ON THE BRUDNYĬ-KRUGLJAK DUALITY THEORY OF SPACES FORMED BY THE $\mathcal{K}$-METHOD OF INTERPOLATION

Ron Kerman, Mario Milman and Gord Sinnamon

Brock University, Florida Atlantic University
and The University of Western Ontario

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Abstract. The Brudnyĭ-Krugljak duality theory for the $\mathcal{K}$-method is elaborated for a class of parameters derived from rearrangement-invariant spaces. As examples, concrete expressions are given for the norms dual to certain interpolation spaces between two rearrangement-invariant spaces. These interpolation spaces are formed by the $\mathcal{K}$-method using parameters related to classical Lorentz spaces or Orlicz spaces.

1. Introduction

The $\mathcal{K}$-method of interpolation is a powerful tool for constructing spaces that lie between a given pair of Banach spaces. The spaces so built are interpolation spaces, they have the property that any operator bounded on both original spaces is also bounded on them. The construction is based on a Banach lattice of real-valued functions defined on the half line, called a parameter. A major advantage of the $\mathcal{K}$-method is that the norm in the new space is given by a simple formula expressed in terms of the norm in the parameter lattice and the Peetre K-functional of the original pair of spaces.

There is a natural way to define dual spaces relative to a given pair of Banach spaces and one readily sees that the dual space of an interpolation space is itself an interpolation space relative to the dual pair. Brudnyĭ and Krugljak give a general construction for the norm of this dual space and also, under mild conditions, show that the dual norm is also given by the $\mathcal{K}$-method, with a parameter constructed

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from the original parameter. When the parameter for the dual can be given explicitly, this process provides a concrete formula for the dual norm. Duality theory for the K-method, prior to the K-divisibility formula of Brudnyĭ-Krugljak, can be found in [5].

In this paper we work out the consequences of the Brudnyĭ-Krugljak duality theory for the $\mathcal{K}$-method when the parameter is closely related to a rearrangement-invariant Banach function space. To construct the parameter for the dual space we make use of the recent papers [11] and [12] of G. Sinnamon. The motivation for our work comes from certain questions in Sobolev imbedding theory; see [8].

In order to state our main results we recall a few definitions. More detailed background is given in Section 2. Let X_1 and X_2 be Banach spaces imbedded in a common Hausdorff topological vector space, $x \in X_1 + X_2$ and $t > 0$. The *Peetre K-functional* is defined by

$$K(t, x; X_1, X_2) = \inf_{x=x_1+x_2} \{\|x_1\|_{X_1} + t\|x_2\|_{X_2}\}.$$

For fixed $x \in X_1 + X_2$, K is a concave function of t. Hence, we may define the *k-functional* by

$$K(t, x; X_1, X_2) = K(0+, x; X_1, X_2) + \int_0^t k(s, x; X_1, X_2)\, ds.$$

If $X \subset X_1 + X_2$ is a Banach space satisfying

$$X_1 \cap X_2 \hookrightarrow X \hookrightarrow X_1 + X_2,$$

where $X_1 \cap X_2$ has norm $\max(\|x\|_{X_1}, \|x\|_{X_2})$ and $X_1 + X_2$ has norm $K(1, f; X_1, X_2)$, we say X is an *intermediate space* between X_1 and X_2. An intermediate space is called an *exact interpolation space* provided that for any linear operator T, $T : X_i \to X_i$, $i = 1, 2$, implies $T : X \to X$, with $\|T\|_X \le \max(\|T\|_{X_1}, \|T\|_{X_2})$.

When considering the class of spaces X intermediate between a fixed pair X_1 and X_2, the notion of the Banach dual needs to be modified so that the duals of intermediate spaces all lie in a common Hausdorff vector space. The natural space to take is $(X_1 \cap X_2)^*$, the usual Banach dual of the intersection. Accordingly, we define

$$X^{\#} := \{y \in (X_1 \cap X_2)^* : \|y\|_{X^{\#}} := \sup_{\substack{x \in X_1 \cap X_2 \\ \|x\|_X \le 1}} |\langle y, x\rangle| < \infty\}.$$

This is the Banach dual of $X_1 \cap X_2$, viewed as a subspace of X. It is effectively equal to the usual Banach dual of X whenever $X_1 \cap X_2$ is dense in X, since every element of $X^{\#}$ has a unique bounded extension from $X_1 \cap X_2$ to X. See [3, Lemma 2.4.4, p. 175].

Suppose $(\Omega, \mathcal{M}, \mu)$ is a non-atomic, totally σ-finite measure space, $M(\Omega)$ is the vector space of measurable functions on Ω and $M^+(\Omega)$ is the cone of non-negative functions in $M(\Omega)$. If $\omega : M^+(\Omega) \to [0, \infty]$, we define

$$L_\omega := \{f \in M(\Omega) : \omega(|f|) < \infty\},$$

and the Köthe dual functional

$$\omega'(g) = \sup \left\{ \int_\Omega fg\, d\mu : \omega(f) \le 1 \right\}, \quad g \in M^+(\Omega).$$

Let $\mathbf{R}_+$ denote the half line $(0,\infty)$ with the usual Lebesgue measure and fix a rearrangement-invariant (r.i.) Banach function norm $\rho : M^+(\mathbf{R}_+) \to [0,\infty]$. (See Section 3 for definitions.) For convenience, we will often identify a function f with its formula $f(t)$ in the argument of ρ and elsewhere. This will avoid the introduction of unnecessary function names by permitting expressions such as $\rho(\frac{1}{1+t})$.

In addition to the Banach function norm ρ we will consider compositions of ρ with the operators T, P, Q, R defined on $M^+(\mathbf{R}_+)$ as follows,

$$Tf(t) = f(t)/t, \quad Pf(t) = \frac{1}{t}\int_0^t f(s)\,ds, \quad Qf(t) = \int_t^\infty f(s)\frac{ds}{s},$$

and $R = P + Q = P \circ Q = Q \circ P$. We also require the operators P_d, Q_d, R_d defined by

$$P_d f = Pf^*, \quad Q_d f = Qf^*, \quad R_d f = Rf^*,$$

where f^* denotes the non-increasing rearrangement of f.

Our main results concern two Banach spaces X_1 and X_2 imbedded in a common Hausdorff topological vector space and an r.i. norm ρ defined on functions in $M^+(\mathbf{R}_+)$ and satisfying

$$\rho(\tfrac{1}{1+t}) < \infty. \tag{1.1}$$

Let X be the set of all $x \in X_1 + X_2$ such that

$$\|x\|_X := \rho(t^{-1}K(t,x;X_1,X_2)) \tag{1.2}$$

is finite. It is well known that with this norm, X is an exact interpolation space between X_1 and X_2.

Our principal result is

Theorem A. *Let X_1, X_2 and X be as above. If, in addition to* (1.1),

$$\begin{gathered} \rho'(\tfrac{1}{1+t}) < \infty, \ \rho(\chi_{(0,a)}) \downarrow 0 \text{ as } a \downarrow 0 \text{ and} \\ X_1^\# \cap X_2^\# \text{ is dense in } X_2^\#, \end{gathered} \tag{1.3}$$

then

$$\|y\|_{X^\#} \approx (\rho \circ P_d)'(k(t,y;X_2^\#,X_1^\#)), \quad y \in X^\#. \tag{1.4}$$

As we show later on, the expression equivalent to the norm of $X^\#$ in (1.4) can be given concrete form for a large class of spaces X_1 and X_2 and some r.i. norms ρ; for example, when $K(t,y;X_2^\#,X_1^\#)$ is known to within multiplicative constants and ρ is a classical Lorentz norm.

Now, in general, the k-functional can be computed only in the rare cases when the K-functional is known exactly, whereas, more often the latter is only known to within constant multiples. The following result takes these facts into account.

