A Calderón couple of down spaces

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Abstract

The down space construction is a variant of the Köthe dual, restricted to the cone of non-negative, non-increasing functions. The down space corresponding to L^1 is shown to be L^1 itself. An explicit formula for the norm of the down space D^∞ corresponding to L^∞ is given in terms of the Hardy averaging operator. A formula for the Peetre K-functional follows and is used to show that (L^1, D^∞) is a uniform Calderón couple with constant of K-divisibility equal to one. As a consequence a complete description of all exact interpolation spaces between L^1 and D^∞ is obtained. These interpolation spaces are shown to be closely related to the rearrangement invariant spaces via the down space construction.

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1 Introduction

For a normed vector space X of λ-measurable functions on $\mathbb{R}$, the space $X^{\downarrow}$ ("X-down") is the collection of all functions f for which

$$\|f\|_{X^{\downarrow}} = \sup \int |f| g \, d\lambda < \infty.$$

The supremum is taken over all non-negative, non-increasing λ-measurable functions g such that $\|g\|_{X'} \leq 1$, where X' denotes the Köthe dual space of X. We write L^1_λ for $L^1(\mathbb{R}, \lambda)$, L^∞_λ for $L^\infty(\mathbb{R}, \lambda)$, and adopt the notation D^∞_λ for the space $(L^\infty_\lambda)^{\downarrow}$. No short form is required for $(L^1_\lambda)^{\downarrow}$ because it is identical with L^1_λ, as we see the end of this section.

The restricted supremum that defines the norm in the down spaces arises naturally in several contexts. Halperin [6] and Lorentz [11] first considered properties of such suprema, with a weighted Lebesgue space for X, in order to describe the dual of the classical Lorentz space $\Lambda_p(w)$. Halperin's investigation of "D-type Hölder inequalities" used the norm to improve the usual Hölder inequality when one factor is monotone. Later, down spaces and the related level function construction were studied in [8,16–22] and applied to prove weighted Hardy inequalities, to prove general versions of Sawyer's duality theorem, to study Banach envelopes of Orlicz-Lorentz spaces, to characterize the dual of the Lorentz spaces $\Gamma_p(w)$, and to give a weight characterization for the boundedness of the Fourier Transform on weighted Lorentz spaces.

Interpolation properties for these spaces have been touched on in [17] but have not been carefully studied. We show that they have a very strong interpolation property; the couple $(L^1_\lambda, D^\infty_\lambda)$ is a uniform Calderón couple. As a consequence we are able to give a complete description of all interpolation spaces for the couple, to make clear connections with the theory of rearrangement-invariant spaces, and to clarify the role of the level function construction.

The main result is presented in two cases. In Sections 2 and 3 we consider the case in which the underlying measure is just the Lebesgue measure on $(0, \infty)$ and in Section 4 the transition to the case of general measures is made. The level function is introduced in Section 5. Section 6 contains a description of all exact interpolation spaces between L^1_λ and D^∞_λ, essentially they are the down spaces of rearrangement-invariant spaces.

Although the level function construction is not used to prove the main result, the techniques used are similar. Notably, we rely on classes of averaging operators that "level" a function out on a given collection of intervals. The heart of the proof is the ability to perform this leveling operation using operators that are bounded on both L^1_λ and D^∞_λ.

The transition to general measures is inspired by the use of measure-preserving transformations in the theory of rearrangements. When applied to monotone functions these transformations simplify considerably.

Definitions and basic properties of the rearrangement of a λ-measurable function, rearrangement-invariant spaces, Banach couples, K-functionals, and interpolation spaces may be found in [1] or [2].

If $L^0(\lambda)$ denotes the vector space of all (equivalence classes of) real-valued λ-measurable functions, then a Banach space $X \subset L^0(\lambda)$ is called a *Banach function space* provided that for all $f \in L^0(\lambda)$ and $g \in X$, if $|f| \le |g|$ then $f \in X$ and $\|f\|_X \le \|g\|_X$. Properties of Banach function spaces and their associate spaces (Köthe duals) may be found in [23]. (See also [1], [9], or [10].)

Throughout the paper, expressions of the form $0/0$, ∞/∞, and $0\cdot\infty$ are taken to be 0.

