be finite). A character of G is defined as when G is finite. If G happens to have a natural topology, then we insist that the character $\chi : G \to \mathbb{T}$ be a continuous map. Find all the characters of the additive group of integers $\mathbb{Z}$, $\mathbb{Z}^n$, and the real numbers $\mathbb{R}$. Notice that the isomorphism $G \simeq \hat{G}$ is not necessarily true anymore, but verify that the Pontryagin duality $G \simeq \hat{\hat{G}}$ holds for these examples.

5. (**Abelian characters for non-abelian groups**). Let G be a group (need not be finite or abelian). An *abelian, or 1-dimensional*, unitary character of G is a homomorphism $\chi : G \to \mathbb{T}$. Show that these abelian characters form an abelian group isomorphic to the dual of the abelianization

of G, $G_{ab} := G/[G, G]$. Find all abelian characters of the symmetric group S_n , and the dihedral group D_n. Find examples of non-isomorphic groups with isomorphic groups of abelian characters. Notice that in the non-abelian case, there are in general far fewer abelian characters than in the abelian case.

6. (**Group determinant for abelian groups**). Let G be a finite abelian group. Define a polynomial in variables $X_g, g \in G$, called the *group determinant* of G, by

$$\Theta(G) = \det(X_{gh^{-1}}).$$

Show that it decomposes into linear factors

$$\det(X_{gh^{-1}}) = \prod_{\chi \in \hat{G}} (\sum_{g \in G} \chi(g) X_g) \qquad (*).$$

(Hint: work inside the group algebra $\mathbb{C}[G]$. consider the linear map $L : \mathbb{C}[G] \to \mathbb{C}[G]$ which is left multiplication by $\sum a_g g$ for fixed numbers $a_g \in \mathbb{C}$. Its matrix in the basis $g \in G$ is $(a_{gh^{-1}})$. Compute its determinant by working in the basis

$$\sum_{g \in G} \chi(g) g, \quad \chi \in \hat{G}.$$

Conclude that the two sides of $*$ are equal for all values of $a_g \in \mathbb{C}$, hence they are equal as polynomials in $X_g, g \in G$. Notice that

$$\prod_{\chi \in \hat{G}} (\sum_{g \in G} \chi(g) X_g) = \prod_{\chi \in \hat{G}} (\sum_{g \in G} \chi^{-1}(g) X_g).$$

7. (**Circulants**) The determinant of the n by n matrix

$$\begin{pmatrix} X_0 & X_1 & X_2 & \dots & X_{n-1} \\ X_{n-1} & X_0 & X_1 & \dots & X_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & X_3 & \dots & X_0 \end{pmatrix}$$

is called a *circulant*. By applying the above result to $G = \mathbb{Z}/n\mathbb{Z}$, the cyclic group of order n, show that this determinant is given by

$$\begin{vmatrix} X_0 & X_1 & X_2 & \dots & X_{n-1} \\ X_{n-1} & X_0 & X_1 & \dots & X_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & X_3 & \dots & X_0 \end{vmatrix} = \prod_{j=0}^{n-1} (X_0 + \zeta^j X_1 + \dots + \zeta^{(n-1)j} X_{n-1})$$

where $\zeta \in \mathbb{C}$ is a primitive n-th root of unity.

8. Use the *Convolution Theorem* to define an explicit algebra isomorphism between the truncated polynomial algebra and the algebra $\mathbb{C}^3$ with pointwise multiplication:

$$\mathbb{C}[x]/(x^3 - 1) \simeq \mathbb{C}^3.$$

Then generalize this result to algebras $\mathbb{C}[x]/(x^n - 1)$ and $\mathbb{C}^n$.