Theorem B. *Suppose* X_1, X_2, *and* ρ *satisfy* (1.3) *and* $\rho(R\chi_{(0,1)}) < \infty$ *or, equivalently,*

$$\rho\left(\frac{1+\log_+(1/t)}{1+t}\right) < \infty. \tag{1.5}$$

Then, $(\rho \circ P_d)'$ *is equivalent to* $\sigma \circ P_d$ *for some r.i. norm* σ *on* $M^+(\mathbf{R}_+)$ *if and only if*

$$(\rho \circ P_d)' \approx (\rho \circ R_d)' \circ P_d. \tag{1.6}$$

In that case, the space X *defined in* (1.2) *satisfies*

$$\|y\|_{X^\#} \approx (\rho \circ R_d)'(t^{-1}K(t, y; X_2^\#, X_1^\#)), \quad y \in X^\#.$$

The applications we have in mind require X_1 and X_2 to be r.i. function spaces in the sense of [1]. See Definition 3.1, below, for the definition of an r.i. norm. Theorems A and B can be combined to yield in this context

Theorem C. *Suppose* ω_1 *and* ω_2 *are r.i. norms on the class* $M^+(\Omega)$, *where* $(\Omega, \mathcal{M}, \mu)$ *is a non-atomic, totally* σ*-finite measure space. Assume, further,*

$$\begin{gathered} L_{\omega_1'} \cap L_{\omega_2'} \text{ is dense in } L_{\omega_2'} \text{ and} \\ \omega_2'(\chi_{E_n}) \downarrow 0 \text{ as } E_n \downarrow \emptyset, E_n \in \mathcal{M}, n = 1, 2, \ldots \end{gathered} \tag{1.7}$$

Then, for any r.i. norm ρ *on* $M^+(\mathbf{R}_+)$ *such that* $\rho(\frac{1}{1+t}) < \infty$, *the functional*

$$\omega(f) := \rho(t^{-1}K(t, f; L_{\omega_1}, L_{\omega_2})), \quad f \in M^+(\Omega),$$

is an r.i. norm.

Moreover, if, also,

$$\rho'(\tfrac{1}{1+t}) < \infty \text{ and } \rho(\chi_{(0,a)}) \downarrow 0 \text{ as } a \downarrow 0,$$

then,

$$\omega'(g) \approx (\rho \circ P_d)'(k(t, g; L_{\omega_2'}, L_{\omega_1'})), \quad g \in M^+(\Omega). \tag{1.8}$$

Finally, the additional requirements (1.5) *and* (1.6) *on* ρ *ensure*

$$\omega'(g) \approx (\rho \circ R_d)'(t^{-1}K(t, g; L_{\omega_2'}, L_{\omega_1'})), \quad g \in M^+(\Omega).$$

The proofs of the above theorems ultimately depend on our next result which is itself of independent interest. Formula (1.9) below, in particular, generalizes the one obtained by Grahame Bennett in [2, (21.13)] for the dual of the so-called Cesaro norm,

$$\rho_{\mathrm{Ces}(p)} := \left(\int_{\mathbf{R}_+} \left(\frac{1}{t}\int_0^t |f(s)|\, ds\right)^p dt\right)^{1/p}, \quad 1 < p < \infty, f \in M(\mathbf{R}_+).$$

Theorem D. *Suppose ρ is an r.i. norm on $M^+(\mathbf{R}_+)$ satisfying $\rho(\frac{1}{1+t}) < \infty$. Then, $\rho \circ P$ is a Banach function norm on $M^+(\mathbf{R}_+)$ and $\rho \circ R$ satisfies all axioms but (A_6) in Definition 3.1. Their Köthe duals are such that*

$$(\rho \circ P)'(g) \approx (\rho \circ P_d)'\big(\sup_{t \le s} g(s)\big) \tag{1.9}$$

and

$$(\rho \circ R)'(g) \approx (\rho \circ P_d)'\left(\frac{d\hat{G}}{dt}\right) + G(0+)\gamma_\rho, \quad g \in M^+(\mathbf{R}_+); \tag{1.10}$$

here,

$$G(t) := \sup_{s \in \mathbf{R}_+} \min(t,s) g(s), \quad t \in \mathbf{R}_+,$$

is the least quasiconcave majorant of $tg(t)$, $\hat{G}$ is the least concave majorant of G and

$$\gamma_\rho := \sup_{f \ne 0} \frac{f^*(0+)}{\rho(Pf^*)}.$$

The outline of the paper is as follows. In Section 2 we sketch the necessary background on the Brudnyĭ-Krugljak duality theory. Section 3 introduces Banach function norms with special attention paid to rearrangement-invariant Banach function norms. The proofs of Theorem D, Theorem A, Theorem B and Theorem C are given in Sections 4, 5, 6, and 7, respectively. Concrete examples involving classical Lorentz and Orlicz norms are presented in Section 8.

2. General Background

Suppose Φ is a Banach space of functions in $M(\mathbf{R}_+)$ which is a Banach function lattice in the sense that $|f| \le |g|$ a.e. and $g \in \Phi$ imply $f \in \Phi$, with $\|f\|_\Phi \le \|g\|_\Phi$. This, of course, implies $\|f\|_\Phi = \||f|\|_\Phi$. When $\min(1,t) \in \Phi$, we say Φ is a parameter of the $\mathcal{K}$-method. In this case the class

$$\mathcal{K}_\Phi = \mathcal{K}_\Phi(X_1, X_2) := \{x \in X_1 + X_2 : \|x\|_{\mathcal{K}_\Phi} = \|K(t,x;X_1,X_2)\|_\Phi < \infty\}$$

turns out to be a Banach space with norm $\| \ \|_{\mathcal{K}_\Phi}$, this space being said to be formed by the $\mathcal{K}$-method of interpolation. It is an exact interpolation space between X_1 and X_2. See [3, Proposition 3.31, p. 338]. The Brudnyĭ-Krugljak description of $\mathcal{K}_\Phi^\#$ involves the construction of a new parameter which is related to Φ and is itself constructed by interpolation methods from a pair of weighted Lebesgue spaces.

Given a non-negative (weight) function $w \in M^+(\mathbf{R}_+)$ and an index p, $1 \le p \le \infty$, set

$$L_p(w) := \{f \in M(\mathbf{R}_+) : \|f\|_{p,w} := \|fw\|_p < \infty\};$$

here, as usual,

$$\|g\|_p = \begin{cases} \left(\int_{\mathbf{R}_+} |g(t)|^p \, dt\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t\in\mathbf{R}_+} |g(t)|, & p = \infty. \end{cases}$$

The Banach lattice $\hat{\Phi}$ is defined by

$$\hat{\Phi} := \mathcal{K}_\Phi(L_\infty(1), L_\infty(t^{-1})).$$

According to [3, Proposition 3.1.17, p. 298],

$$K(t, f; L_\infty(1), L_\infty(t^{-1})) = \hat{f}(t), \quad f \in M(\mathbf{R}_+),$$

where $\hat{f}$ is the least concave majorant of $|f|$, so

$$\|f\|_{\hat{\Phi}} = \|\hat{f}\|_\Phi, \quad f \in M(\mathbf{R}_+). \tag{2.1}$$

Since $K(t, f; X_1, X_2)$ is concave, we conclude, from (2.1),

$$\mathcal{K}_\Phi = \mathcal{K}_{\hat{\Phi}}.$$

The Banach lattice Φ^+ associated to Φ has

$$\|g\|_{\Phi^+} := \sup_{\|f\|_\Phi \le 1} \int_{\mathbf{R}_+} |g(t)f(1/t)| \, \frac{dt}{t}, \quad f, g \in M(\mathbf{R}_+).$$

We observe that $\min(1, t) \in \Phi$ implies $L_\infty(1) \cap L_\infty(t^{-1}) \hookrightarrow \hat{\Phi}$, which, in turn, gives

$$\hat{\Phi}^+ \hookrightarrow [L_\infty(1) \cap L_\infty(t^{-1})]^+ = L_1(t^{-1}) + L_1(t^{-2}).$$

The fundamental result concerning $\mathcal{K}_\Phi^\#$ is given in terms of the so-called $\mathcal{J}$ functional.