In any normed Banach function space X the homogeneity of the norm in X' shows that

$$\|f\|_{X^\downarrow} = \sup_{0\le g\downarrow} \frac{\int |f| g \, d\lambda}{\|g\|_{X'}}, \tag{1}$$

a slightly different form of the norm than the one given above. It is routine to check that this expression defines a seminorm. It is a norm provided $\chi_{(-\infty,x]} \in X'$ for each $x \in \mathbb{R}$. It is also routine to check that the space $X^\downarrow$ has the Fatou property, that is, if $0 \le f_n$ increases to f pointwise λ-almost everywhere then $\|f_n\|_{X^\downarrow}$ increases to $\|f\|_{X^\downarrow}$.

For a general space X, it may be difficult to find a more concrete expression for the norm in $X^\downarrow$. However, it is a simple matter to give formulas for the down norms corresponding to L^1_λ and L^∞_λ provided the measure λ satisfies $\Lambda(x) \equiv \lambda(-\infty, x] < \infty$ for all $x \in \mathbb{R}$. The simpler case is L^1_λ where we have $(L^1_\lambda)^\downarrow = L^1_\lambda$ with equality of norms. To see this, observe that since $(L^1_\lambda)' = L^\infty_\lambda$ with equality of norms,

$$\|f\|_{L^1_\lambda} = \sup_{0\le g} \frac{\int |f| g \, d\lambda}{\|g\|_{L^\infty_\lambda}} \ge \sup_{0\le g\downarrow} \frac{\int |f| g \, d\lambda}{\|g\|_{L^\infty_\lambda}} \ge \frac{\int |f| \, d\lambda}{\|1\|_{L^\infty_\lambda}} = \|f\|_{L^1_\lambda}.$$

Thus $\|f\|_{(L^1_\lambda)^\downarrow} = \|f\|_{L^1_\lambda}$. From now on we will avoid writing the expression $(L^1_\lambda)^\downarrow$.

The space $(L^\infty_\lambda)^\downarrow$ is a much larger space than L^∞_λ in general. To find its norm we define the P and Q by

$$Pf(x) = \frac{1}{\Lambda(x)} \int_{(-\infty,x]} f \, d\lambda \quad \text{and} \quad Qh(x) = \int_{[x,\infty)} \frac{h}{\Lambda} \, d\lambda.$$

Note that $\int (Pf) h \, d\lambda = \int f (Qh) \, d\lambda$ whenever both f and h are non-negative λ-

measurable functions on $\mathbb{R}$. Lemma 1.2 of [22] shows that every non-negative, non-increasing function g is λ-almost everywhere the pointwise limit of an increasing sequence of functions of the form Qh for $h \geq 0$.

Since $(L^\infty_\lambda)' = L^1_\lambda$, with equality of norms, and $P1 = 1$,

$$\|f\|_{(L^\infty_\lambda)^\downarrow} = \sup_{0\leq g\downarrow} \frac{\int |f| g\, d\lambda}{\|g\|_{L^1_\lambda}} = \sup_{0\leq h} \frac{\int |f|(Qh)\, d\lambda}{\int Qh\, d\lambda} = \sup_{0\leq h} \frac{\int (P|f|)h\, d\lambda}{\int (P1)h\, d\lambda} = \|P|f|\|_{L^\infty_\lambda}.$$

Thus

$$\|f\|_{(L^\infty_\lambda)^\downarrow} = \|P|f|\|_{L^\infty_\lambda} = \sup_{x\in\mathbb{R}} \frac{1}{\Lambda(x)} \int_{(-\infty,x]} |f|\, d\lambda. \tag{2}$$

As mentioned above we will shorten $(L^\infty_\lambda)^\downarrow$ to D^∞_λ in the remainder of the paper.

The example $X = L^\infty_\lambda$ shows that, in general, $X^\downarrow$ need not be rearrangement invariant even when the original space X is.

Note that the norm in D^∞_λ is generated by the sublinear operator $f \mapsto P|f|$. Banach spaces generated by sublinear operators arise naturally in many problems. Some topological properties of spaces of this type were studied in [13] and interpolation for these spaces was investigated in [12].

2 The K-functional

In this section we restrict ourselves to the case that λ is the Lebesgue measure on $(0,\infty)$ and drop the subscript λ when referring to the spaces L^1, L^∞ and D^∞. Fix a function $f \in L^1 + D^\infty$ and set

$$F(t) = \int_0^t |f| \quad \text{and} \quad K(t) = K(t, f; L^1, D^\infty) \equiv \inf_{f=f_0+f_1} \|f_0\|_{L^1} + t\|f_1\|_{D^\infty}.$$

Lemma 2.1 *For all $t > 0$,*

$$K(t) = \inf_{x>0} \sup_{y>x} \left(F(x) + \frac{t}{y}(F(y) - F(x)) \right).$$

Consequently, K is the least concave majorant of F.

