The Laplace Operator

In mathematics and physics, the Laplace operator or Laplacian, named after Pierre-Simon de Laplace, is an unbounded differential operator, with many applications. However, in describing application of spectral theory, we restrict the attention to an open subset of Euclidean space R^d.

Definition and Self Adjointness

Let, $U \subset R^d$ be open and non empty set and $H = L^2(U)$ be the space of all square integrable functions on U. Then we define the Laplace operator $(D(\Delta), \Delta)$ as follows. The domain $D(\Delta) = C_c^\infty(U)$ is the space of smooth and compactly supported functions on U and assumed to be dense in H.

For $\phi \in D(\Delta)$, put

$$\Delta\phi = -\sum_{j=1}^{n} \frac{\partial^2 \phi}{\partial x_j^2}$$

which is again smooth, compactly supported function and hence lies in $L^2(U)$.

For differential operators in such a domain, we can not expect their self adjointness. However, we proceed to show that Laplace operator has at least self adjoint extension.

Proposition :

Let U be open and non empty open subset of R^d, and $(D(\Delta), \Delta)$ be the Laplace operator defined above. Then,

1. The Laplace operator is symmetric on $D(\Delta)$ i.e.

$$\langle \Delta\phi, \psi \rangle = \langle \phi, \Delta\psi \rangle \quad \forall\ \phi, \psi \in C_c^\infty(U)$$

2. The Laplace operator is positive i.e.

$$\langle \Delta\phi, \phi \rangle \geq 0 \ \ \forall\ \phi \in C_c^\infty(U)$$

Proof. Using integration by parts twice and using the fact that compactly supported functions are zero in an open set, we get

$$\begin{aligned}\langle \Delta\phi, \psi\rangle &= \int_U \Delta\phi(x)\overline{\psi(x)}dx \\ &= \sum_{j=1}^{n} \int_U -\partial_{x_j}^2 \phi(x)\overline{\psi(x)}dx \\ &= \sum_{j=1}^{n} \int_U -\phi(x)\overline{\partial_{x_j}^2 \psi(x)}dx \\ &= \langle \phi, \Delta\psi\rangle\end{aligned}$$

for every $\phi, \psi \in C_c^\infty(U)$ Hence Δ is symmetric.

Again, for any $\phi \in D(\Delta)$, we have

$$\begin{aligned}\langle \Delta\phi, \phi\rangle &= \int_U \Delta\phi(x)\overline{\phi(x)}dx \\ &= \sum_{j=1}^{n} \int_U -\partial_{x_j}^2 \phi(x)\overline{\phi(x)}dx \\ &= \sum_{j=1}^{n} \int_U \partial_{x_j}\phi(x)\overline{\partial_{x_j}\phi(x)}dx \\ &= \sum_{j=1}^{n} \int_U \left|\partial_{x_j}\phi(x)\right|^2 dx \\ &\geq 0 \Rightarrow \Delta \text{ is positive}\end{aligned}$$

Note.

1. Every self adjoint linear $T : H \to H$ operator is symmetric. On the other hand, symmetric linear operators need not be self adjoint. The reason is that T^* may be a proper extension if T i.e $D(T) \neq D(T^*)$. Clearly, this can not happen if $D(T)$is all of H. So

For linear operators $T : H \to H$ on complex Hilbert space H, the concepts of symmetry and self adjointness are identical.

2. If there were no negative sign in the definition, the Laplace operator would have been negative.

Theorem. Let $U \subset R^d$ be a non empty open subset. Then the Laplace operator admits self adjoint extensions. Proof : This follows at once from the general theorem of Friedrichs.

Theorem. (Friedrichs) Let, H be a Hilbert space,.$(D(T), T) \in DD^*(H)$ a symmetric operator which is positive i.e. such that

$$\langle T(v), v\rangle \geq 0,$$

for all $v \in D(T)$. Then T admits at least one self adjoint extension, called the Friedrichs Extension $(D(S), S)$ such that

$$\langle S(v), v\rangle \geq 0, \text{ for all} v \in D(S).$$

Two Basic Examples

In this unit, we will discuss two examples of Laplace operators acting on the whole space R^n and on the open cube $(0,1)^n$ and discuss their spectral properties by finding the explicit representation of self adjoint extension of Δ as multiplication operators. These operators differ in two aspects. On R^n , Δ is essentially self adjoint i.e. the closure is self adjoint and is the unique self adjoint extension of Δ. Its spectrum is purely continuous and $\sigma(\Delta) = [0, \infty)$. On the other hand, Laplace operator on cube $(0,1)^n$ has distinct extensions and in particular, we discuss two of them, the Dirichlet and the Neumanns extension. These two extensions are distinguished by specific boundary conditions. In this case,Δ turns out to have purely point spectrum. Also, the eigenvalues can be computed fairly explicitly.