Theorem 2.1. *(*[3, Theorem 3.7.6, pp. 426–427]*) If Φ is a parameter of the $\mathcal{K}$-method, then*

$$\mathcal{K}_\Phi(X_1, X_2)^\# \approx \mathcal{J}_{\hat{\Phi}^+}(X_1^\#, X_2^\#)$$

if and only if

$$\hat{\Phi} \setminus (L_\infty(1) \cup L_\infty(t^{-1})) \ne \emptyset.$$

The norm of $\mathcal{J}_{\hat{\Phi}^+}$ is not easy to work with. Thus, one seeks an equivalent norm given in terms of the more tractable K-functional.

Theorem 2.2. *(*[3, Theorem 3.5.5, pp. 389–390]*) Suppose $\Phi \setminus (L_1(t^{-1}) \cup L_1(t^{-2}))$ is not empty and let $\Psi = \mathcal{J}_\Phi(L_\infty(1), L_\infty(t^{-1}))$. Then,*

$$\mathcal{J}_\Phi(X_1, X_2) \approx \mathcal{K}_\Psi(X_1, X_2).$$

Putting Theorems 2.1 and 2.2 together, we obtain

Theorem 2.3. *Assume the Banach lattice* Φ *satisfies*

$$\hat{\Phi} \setminus (L_\infty(1) \cup L_\infty(t^{-1})) \neq \emptyset$$

and

$$\hat{\Phi}^+ \setminus (L_1(t^{-1}) \cup L_1(t^{-2})) \neq \emptyset.$$

Then,

$$\mathcal{K}^\#_\Phi(X_1, X_2) \approx K_\Psi(X_1^\#, X_2^\#),$$

where $\Psi = \mathcal{J}_{\hat{\Phi}^+}(L_\infty(1), L_\infty(t^{-1}))$*.*

3. Specific Background

We now focus on a special class of Banach lattice norms.

Definition 3.1. *Suppose* $(\Omega, \mathcal{M}, \mu)$ *is a totally* σ*-finite measure space. Let* $M(\Omega)$ *be the set of* μ*-measurable functions on* Ω *and* $M^+(\Omega)$ *the non-negative functions in* $M(\Omega)$*. A Banach function norm is a functional* $\omega : M^+(\Omega) \to [0, \infty]$ *satisfying*

$$(A_1) \qquad \omega(f) = 0 \text{ if and only if } f = 0 \ \mu\text{-a.e.};$$

$$(A_2) \qquad \omega(cf) = c\omega(f), \quad c \geq 0;$$

$$(A_3) \qquad \omega(f+g) \leq \omega(f) + \omega(g);$$

$$(A_4) \qquad 0 \leq f_n \uparrow f \text{ implies } \omega(f_n) \uparrow \omega(f);$$

$$(A_5) \qquad \mu(E) < \infty \text{ implies } \omega(\chi_E) < \infty;$$

$$(A_6) \qquad \mu(E) < \infty \text{ implies } \int_E f \, d\mu \leq c_E(\omega)\omega(f)$$

for some constant $c_E(\omega)$ *depending on* E *and* ω *but not on* $f \in M^+(\Omega)$*. (Notice that* A_4 *implies* $\omega(f) \leq \omega(g)$ *whenever* $0 \leq f \leq g$*.)*

Further, such a Banach function norm is said to be a rearrangement-invariant (r.i.) Banach function norm if

$$\omega(f) = \omega(g)$$

whenever f *and* g *are equimeasurable; that is, whenever*

$$\mu_f(t) = \mu_g(t), \quad t \in \mathbf{R}_+,$$

where

$$\mu_h(t) := \mu(\{x \in \Omega : |h(x)| > t\}), \quad h \in M(\Omega), t \in \mathbf{R}_+.$$

Luxemburg has shown that if $(\Omega, \mathcal{M}, \mu)$ is non-atomic, then corresponding to any r.i. norm ω on $M^+(\Omega)$ there is an r.i. norm, $\bar{\omega}$, on $M^+(\mathbf{R}_+)$ for which

$$\omega(f) = \bar{\omega}(f^*), \quad f \in M^+(\Omega). \tag{3.1}$$

Here, f^* is the nonincreasing rearrangement of f on $\mathbf{R}_+$ given by $f^* := \mu_f^{-1}$. We observe that although the operation $f \mapsto f^*$ is not subadditive, the operation $f \mapsto t^{-1} \int_0^t f^*$ is; explicitly,

$$t^{-1} \int_0^t (f+g)^*(s)\, ds \leq t^{-1} \int_0^t f^*(s)\, ds + t^{-1} \int_0^t g^*(s)\, ds,$$

for all $f, g \in M(\Omega)$ and $t \in \mathbf{R}_+$.

The Köthe dual of a Banach function norm ω is another such norm, ω', with

$$\omega'(g) := \sup_{\omega(f) \leq 1} \int_\Omega gf\, d\mu, \quad f, g \in M^+(\Omega).$$

(Indeed, one readily shows

$$\omega'(g) = \sup_{\omega(f) \leq 1} \left| \int_\Omega gf\, d\mu \right|, \quad g \in M(\Omega), f \in S(\Omega, \mu),$$

$S(\Omega, \mu)$ being the set of simple (μ-integrable) functions in $M(\Omega)$.) It obeys the Principle of Duality; that is,

$$\omega'' := (\omega')' = \omega.$$

Moreover, the Hölder inequality

$$\int_\Omega fg\, d\mu \leq \omega(f)\omega'(g)$$

holds for all $f, g \in M^+(\Omega)$ and this inequality is saturated, in the sense that, given $f \in M^+(\Omega)$ and $\varepsilon > 0$, there exists a $g_0 \in M^+(\Omega)$, $\omega'(g_0) \leq 1$, such that

$$\int_\Omega fg\, d\mu > (1 - \varepsilon)\omega(f).$$

The Hardy-Littlewood-Pólya inequality

$$\int_\Omega fg\, d\mu \leq \int_{\mathbf{R}_+} f^* g^*, \quad f, g \in M^+(\Omega),$$

holds for any σ-finite μ and ensures the Köthe dual of an r.i. norm is also an r.i. norm when μ is non-atomic.

The space $L_\omega = L_\omega(\Omega, \mu)$ is the vector space

$$\{f \in M(\Omega) : \omega(|f|) < \infty\},$$

together with the norm

$$\|f\|_{L_\omega} := \omega(|f|).$$

The normed space L_ω is called a Banach function space provided ω is a Banach function norm and is called an r.i. space provided ω is an r.i. Banach function norm.

If ω' is the Köthe dual of the Banach function norm ω, then $L_{\omega'}$ is referred to as the Köthe dual space of L_ω. Sections 3 and 4 in Chapter 1 of [1] yield

Theorem 3.2. *Let $(\Omega, \mathcal{M}, \mu)$ be a non-atomic, totally σ-finite measure space. Suppose ω is an r.i. norm on $M^+(\Omega)$, as in (3.1). Assume a closed linear subspace, X, of L_ω is a Banach lattice containing the class $S(\Omega, \mu)$. Then, the Banach dual, X^*, of X is isometrically isomorphic to the Köthe dual space $L_{\omega'}$ if and only if*

$$\omega(\chi_{E_n}) \downarrow 0 \text{ whenever } E_n \downarrow \emptyset, E_n \in \mathcal{M}, n = 1, 2, \dots .$$

The basic example of an r.i. space is L_p, $1 \le p \le \infty$, where, given $f \in M^+(\Omega)$,

$$\omega_p(f) := \left(\int_\Omega f^p \, d\mu \right)^{1/p} = \left(\int_0^\infty f^*(t)^p \, dt \right)^{1/p}, \quad 1 \le p < \infty,$$

and

$$\omega_\infty(f) := \operatorname{ess\,sup}_{x \in \Omega} f(x) = f^*(0+), \quad p = \infty.$$

One readily shows the smallest r.i. space is $L_1 \cap L_\infty$, while the largest is $L_1 + L_\infty$. Further, there is the following characterization of the r.i. spaces due to Calderón; see [1, Theorem 2.12, p. 116].