Proof. Fix $t > 0$. If $x > 0$ then $f = f\chi_{(0,x]} + f\chi_{(x,\infty)}$ so by (2)

$$K(t) \leq \|f\chi_{(0,x]}\|_{L^1} + t\|f\chi_{(x,\infty)}\|_{D^\infty} = \sup_{y>x} \left(F(x) + \frac{t}{y}(F(y) - F(x)) \right).$$

In order to prove the reverse inequality, suppose $f = f_0 + f_1$ and choose $x \in [0, \infty]$ such that

$$\int_0^x |f| = \int_0^\infty \min\{|f|, |f_0|\}.$$

Clearly $\|f_0\|_{L^1} \geq \|f\chi_{(0,x]}\|_{L^1}$. Also

$$|f_1| \geq \max\{0, |f| - |f_0|\} = |f| - \min\{|f|, |f_0|\}$$

so for $y > x$ we have

$$\int_0^y |f_1| \geq \int_0^y |f| - \min\{|f|, |f_0|\} \geq \int_0^y |f| - \int_0^\infty \min\{|f|, |f_0|\} = \int_0^y |f|\chi_{(x,\infty)}.$$

It follows from the formula (2) that $\|f_1\|_{D^\infty} \geq \|f\chi_{(x,\infty)}\|_{D^\infty}$ and therefore

$$\begin{aligned} \|f_0\|_{L^1} + t\|f_1\|_{D^\infty} &\geq \|f\chi_{(0,x]}\|_{L^1} + t\|f\chi_{(x,\infty)}\|_{D^\infty} \\ &= \sup_{y>x} \left(F(x) + \frac{t}{y}(F(y) - F(x)) \right). \end{aligned}$$

Taking the infimum over all decompositions $f = f_0 + f_1$ completes the proof of the first statement.

The proof of the second statement is standard but is included here because of its essential role in the sequel. Since K is concave, to show that it is the least concave majorant of F it is enough to show that $K \geq F$ and that K lies under any line that lies above F.

Fix $t > 0$. If $x \geq t$ then

$$\sup_{y>x} \left(F(x) + \frac{t}{y}(F(y) - F(x)) \right) \geq \sup_{y>x} F(x) \geq F(t).$$

If $x < t$ then

$$\sup_{y>x} \left(F(x) + \frac{t}{y}(F(y) - F(x)) \right) \geq F(x) + \frac{t}{t}(F(t) - F(x)) = F(t).$$

Taking the infimum over all x yields $K(t) \geq F(t)$.

Now suppose that F lies under some line, say $F(t) \leq r + st$ for some $r, s \in \mathbb{R}$. If $r = F(x)$ for some x then for any $t > 0$,

$$K(t) \leq \sup_{y>x} \left(r + \frac{t}{y}(F(y) - r) \right) \leq r + st.$$

If $r \neq F(x)$ for any x then, since $r \geq F(0) = 0$, the only other possibility is that $F(x) < r$ for all $x \geq 0$. Since F is non-decreasing $F(t) \leq r + st$ implies

$s \geq 0$. Thus,

$$K(t) \leq \lim_{x\to\infty} \sup_{y>x} \left(F(x) + \frac{t}{y}(F(y) - F(x)) \right) \leq \lim_{x\to\infty} r + \frac{t}{x}(r - 0) = r \leq r + st$$

for any $t > 0$. This completes the proof.

Since K is concave its derivative, K', exists almost everywhere. The following property of the derivative of the least concave majorant is needed in Theorem 2.3.

Lemma 2.2 *If $0 \leq a < b \leq \infty$ and $F < K$ on (a, b) then K' is constant on (a, b).*

Proof. We are free to suppose that $0 < a < b < \infty$ since the general case follows readily from that one. Let ℓ be the line through $(a, K(a))$ and $(b, K(b))$. Since K is concave we have $K \geq \ell$ on $[a, b]$ and $F \leq K \leq \ell$ on the complement of (a, b). Next we show that $K \leq \ell$ on $(0, \infty)$. This will complete the proof since then K and ℓ coincide on (a, b).