Proposition :

The closure of Laplace operator on R^n is unitarily equivalent with the multiplication operator (D, T) acting on $L^2(R^n)$, where

$$D = \{\phi \in L^2(R^n)|x \to \|x\|^2 \phi(x) \in L^2(R^n)\}$$

$$T\phi(x) = (2\pi)^2 \|x\|^2 \phi(x)$$

where $\|x\|^2$ is the Euclidean norm on R^n.

In particular, Δ is essentially self adjoint on R^n, its spectrum is equal to $[0, \infty)$ and is entirely continuous spectrum.

Proof.

The main tool for proving this argument is the Fourier Transform,

$$U : L^2(R^n) \to L^2(R^n)$$

given by

$$U\phi(x) = \int_{R^n} \phi(t)e^{-2\pi i\langle x,t\rangle}dt$$

Integrating by parts, we get

$$U(\partial_{x_j}\phi)(x) = 2\pi x_j U\phi(x)$$

and from this we derive that

$$U(\Delta\phi)(x) = (2\pi)^2 \|x\|^2 U\phi(x) \text{ for} \phi \in D(\Delta)$$

We first prove that Δ is essentially self adjoint by showing $\mathrm{Im}(\Delta+i)$ is dense in $L^2(R^n)$. Let $z = i$ or $-i$. For showing $\mathrm{Im}(\Delta + z)$ is dense in $L^2(R^n)$, we shall prove its orthogonal complement

$$\mathrm{Im}(\Delta + z)^{\perp} = 0$$

Let $\phi \in L^2(R^n)$ be such that

$$\langle \phi, (\Delta + z)\psi \rangle = 0 \text{for all} \psi \in D(\Delta) = C_c^{\infty}(R^n)$$

Since, Fourier transform is unitary,

$$\begin{aligned} 0 = \langle \phi, (\Delta + z)\psi \rangle &= \langle U\phi, U(\Delta + z)\psi \rangle \\ &= \left\langle U\phi, (4\pi^2 \|x\|^2 + z)U\psi \right\rangle \\ &= \left\langle (4\pi^2 \|x\|^2 + \bar{z})U\phi, U\psi \right\rangle \text{ for all} \psi \in D(\Delta) \end{aligned}$$

Since $D(\Delta)$ is dense in $L^2(R^n)$, so is $UD(\Delta)$ and thus we have

$$\begin{aligned} &(4\pi^2 \|x\|^2 + \bar{z})U\phi = 0 \\ &\Rightarrow U\phi = 0 \\ &\Rightarrow \phi = 0 \end{aligned}$$

Thus, Δ is essentially self adjoint.

Also, the formula above shows that Laplace operator $D((\Delta), \Delta)$ is unitarily equivalent with the multiplication operator.

$$M_{4\pi^2\|.\|^2} : \phi \longmapsto 4\pi^2 \|x\|^2 \phi \quad \text{on } UD(\Delta)$$

This is essentially self adjoint.

The multiplication operator can be defined on

$$D = \{\phi \in L^2(R^n) | \, \|x\|^2 \phi(x) \in L^2(R^n)\}$$

Indeed, $(D, M_{4\pi^2\|x\|^2})$ is self adjoint and so is the closure of $(UD(\Delta), M_{4\pi^2\|x\|^2})$.

Now, from inverse Fourier transform, it follows that closure of Δ is unitarily equivalent with $(D, 4\pi^2 \|x\|^2)$.

Finally, the essential range of $M_{4\pi^2\|x\|^2}$ is $[0,\infty)$, it follows that $\sigma(\Delta) = [0,\infty)$.

Also, since there is no eigen value of the multiplication operator, the spectrum is purely continuous.

Example :

Next, we see for U=$(o,1)^n$ The main idea corresponds to expanding of function $\phi \in L^2(R^n)$ in terms of orthonormal basis of $H = L^2(U)$ formed by complex exponentials

$$e_k : x \rightarrow e^{2\pi i\langle x,k\rangle} \text{ for } k = (k_1, k_2, ..., k_n) \in Z^n$$

Note that ($e_k \in C^\infty(U)$). Moreover, viewing Δ as an differential operator and using the relation

$$e_k(x_1, x_2, ..., x_n) = e^{2\pi i k_1 x_1}.e^{2\pi i k_2 x_2}....e^{2\pi i k_n x_n}$$

and $y^{"} = \alpha^2 y$ for $y(x) = e^{\alpha x}$,we see that

$$\Delta e_k = (2\pi)^2 \|k\|^2 e_k$$

However,e_k is not compactly supported. So, the domain of this eigenfunction is different from Δ and such eigenfunction could be many.

For any

$$\alpha = (\alpha_1, ..., \alpha_n) \in C^n$$
$$f_\alpha(x) = e^{\alpha_1 x_1 + ... + \alpha_n x_n},$$

we have

$$\Delta f_\alpha = \left(\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2\right) f_\alpha$$

Using this, we confirm that Δ is not essentially self adjoint. Namely, for any $\alpha \in C^n$ with

$$\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 = \pm i \left(\text{ for instance } \alpha = \left(e^{i\pi/4}, 0, 0....0\right)\right)$$

the function f_α is an eigenfunction of Δ with eigenvalue $\pm i$.