Theorem 3.3. *Let $(\Omega, \mathcal{M}, \mu)$ be a non-atomic, totally σ-finite measure space and suppose ω is a Banach function norm on $M^+(\Omega)$. Then, ω is an r.i. norm if and only if L_ω is an exact interpolation space between L_1 and L_∞.*

Let $(\Omega, \mathcal{M}, \mu)$, ω and $\bar{\omega}$ be as in Theorem 3.2. The dilation operator E_s, $s \in \mathbf{R}_+$, given at $f \in M^+(\mathbf{R}_+)$ by

$$(E_s f)(t) := \begin{cases} f(t/s), & 0 < t < s, \\ 0, & s < t, \end{cases}$$

is bounded on $L_{\bar{\omega}}$. With the norm of E_s on $L_{\bar{\omega}}$ denoted by $h_\omega(s)$, we define the lower and upper Boyd indices of L_ω as

$$i_\omega := \lim_{s \to 0+} \frac{\log(1/s)}{\log(h_\omega(s))} \quad \text{and} \quad I_\omega := \lim_{s \to \infty} \frac{\log(1/s)}{\log(h_\omega(s))},$$

respectively. They satisfy

$$1 \le i_\omega \le I_\omega \le \infty;$$

moreover,

$$i_{\omega'} = \frac{I_\omega}{I_\omega - 1} \quad \text{and} \quad I_{\omega'} = \frac{i_\omega}{i_\omega - 1}.$$

See [9, V. II, pp. 131–132].

A generalization of L_p, due to Lorentz, is the space $L_{p\,q}$. For $1 < p < \infty$, $1 \le q \le \infty$ and $f^{**}(t) := t^{-1} \int_0^t f^*(s) \, ds$,

$$\omega_{p\,q}(f) := \left(\int_0^\infty (f^{**}(t) t^{1/p - 1/q})^q \, dt \right)^{1/q},$$

when $q < \infty$, and

$$\omega_{p\,\infty}(f) := \sup_{0<t<\infty} t^{1/p} f^{**}(t).$$

It follows from a well-known inequality of Hardy that

$$\omega_{p\,p}(f) \approx \omega_p(f), \quad f \in M^+(\Omega).$$

We conclude this section with an example of pairs of spaces for which the K-functional is known, but only up to equivalence. In this situation Theorem B gives a computable result but Theorem A need not.

Theorem 3.4. *(Holmstedt's formulas,* [7, Theorem 4.1]*) Let $(\Omega, \mathcal{M}, \mu)$ be a non-atomic, totally σ-finite measure space. Fix p_1, p_2, q_1, q_2, $1 < p_1 < p_2 < \infty$, $1 \le q_1, q_2 < \infty$ and set $\frac{1}{\alpha} = \frac{1}{p_1} - \frac{1}{p_2}$. Then, with $f \in M^+(\Omega)$, $t \in \mathbf{R}_+$,*

$$\begin{aligned} &K(t, f; L_{p_1\,q_1}, L_{p_2\,q_2}) \\ &\quad\approx \left(\int_0^{t^\alpha} (f^*(s)s^{1/p_1-1/q_1})^{q_1}\,ds\right)^{1/q_1} + t\left(\int_{t^\alpha}^\infty (f^*(s)s^{1/p_2-1/q_2})^{q_2}\,ds\right)^{1/q_2} \end{aligned}$$

Further, for $1 < p_1 < \infty$, $1 \le q_1 \le \infty$,

$$K(t, f; L_{p_1\,q_1}, L_\infty) \approx \left(\int_0^{t^{p_1}} (f^*(s)s^{1/p_1-1/q_1})^{q_1}\,ds\right)^{1/q_1}, \quad q_1 < \infty$$

and

$$K(t, f; L_{p_1\,\infty}, L_\infty) \approx \sup_{0<s<t^{p_1}} f^*(s)s^{1/p_1}, \quad q_1 = \infty.$$

4. Proof of Theorem D.

It is a straightforward exercise to verify the $\rho \circ T$ and $\rho \circ P$ are Banach function norms, given the conditions on ρ. It is also routine to check that $\rho \circ R$ satisfies conditions (A_1)-(A_5) and satisfies (A_6), provided the set E is required to have compact support in $\mathbf{R}_+$.

Since the kernel of P, namely, $k(t,s) = t^{-1}\chi_{(0,t)}(s)$ is nonincreasing in s for each t, we obtain, from [12, Theorem 3.3],

$$(\rho \circ P)'(g) = \sup_{f \ge 0} \frac{\int_{\mathbf{R}_+} f(t)g(t)\,dt}{(\rho \circ P)(f)} = \sup_{f \ge 0} \frac{\int_{\mathbf{R}_+} f(t)\sup_{t \le s} g(s)\,dt}{(\rho \circ P)(f)}. \tag{4.1}$$

Also, according to [11, Proposition 2.1 and Lemma 3.2], the level function, $f^o := \frac{d\hat{F}}{dt}$, of $f \ge 0$ ($F(t) = \int_0^t f(s)\,ds$) is nonincreasing and satisfies

$$\int_{\mathbf{R}_+} f(t)h(t)\,dt \le \int_{\mathbf{R}_+} f^o(t)h(t)\,dt, \quad 0 \le h \downarrow,$$

and

$$(\rho \circ P)(f^o) \leq 3(\rho \circ P)(f), \quad \rho \text{ an r.i. norm.}$$

Therefore,

$$\begin{aligned}
\sup_{f\geq 0} \frac{\int_{\mathbf{R}_+} f(t) \sup_{t\leq s} g(s)\, dt}{(\rho \circ P)(f)} &\leq 3 \sup_{f\geq 0} \frac{\int_{\mathbf{R}_+} f^o(t) \sup_{t\leq s} g(s)\, dt}{(\rho \circ P)(f^o)} \\
&\leq 3 \sup_{0\leq f\downarrow} \frac{\int_{\mathbf{R}_+} f(t) \sup_{t\leq s} g(s)\, dt}{(\rho \circ P)(f)} \\
&\leq 3 \sup_{f\geq 0} \frac{\int_{\mathbf{R}_+} f(t) \sup_{t\leq s} g(s)\, dt}{(\rho \circ P)(f)}.
\end{aligned} \tag{4.2}$$

We conclude from (4.1) and (4.2) that

$$(\rho \circ P)'(g) \approx (\rho \circ P_d)'(\sup_{t\leq s} g(s)), \quad g \in M^+(\mathbf{R}_+).$$

Next, if $h(s) = f(s)/s$, then,

$$(Rf)(t) = \int_{\mathbf{R}_+} \min(1/t, 1/s) f(s)\, ds = \int_{\mathbf{R}_+} \min(s/t, 1) h(s)\, ds := (R_1 h)(t).$$

But, the kernel $k(t,s) = \min(s/t, 1)$ of R_1 is a quasiconcave function of s for each t, so, by [12, Theorem 4.1],

$$\begin{aligned}
(\rho \circ R)'(g) &= \sup_{f\geq 0} \frac{\int_{\mathbf{R}_+} f(t) g(t)\, dt}{(\rho \circ R)(f)} \\
&= \sup_{h\geq 0} \frac{\int_{\mathbf{R}_+} h(t) t g(t)\, dt}{(\rho \circ R_1)(h)} \\
&= \sup_{h\geq 0} \frac{\int_{\mathbf{R}_+} h(t) G(t)\, dt}{(\rho \circ R_1)(h)} \\
&= \sup_{f\geq 0} \frac{\int_{\mathbf{R}_+} f(t) G(t)/t\, dt}{(\rho \circ R)(f)}.
\end{aligned}$$

Since G is quasiconcave, we have $G \leq \hat{G} \leq 2G$ and $\hat{G}(0+) = G(0+)$. Therefore, the last expression is equivalent to

$$\begin{aligned}
&\sup_{f\geq 0}\frac{\int_{\mathbf{R}_+} f(t)((P\frac{d\hat{G}}{dt})(t)+G(0+)/t)\,dt}{(\rho\circ R)(f)}\\
\approx&\sup_{f\geq 0}\frac{\int_{\mathbf{R}_+} f(t)(P\frac{d\hat{G}}{dt})(t)\,dt}{(\rho\circ R)(f)}+G(0+)\sup_{f\geq 0}\frac{\int_{\mathbf{R}_+} f(t)/t\,dt}{(\rho\circ R)(f)}\\
=&\sup_{f\geq 0}\frac{\int_{\mathbf{R}_+} (Qf)(t)\frac{d\hat{G}}{dt}(t)\,dt}{(\rho\circ P)(Qf)}+G(0+)\sup_{f\geq 0}\frac{\int_{\mathbf{R}_+} f(t)/t\,dt}{(\rho\circ P)(Qf)},\ \text{by Fubini's Theorem,}\\
=&\sup_{0\leq F\downarrow}\frac{\int_{\mathbf{R}_+} F(t)\frac{d\hat{G}}{dt}(t)\,dt}{(\rho\circ P)(F)}+G(0+)\sup_{0\leq F\downarrow}\frac{F(0)}{(\rho\circ P)(F)}\\
\approx&(\rho\circ P_d)'\left(\frac{d\hat{G}}{dt}\right)+G(0+)\gamma_\rho.
\end{aligned}$$

The last line follows from (4.1), $0\leq \frac{d\hat{G}}{dt}\downarrow$ and $(Pf^*)(0+)=f^*(0+)$.

5. Proof of Theorem A

We will need two preliminary results.

Lemma 5.1. *Let ρ be an r.i. norm on $M^+(\mathbf{R}_+)$ for which $\rho(\frac{1}{1+t})<\infty$ and $\rho(\chi_{(0,a)})\downarrow 0$ as $a\downarrow 0$. If $\Phi=L_{\rho\circ T}$ then, $\hat{\Phi}\setminus(L_\infty(1)\cup L_\infty(t^{-1}))\neq\emptyset$.*

Proof. We construct a function g on $\mathbf{R}_+$ satisfying $g(t)\uparrow\infty$, $g(t)/t\downarrow$, $\rho(g(t)/(1+t))<\infty$ and $\rho(g(t^{-1})\chi_{(0,1)}(t))<\infty$. This yields the quasiconcave function

$$f(t):=tg(t^{-1})\chi_{(0,1)}(t)+g(t)\chi_{(1,\infty)}(t)$$

in $\hat{\Phi}\setminus(L_\infty(1)\cup L_\infty(t^{-1})$.

Using the hypothesis $\rho(\chi_{(0,a)})\downarrow 0$ as $a\downarrow 0$, it is a simple matter to construct an unbounded function $h\in L_\rho$, with $h^*(1)=1$. Let $t_0=0$ and define t_n, inductively, to be the least $t>2t_{n-1}+1$ such that

$$\rho\left(\frac{\chi_{(t_n,\infty)}(t)}{1+t}\right)<\frac{1}{2^{2n}}\rho(\tfrac{1}{1+t})\quad\text{and}\quad h^*(t_n^{-1})\geq 2^n.$$

Set

$$g(t):=\begin{cases}2^{n-1}, & 2t_{n-1}<t<t_n,\\ 2^{n-1}t/t_n, & t_n<t<2t_n,\quad n=1,2,\ldots.\end{cases}$$

Then, $g(t) \uparrow \infty$, $g(t)/t \downarrow$, $g(t) \le h^*(t^{-1})$ on $(1,\infty)$ (since, on $t_n < t < 2t_n$, $2^{n-1}t/t_n \le 2^n \le h^*(t_n^{-1}) \le h^*(t^{-1})$) and

$$\begin{aligned}\rho\left(\frac{g(t)}{1+t}\right) &\le \sum_{n=1}^{\infty} \rho\left(\frac{g(t)\chi_{(t_{n-1},t_n)}(t)}{1+t}\right)\\ &\le \sum_{n=1}^{\infty} \rho\left(\frac{2^n\chi_{(t_{n-1},t_n)}(t)}{1+t}\right)\\ &\le \sum_{n=1}^{\infty} 2^n\rho\left(\frac{\chi_{(t_{n-1},\infty)}(t)}{1+t}\right)\\ &\le \sum_{n=1}^{\infty} 2^n\frac{1}{2^{2(n-1)}}\rho\left(\frac{1}{1+t}\right)\\ &\le 4\rho(\tfrac{1}{1+t}) < \infty\end{aligned}$$

Finally,

$$\rho(g(t^{-1})\chi_{(0,1)}(t)) \le \rho(h^*(t)\chi_{(0,1)}(t)) \le \rho(h^*) < \infty.$$

Lemma 5.2. *Suppose ρ is an r.i. norm on $M^+(\mathbf{R}_+)$ for which $\rho'(\frac{1}{1+t}) < \infty$. If $\Phi = L_{\rho\circ T}$ then, $\hat{\Phi}^+ \setminus (L_1(t^{-1}) \cup L_1(t^{-2})) \ne \emptyset$.*

Proof. Set $f(t) = \min(1,t)$. Then, $f \notin L_1(t^{-1}) \cup L_1(t^{-2})$. But,

$$\|f\|_{\hat{\Phi}^+} = \sup_{g\ge 0, \|g\|_{\hat{\Phi}}\le 1} \int_{\mathbf{R}_+} g(t^{-1})f(t)\,\frac{dt}{t}$$

and

$$\begin{aligned}\int_{\mathbf{R}_+} g(t^{-1})f(t)\,\frac{dt}{t} &= \int_{\mathbf{R}_+} g(t)f(t^{-1})\,\frac{dt}{t}\\ &\le \int_{\mathbf{R}_+} \frac{\hat{g}(t)}{t}\min(1,t^{-1})\,dt\\ &\le \rho\left(\frac{\hat{g}(t)}{t}\right)\rho'(\min(1,t^{-1}))\\ &\le 2\rho'(\tfrac{1}{1+t}).\end{aligned}$$

We are now ready to prove Theorem A. Set $\Phi = L_{\rho\circ T}$. The space X is $\mathcal{K}_\Phi(X_1, X_2)$ and, since $\rho\circ T(\min(1,t)) \le 2\rho(\frac{1}{1+t}) < \infty$, it is an exact interpolation space between X_1 and X_2.

Now, Lemmas 5.1 and 5.2 ensure the hypotheses of Theorem 2.3 hold for our Banach lattice Φ, so we have

$$X^\# \approx \mathcal{K}_\Psi(X_1^\#, X_2^\#),$$

where $\Psi = J_{\hat{\Phi}+}(L_\infty(1), L_\infty(t^{-1}))$.

The discussion following Definition 3.7.1 on page 422 of [3] identifies elements of $(L_1(t^{-1}) \cap L_1(t^{-2}))^*$ with functions in such a way that

$$\begin{gathered} L_1(t^{-1})^{\#} = L_1(t^{-1})^{+} = L_\infty(1), \\ L_1(t^{-2})^{\#} = L_1(t^{-2})^{+} = L_\infty(t^{-1}), \text{ and} \\ \mathcal{K}_\Phi(L_1(t^{-1}), L_1(t^{-2}))^{\#} = \mathcal{K}_\Phi(L_1(t^{-1}), L_1(t^{-2}))^{+}. \end{gathered}$$

Using these identifications, and applying Theorem 2.1 with X_1 and X_2 replaced by $L_1(t^{-1})$ and $L_1(t^{-2})$, respectively, we obtain

$$\Psi \approx \mathcal{K}_\Phi(L_1(t^{-1}), L_1(t^{-2}))^{+}.$$

According to [3, Proposition 3.1.17, pp 298f] (note the misprints in the statement and the proof)

$$K(s, f; L_1(t^{-1}), L_1(t^{-2})) = \int_{\mathbf{R}_+} |f(t)| \min(1, s/t) \frac{dt}{t} = sR \circ T(|f|)(s).$$