We now check $f_\alpha \in D(\Delta^*)$and satisfies

$$\Delta^* f_\alpha = \pm i f_\alpha$$

In fact, for $\phi \in D(\alpha)$, we have

$$\langle \Delta\phi, f_\alpha \rangle = \int_U \Delta\phi(x) e^{\langle \alpha,x\rangle} dx$$

Integrating by parts and using the fact that ϕ is compactly supported (despite the fact that it is not for f_α shows that Δ is symmetric. However, by definition this proves that $(D(\Delta), \Delta)$ is not self adjoint.

The computation gives a rough idea of self adjoint extension of Δ by allowing more general functions rather than compactly supported one so that boundary conditions will force the integration by parts to get symmetric operators. But, the domain of such may be delicate to describe.

It is easy to think the situation for n=1. Define three subspaces of $L^2([0,1])$ containing $D(\Delta) = C_C^\infty((0,1))$ as follows.

First, define $\tilde{D}$ to be space of function of $\phi \in C^\infty((0,1))$ for which every derivative $\phi^{(j)}, j \geq 0$ extends to continuous function on [0,1], where we denote by $\phi^{(j)}(0)$ and $\phi^{(j)}(1)$ the corresponding value at the boundary points.

Then let

$$D_1 = \left\{\phi \in \ \tilde{D} | \phi(0) = \phi(1), \phi^{'}(0) = \phi^{'}(1), ...\right\}$$

$$D_2 = \left\{\phi \in \ \tilde{D} | \phi(0) = \phi(1) = 0\right\}$$

$$D_3 = \left\{\phi \in \ \tilde{D} | \phi^{'}(0) = \phi^{'}(1) = 0\right\}$$

These spaces are distinct and have the laplace operators (D_i, Δ), each extending to the laplace operator $(D(\Delta), \Delta)$.

Then integrating by parts

$$\begin{aligned} \langle \Delta\phi, \psi \rangle &= \int_0^1 -\phi^{"}(x)\overline{\psi(x)}dx \\ &= \int_0^1 -\phi(x)\overline{\psi^{"}(x)}dx \\ &= \langle \phi, \Delta\psi \rangle \, \forall \phi, \psi \in D_j \end{aligned}$$

So, Δ is symmetric.

Here,D_1, D_2 and D_3 are called respectively the Laplace operator with periodic, Dirichlet and Neumann's boundary conditions.

Proposition :

The three operators $(D_j, \Delta), 1 \leq j \leq 3$ are essentially self-adjoint extensions of the Laplace operator $(D(\Delta), \Delta)$ on $D(\Delta)$. Moreover, all three $\sigma(\Delta_j) = \sigma_p(\Delta_j)$-(pure point spectrum) and the eigenvalues are given by the following:

$$\sigma(D_1) = \left\{0, 4\pi^2, 16\pi^2, ..., 4\pi^2 k^2, ...\right\}$$

$$\sigma(D_2) = \left\{\pi^2, 4\pi^2, ..., k^2\pi^2, ...\right\}$$

$$\sigma(D_1) = \left\{0, \pi^2, 4\pi^2, ..., k^2\pi^2, ...\right\}$$

The spectrum is simple, i.e., the eigen spaces have dimension 1, for D_2 and D_3. For D_1, we have

$$\text{dim ker } (D_1) = 1, \text{ dim ker } \left(D_1 - (2\pi k)^2\right) = 2, \text{ for } k \geq 1$$

Corollary :

Let $U = (0,1)^n$ with $n \geq 1$. Consider the operators $\Delta_p = (D_p, \Delta)$ and $\Delta_d = (D_d, \Delta)$ extending $(D(\Delta), \Delta)$ with domains given, respectively, by D_p, which is the space of restrictions of C^∞ functions on R which are Z^n periodic, and D_d which is the space of function $\phi \in C^\infty(U)$ for which every partial derivative of any order $\partial_\alpha \phi$ extends to a continuous function on $\bar{U}$, and moreover such that $\phi(x) = 0$for $x \in \partial U$, where we use the same notation ϕ for the function and its extension to $\bar{U}$. Then Δ_p and Δ_d are essentially self-adjoint. Their closures have pure point spectra, given by the real numbers of the form

$$\lambda = 4\pi^2 \left(k_1^2 + ... + k_n^2\right), k_i \in Z$$

with the condition $k_i \geq 1$ for D_d The multiplicity of a given λ is the number of $k_1, ..., k_n \in Z^n$ with

$$\lambda = 4\pi^2 \left(k_1^2 + ... + k_n^2\right)$$

for Δ_p, and is 2^{-n} times that number for Δ_d .

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