Hence, from (1.2),

$$\|f\|_{\mathcal{K}_\Phi(L_1(t^{-1}), L_1(t^{-2}))} = \rho \circ R(T(|f|)).$$

Combining these to eliminate Ψ, and applying Theorem D, yields

$$\begin{aligned} \|y\|_{X^{\#}} &\approx \|K(t, y, X_1^{\#}, X_2^{\#})\|_\Psi \\ &= \sup_{\rho \circ R(T(|f|)) \le 1} \int_{\mathbf{R}_+} K(t, y; X_1^{\#}, X_2^{\#}) |f(1/t)| \frac{dt}{t} \\ &= \sup_{\rho \circ R(T(|f|)) \le 1} \int_{\mathbf{R}_+} K(t^{-1}, y; X_1^{\#}, X_2^{\#})(T|f|)(t)\, dt \\ &\approx (\rho \circ P_d)' \left(\frac{d\hat{G}}{dt} \right) + G(0+)\gamma_p. \end{aligned}$$

Here

$$\begin{aligned} \hat{G}(t) &= \sup_{s>0} \min(t, s) K(s^{-1}, y; X_1^{\#}, X_2^{\#}) \\ &= K(t, y; X_2^{\#}, X_1^{\#}), \quad \text{[1, Proposition 1.2, p. 294]} \end{aligned}$$

so, from (1.3), $G(0+) = 0$ and $\frac{d\hat{G}}{dt}(t) = k(t, y; X_2^{\#}, X_1^{\#})$. Therefore,

$$\|y\|_{X^{\#}} \approx (\rho \circ P_d)'(k(t, y; X_2^{\#}, X_1^{\#})), \quad y \in X^{\#}.$$

This completes the proof.

6. Proof of Theorem B

One readily verifies that $\rho\circ P_d$ and $\rho\circ R_d$ are r.i. norms, condition (1.5) ensuring both satisfy (A_5). Also, $(\rho\circ R_d)'\circ P_d$ is seen to be an r.i. norm, since

$$(\rho\circ R_d)'\circ P_d(\chi_{(0,1)}) = \sup_{g\geq 0}\frac{\int_0^1 Qg}{\rho\circ R_d(g)} \leq \sup_{g\geq 0}\frac{\int_0^1 R_d g}{\rho(R_d g)} \leq \rho'(\chi_{(0,1)}) < \infty.$$

If $f\geq 0$,

$$(\rho\circ R_d)'(P_d f) \leq \sup_{g\geq 0}\frac{\int (Pf^*)g^*}{\rho\circ R(g^*)} = \sup_{g\geq 0}\frac{\int f^*(Qg^*)}{\rho\circ P_d(Qg^*)} \leq (\rho\circ P_d)'(f^*) = (\rho\circ P_d)'(f).$$

For the reverse inequality we observe that for all $g\geq 0$,

$$\rho\circ R_d(g) = \rho\circ P_d(Qg^*) \leq \sup_{h\geq 0}\frac{\int (Qg^*)h^*}{(\rho\circ P_d)'(h)} \approx \sup_{h\geq 0}\frac{\int g^*(Ph^*)}{\sigma\circ P_d(h)} \leq \sigma'(g^*) = \sigma'(g)$$

and, therefore,

$$\begin{aligned}(\rho\circ P_d)'(f) \approx \sigma(Pf^*) &= \sup_{g\geq 0}\frac{\int (Pf^*)g}{\sigma'(g)}\\ &\lessapprox \sup_{g\geq 0}\frac{\int (Pf^*)g}{\rho\circ R_d(g)} = (\rho\circ R_d)'(Pf^*) = (\rho\circ R_d)'(P_d f).\end{aligned}$$

When, in (1.4), $(\rho\circ P_d)'$ can be replaced by $(\rho\circ R_d)'\circ P_d$, we have,

$$\|y\|_{X^\#} \approx (\rho\circ R_d)'(P_d(k(t,y;X_2^\#,X_1^\#))) = (\rho\circ R_d)'(t^{-1}K(t,y;X_2^\#,X_1^\#)),$$

for all $y\in X^\#$.

Corollary 6.1. *Let X_1, X_2, ρ and X be as in Theorem B. Assume, in addition, the upper Boyd index, I_ρ, of L_ρ is finite. Then,*

$$\|y\|_{X^\#} \approx (\rho\circ R_d)'(t^{-1}K(t,y;X_2^\#,X_1^\#)),\quad y\in X^\#.$$

Proof. We have $(\rho\circ R_d)'\circ P_d(f) \leq (\rho\circ P_d)'(f)$ just as in the proof of Theorem B. According to [9], $I_\rho<\infty$ if and only if $Q: L_\rho\to L_\rho$. Thus,

$$(\rho\circ P_d)'(f) \leq \sup_{g\geq 0}\frac{\int f^*g^*}{\rho(Pg^*)} \lessapprox \sup_{g\geq 0}\frac{\int f^*g^*}{\rho(Q(Pg^*))} \leq \sup_{g\geq 0}\frac{\int (P_d f)g^*}{\rho\circ R_d(g^*)} \leq (\rho\circ R_d)'(P_d f).$$

7. Proof of Theorem C

As a special case of [3, Proposition 3.3.1, p. 338] we obtain that L_ω is an exact interpolation space between L_{ω_1} and L_{ω_2}. In particular, ω satisfies (A_1), (A_2), and (A_3) in Definition 3.1.

Consider $E \subset \Omega$, $\mu(E) < \infty$. We have

$$K(t, \chi_E; L_{\omega_1}, L_{\omega_2}) \leq \min(\omega_1(\chi_E), t\omega_2(\chi_E)),$$

so

$$\omega(\chi_E) \leq 2\max(\omega_1(\chi_E), \omega_2(\chi_E))\rho(\tfrac{1}{1+t}) < \infty.$$

For $f \in M^+(\Omega)$, with $f = f_1 + f_2$, $0 \leq f_i \in L_{\omega_i}$, $i = 1, 2$,

$$\begin{aligned}
\omega(f) &= \rho(t^{-1} \inf_{f=f_1+f_2} \omega_1(f_1) + t\omega_2(f_2)) \\
&\geq \rho\left(t^{-1} \inf_{f=f_1+f_2} c_E(\omega_1)^{-1} \int_E f_1\, d\mu + tc_E(\omega_2)^{-1} \int_E f_2\, d\mu\right) \\
&\geq \rho(t^{-1} \min(c_E(\omega_1)^{-1}, tc_E(\omega_2)^{-1})) \int_E f\, d\mu.
\end{aligned}$$

This gives us (A_5) and (A_6) for ω.

As for (A_4), [1, Exercise 5, p. 175] and (1.7) guarantee $K(t, f; L_{\omega_1}, L_{\omega_2})$ is a Banach function norm for each $t > 0$, so for each $t > 0$, $f, f_n \in M^+(\Omega)$, $n = 1, 2, \ldots$, $0 \leq f_n \uparrow f$ implies

$$t^{-1}K(t, f_n; L_{\omega_1}, L_{\omega_2}) \uparrow t^{-1}K(t, f; L_{\omega_1}, L_{\omega_2})$$

and hence

$$\omega(f_n) \uparrow \omega(f).$$

Thus, ω is a Banach function norm.

Now, Theorem 3.3 tells us that L_{ω_1} and L_{ω_2} are exact interpolation spaces between L_1 and L_∞. Since we know L_ω is an exact interpolation space between L_{ω_1} and L_{ω_2} it follows that L_ω is an exact interpolation space between L_1 and L_∞. We conclude, from Theorem 3.3 again, that ω is an r.i. norm on $M^+(\Omega)$.

Next, $\omega_2(E_n) \downarrow 0$ as the measurable sets $E_n \downarrow \emptyset$ and so

$$0 \leq \omega(E_n) \leq t\omega_2(E_n) \downarrow 0 \text{ as } E_n \downarrow \emptyset.$$

Since $L_{\omega_1} \cap L_{\omega_2}$ contains the simple functions, [1, Theorem 4.1 p. 20] shows that $L^{\#}_{\omega_2}$ and $L^{\#}_{\omega}$ are isometrically isomorphic to $L_{\omega_2'}$ and $L_{\omega'}$ respectively. Fix a functional $y \in L^{\#}_{\omega}$ and its corresponding function $g \in L_{\omega'}$. If $g = g_1 + g_2$ with $g_1 \in L_{\omega_1'}$ and $g_2 \in L_{\omega_2'}$ then g_1 and g_2 determine functionals $y_1 \in L^{\#}_{\omega_1}$ and $y_2 \in L^{\#}_{\omega_2}$ that have the same norms as g_1 and g_2 and satisfy $y = y_1 + y_2$. On the other hand, if $y = y_1 + y_2$

with $y_1 \in L^{\#}_{\omega_1}$ and $y_2 \in L^{\#}_{\omega_2}$ then y_2 corresponds to a function $g_2 \in L_{\omega_2'}$ with the same norm as y_2. A calculation shows that the functional corresponding to $g - g_2$ coincides with y_1. Since the decompositions $y = y_1 + y_2$ and $g = g_1 + g_2$ correspond isometrically, we readily obtain

$$K(t, y; L^{\#}_{\omega_2}, L^{\#}_{\omega_1}) = K(t, g; L_{\omega_2'}, L_{\omega_1'})$$

for all $t > 0$. Differentiation shows that the same relationship holds for the k-functional.

Theorems A and B yield

$$\omega'(g) = \|y\|_{L^{\#}_{\omega}} \approx (\rho \circ P_d)'(k(t, y; L^{\#}_{\omega_2}, L^{\#}_{\omega_1})) = (\rho \circ P_d)'(k(t, g; L_{\omega_2'}, L_{\omega_1'}))$$

and, under the additional assumptions,

$$\omega'(g) = \|y\|_{L^{\#}_{\omega}} \approx (\rho \circ R_d)'(t^{-1}K(t, y; L^{\#}_{\omega_2}, L^{\#}_{\omega_1})) = (\rho \circ R_d)'(t^{-1}K(t, g; L_{\omega_2'}, L_{\omega_1'})).$$

8. Examples

1. Classical Lorentz Spaces. Fix p, $1 \le p < \infty$ and let φ be a non-negative, locally integrable (weight) function on $\mathbf{R}_+$. At $f \in M^+(\mathbf{R}_+)$, the classical Lorentz functional $\rho = \rho_{\varphi,p}$ is given by

$$\rho(f) := \left(\int_{\mathbf{R}_+} f^*(t)^p \varphi(t)\, dt \right)^{1/p}.$$

This functional is equivalent to an r.i. norm on $M^+(\mathbf{R}_+)$ if and only if there exists a $C > 0$ such that

$$t^{-1} \int_0^t \varphi(y)\, dy \le C s^{-1} \int_0^s \varphi(y)\, dy, \quad 0 < s \le t, \text{ when } p = 1, \tag{8.1i}$$

$$t^p \int_t^\infty s^{-p} \varphi(s)\, ds \le C \int_0^t \varphi(s)\, ds, \quad t \in \mathbf{R}_+, \text{ when } 1 < p < \infty. \tag{8.1ii}$$

See [4, Theorem 2.3] for $p = 1$ and [10, Theorem 4] for $1 < p < \infty$. By [10, Theorem 4], the condition in (8.1ii) is equivalent to

$$\rho(f) = \rho(f^*) \approx \rho(Pf^*) = \rho \circ P_d(f), \quad f \in M^+(\mathbf{R}_+).$$

So, given $g \in M^+(\mathbf{R}_+)$,

$$(\rho \circ P_d)'(g) \approx \sup_{t \in \mathbf{R}_+} \frac{(Pg^*)(t)}{(R\varphi)(t)},$$

when $p = 1$ by [4, Theorem 2.3], and

$$(\rho \circ P_d)'(g) \approx \rho'(g) \approx \left(\int_{\mathbf{R}_+} \left(\frac{(Pg^*)(t)}{(P\varphi)(t)} \right)^{p'} \varphi(t)\, dt \right)^{1/p'},$$

when $1 < p < \infty$, $p' = p/(p-1)$ and $\int_{\mathbf{R}_+} \varphi(t)\, dt = \infty$ by [10, Theorem 4].

Accordingly, Theorem C becomes

Theorem 8.1. *Suppose ω_1 and ω_2 are r.i. norms on $M^+(\Omega)$, where $(\Omega,\mathcal{M},\mu)$ is a non-atomic, totally σ-finite measure space. Assume, further,*

$$L_{\omega_1'} \cap L_{\omega_2'} \text{ is dense in } L_{\omega_2'} \text{ and}$$
$$\omega_2'(\chi_{E_n}) \downarrow 0 \text{ as } E_n \downarrow \emptyset, E_n \in \mathcal{M}, n = 1,2,\ldots$$

Fix p, $1 \le p < \infty$, and let φ be a locally integrable function in $M^+(\mathbf{R}_+)$ satisfying

$$\rho_{\varphi,1}(\tfrac{1}{1+t}) = \int_{\mathbf{R}_+} \frac{\varphi(t)}{1+t}\,dt < \infty$$

and (8.1i)*, or, when $1 < p < \infty$,*

$$\rho_{\varphi,p}(\tfrac{1}{1+t})^p = \int_{\mathbf{R}_+} \frac{\varphi(t)}{(1+t)^p}\,dt < \infty$$

and (8.1ii)*. Then, the functional*

$$\omega(f) := \rho_{\varphi,p}(t^{-1}K(t,f;L_{\omega_1},L_{\omega_2})), \quad f \in L_{\omega_1} + L_{\omega_2},$$

is an r.i. norm on $M^+(\Omega)$.

Moreover, if, in addition,

$$\rho'_{\varphi,1}(\tfrac{1}{1+t}) = \sup_{t\in\mathbf{R}_+} \frac{\log(1+t)}{t(R\varphi)(t)} < \infty,$$

or, when $1 < p < \infty$

$$\rho'_{\varphi,p}(\tfrac{1}{1+t})^{p'} = \int_{\mathbf{R}_+} \left(\frac{\log(1+t)}{t(P\varphi)(t)}\right)^{p'} \varphi(t)\,dt < \infty,$$

then, for $g \in M^+(\Omega)$,

$$\omega'(g) \approx \sup_{t\in\mathbf{R}_+} \frac{K(t,g;L_{\omega_2'},L_{\omega_1'})}{t(R\varphi)(t)},$$

when $p = 1$, and

$$\omega'(g) \approx \left(\int_{\mathbf{R}_+} \left(\frac{K(t,g;L_{\omega_2'},L_{\omega_1'})}{t(P\varphi)(t)}\right)^{p'} \varphi(t)\,dt\right)^{1/p'},$$

when $1 < p < \infty$, $p' = p/(p-1)$ and $\int_{\mathbf{R}_+} \varphi(t)\,dt = \infty$.

2. Orlicz Spaces. Consider a Young function given by

$$A(t) = \int_0^t a(s)\,ds, \quad t \in \mathbf{R}_+,$$

in which $a(t)$ is increasing on $\mathbf{R}_+$, $a(0+) = 0$, and $\lim_{t\to\infty} a(t) = \infty$. One defines the Luxemburg-Orlicz norm, ρ_A, of $f \in M^+(\mathbf{R}_+)$ by

$$\rho_A(f) := \inf\left\{\lambda > 0 : \int_{\mathbf{R}_+} A\left(\frac{f(t)}{\lambda}\right) dt \le 1\right\}.$$

The Köthe dual of ρ_A is equivalent to the Orlicz norm $\rho_{\tilde{A}}$, where

$$\tilde{A}(t) := \int_0^t a^{-1}(s)\,ds, \quad t \in \mathbf{R}_+,$$

is called the Young function complementary to A. Indeed,

$$\rho_{\tilde{A}}(g) \le \rho_A'(g) \le 2\rho_{\tilde{A}}(g), \quad g \in M^+(\mathbf{R}_+). \tag{8.2}$$

It is shown by A. Gogatishvili and the first author, in [6] that if, for fixed p, $1 < p < \infty$, and $A(t) = t^p/p$ when $0 < t < 1$, then

$$\rho_A \circ P_d(f) \approx \rho_{\mathcal{A}}(f), \quad f \in M^+(\mathbf{R}_+), \tag{8.3}$$

with $\mathcal{A}$ a Young function satisfying

$$\mathcal{A}(t) = t\int_1^t A(s)\,\frac{ds}{s^2}, \quad t \gg 1.$$

(The result in [6] is essentially one for large values of t, so the requirement $A(t) = t^p/p$ is made only for convenience.)

Observing that for *any* Young function A,

$$\rho_A(\chi_{(0,t)}) = \frac{1}{A^{-1}(1/t)} \downarrow 0 \text{ as } t \downarrow 0,$$

we obtain two theorems from Theorem C. The first one is

Theorem 8.2. *Let ω_1 and ω_2 be as in Theorem 8.1. Assume, further, A is a Young function, with $\rho_A(\frac{1}{1+t}) < \infty$. Then, the functional*

$$\omega(f) := \inf\left\{\lambda > 0 : \int_{\mathbf{R}_+} A((\lambda t)^{-1}K(t, f; L_{\omega_1}, L_{\omega_2}))\,dt \le 1\right\}, \tag{8.4}$$

for $f \in L_{\omega_1} + L_{\omega_2}$, is an r.i. norm on $M^+(\Omega)$.

Moreover, if, also, $A(t) = t^p/p$, where p is fixed, $1 < p < \infty$, and $0 < t < 1$, (so $\tilde{A}(t) = t^{p'}/p'$, $p' = p/(p-1)$, $0 < t < 1$), then, for $g \in M^+(\Omega)$,

$$\omega'(g) \approx \inf\left\{\lambda > 0 : \int_{\mathbf{R}_+} \tilde{A}(\lambda^{-1}k(t, g; L_{\omega_2'}, L_{\omega_1}'))\,dt \leq 1\right\},$$

in which the Young function $\tilde{A}$ has

$$(\tilde{A}')^{-1}(t) \approx \begin{cases} t^{p-1}, & 0 < t < 1, \\ \int_1^t a(s)\,\frac{ds}{s}, & t \gg 1. \end{cases}$$

To prove the second theorem we will need

Lemma 8.3. *Suppose C is a twice continuously differentiable Young function equal to t^p/p for some fixed $p > 1$ and $0 < t < 1$, with $\lim_{t\to\infty} \sup_{1<s\leq t} sC''(s) = \infty$. Assume, in addition, that*

$$C'(t) \approx \int_1^t \sup_{1<y\leq s} yC''(y)\,\frac{ds}{s}, \quad t \gg 1, \tag{8.5}$$

which holds, in particular, if $tC''(t)$ increases for $t \gg 1$. Then,

$$\rho_C \approx \rho_B \circ P_d,$$

where

$$B(t) := \begin{cases} t^p/p, & 0 < t < 1 \\ \int_1^t \sup_{1<y\leq s} yC''(y)\,ds, & t \gg 1, \end{cases} \tag{8.6}$$

and, in particular,

$$B(t) = tC'(t) - C(t), \quad t \gg 1,$$

in case $tC''(t)$ increases when $t \gg 1$.

Proof. According to the result from [6] quoted above, we need only show

$$C'(t) \approx \int_1^t B(s)\,\frac{ds}{s^2} + \frac{B(t)}{t}, \quad t \gg 1.$$

But,

$$\int_1^t B(s)\,\frac{ds}{s^2} = -\frac{B(s)}{s}\bigg|_1^t + \int_1^t B'(s)\,\frac{ds}{s} = -\frac{B(t)}{t} + B(1) + \int_1^t B'(s)\,\frac{ds}{s},$$

so, for $t \gg 1$,

$$\begin{aligned} \int_1^t B(s)\,\frac{ds}{s^2} + \frac{B(t)}{t} \approx \int_1^t B'(s)\,\frac{ds}{s} &= \int_1^t \sup_{1<y\leq s} yC''(y)\,\frac{ds}{s}, \quad \text{by (8.6)} \\ &\approx C'(t), \quad \text{by (8.5)}. \end{aligned}$$

Theorem 8.4. *Let ω_1 and ω_2 be as in Theorem 8.1. Suppose A is a twice continuously differentiable Young function with $A(t) = t^p/p$ for some fixed $p > 1$ and $0 < t < 1$. Define the function C on $\mathbf{R}_+$ by*

$$(C')^{-1}(t) := \begin{cases} t^{p-1}, & 0 < t < 1, \\ \int_1^t a(s)\,\frac{ds}{s}, & t \gg 1. \end{cases}$$

Assume

$$(C')(t) \approx \int_1^t \sup_{1<y\le s} \frac{yC'(y)}{a(C'(y))}\,\frac{ds}{s}, \quad t \gg 1. \tag{8.7}$$

Then, the r.i. norm ω, given in (8.4), satisfies

$$\omega'(g) = \inf\left\{\lambda > 0 : \int_{\mathbf{R}_+} B((\lambda t)^{-1}k(t, g; L_{\omega_2'}, L_{\omega_1'}))\,dt \le 1\right\},$$

for $g \in M^+(\Omega)$, with

$$B(t) := \int_1^t \sup_{1<y\le s} \frac{yC'(y)}{a(C'(y))}\,ds \quad t \gg 1.$$

Proof. In view of (1.8), (8.3) and (8.2), we must show $\rho_{\tilde{\mathcal{A}}}$ equivalent to $\rho_B \circ P_d$. Now, for $0 < y < 1$, $\tilde{\mathcal{A}}'(y) = y^{p'-1}$ while, for $y \gg 1$, $\tilde{\mathcal{A}}'(y)$ is the inverse of

$$\begin{aligned} \mathcal{A}'(y) &= \int_1^y A(s)\,\frac{ds}{s^2} + \frac{A(y)}{y} \\ &= -\frac{A(s)}{s}\Big|_1^y + \int_1^y a(s)\,\frac{ds}{s} + \frac{A(y)}{y} \\ &= A(1) + \int_1^y a(s)\,\frac{ds}{s} \\ &\approx \int_1^y a(s)\,\frac{ds}{s}. \end{aligned}$$

Thus, without loss of generality we may replace $\tilde{\mathcal{A}}$ by C.

But, when $y \gg 1$,

$$yC''(y) = y\frac{dC'}{dy}(y) = \frac{y}{((C')^{-1})'(C'(y))} = \frac{yC'(y)}{a(C'(y))}.$$

Therefore, (8.7) amounts to (8.5) and we are done.

To illustrate the above result we observe that the Young function

$$A(t) = \begin{cases} t^p/p, & 0 < t < 1 \\ te^t, & t \gg 1, \end{cases}$$

yields the Zygmund class, $\exp L$, of exponentially integrable functions and has its C essentially given by

$$(C')^{-1}(t) \approx \begin{cases} t^{p-1}, & 0 < t < 1 \\ e^t, & t \gg 1. \end{cases}$$

so,

$$C'(t) = \log t \approx \int_1^t \sup_{1<y\leq s} \frac{y \log y}{y \log y} \frac{ds}{s} = \int_1^t \sup_{1<y\leq s} \frac{yC'(y)}{a(C'(y))} \frac{ds}{s}, \quad t \gg 1,$$

and, similarly,

$$B(t) = \int_1^t \sup_{1<y\leq s} \frac{yC'(y)}{a(C'(y))} \, ds \approx t, \quad t \gg 1.$$

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Ron Kerman, Department of Mathematics, Brock University, St. Catharines, Ontario, L2S 3A1, CANADA; Mario Milman, Department of Mathematics, Florida Atlantic University, Boca Raton, Florida, U.S.A. 33431 and Gord Sinnamon, Department of Mathematics, University of Western Ontario, London, Ontario, L6A 5B7, CANADA

E-mail address: `rkerman@brocku.ca, milman@fau.edu and sinnamon@uwo.ca`