Let m be the maximum value of the continuous function $F - \ell$ on $[a, b]$ and choose $t \in [a, b]$ such that $m = F(t) - \ell(t)$. If $m \geq 0$ then $F \leq \ell + m$ on $(0, \infty)$ so by Lemma 2.1 we have $K \leq \ell + m$ on $(0, \infty)$. In particular, at the point t,

$$F(t) \leq K(t) \leq \ell(t) + m = F(t)$$

so $F(t) = K(t)$ and by hypothesis, $t \notin (a, b)$. Therefore

$$m = \max\{F(a) - \ell(a), F(b) - \ell(b)\} \leq \max\{K(a) - \ell(a), K(b) - \ell(b)\} = 0.$$

We conclude that $m \leq 0$ and it follows that $F \leq \ell$ on $[a, b]$ and hence on $(0, \infty)$. By Lemma 2.1 we have $K \leq \ell$ on $(0, \infty)$ as required.

Theorem 2.3 *Let $f \in L^1 + D^\infty$ and $K(t) = K(t, f; L^1, D^\infty)$. Then there exists an $a_f \in [0, \infty]$, and a collection $\mathcal{I}_f$ of open subintervals of $(0, a_f)$ such that for almost every t,*

$$K'(t) = \begin{cases} \frac{1}{b-a}\int_a^b |f|, & t \in (a, b) \in \mathcal{I}_f \\ \limsup_{b\to\infty} \frac{1}{b}\int_0^b |f|, & t > a_f \\ f(t), & t \in (0, a_f] \setminus \cup_{I\in\mathcal{I}_f} I. \end{cases}$$

For all $x \in (a, b) \in \mathcal{I}_f$

$$\frac{1}{x - a}\int_a^x |f| \leq \frac{1}{b - a}\int_a^b |f|,$$

and for all $b > a_f$

$$\frac{1}{b-a_f}\int_{a_f}^{b}|f| \leq \limsup_{b\to\infty}\frac{1}{b}\int_0^b |f|.$$

Proof. Both F and K are continuous on $(0,\infty)$ so $U = \{t > 0 : F < K\}$ is an open set. Let $\mathcal{I}_f$ be the collection of bounded connected components of U and let (a_f,∞) be the unbounded connected component of U if there is one. If not, set $a_f = \infty$. The concave function K is differentiable almost everywhere and is the integral of its derivative. Since simple functions are dense in L^1 and contained in $L^1 \cap D^\infty$, [1, Proposition 1.15] shows that setting $K(0) = 0$ makes K continuous at 0.

By Lemma 2.2, K' is constant on each $(a,b) \in \mathcal{I}_f$ and since $a, b \notin U$ the value K' takes on (a,b) is

$$\frac{1}{b-a}\int_a^b K' = \frac{K(b)-K(a)}{b-a} = \frac{F(b)-F(a)}{b-a} = \frac{1}{b-a}\int_a^b |f|.$$

If $(a,b) \in \mathcal{I}_f$, then $F(a) = K(a)$ so for any $x \in (a,b)$,

$$\frac{1}{x-a}\int_a^x |f| = \frac{F(x)-F(a)}{x-a} \leq \frac{K(x)-K(a)}{x-a} = \frac{1}{x-a}\int_a^b K' = \frac{1}{b-a}\int_a^b |f|.$$

If $a_f < \infty$ then Lemma 2.2 shows that K' is constant on (a_f,∞). Denote its value there by $K'(\infty)$. Since $a_f \notin U$, for each $b > a_f$ we have

$$\frac{1}{b-a_f}\int_{a_f}^b |f| = \frac{F(b)-F(a_f)}{b-a_f} \leq \frac{K(b)-K(a_f)}{b-a_f} = \frac{1}{b-a_f}\int_{a_f}^b K' = K'(\infty).$$

Thus

$$K'(\infty) \geq \limsup_{b\to\infty}\frac{1}{b-a_f}\int_{a_f}^b |f| = \limsup_{b\to\infty}\frac{1}{b}\int_0^b |f|. \tag{3}$$

On the other hand, if $s > \limsup_{b\to\infty}\frac{1}{b}\int_0^b |f|$ then $\sup_{t>0} F(t) - st < \infty$. Therefore F lies under some line of slope s and so does K. It follows that $K'(\infty) < s$. This shows that we have equality in (3).

It remains to show that $K'(t) = f(t)$ for almost all $t \notin U$. For such t, $K(t) = F(t)$ so for any $\varepsilon > 0$ we have

$$\frac{K(t)-K(t-\varepsilon)}{\varepsilon} \leq \frac{F(t)-F(t-\varepsilon)}{\varepsilon} = \frac{1}{\varepsilon}\int_{t-\varepsilon}^t f,$$

and

$$\frac{K(t+\varepsilon)-K(t)}{\varepsilon} \geq \frac{F(t+\varepsilon)-F(t)}{\varepsilon} = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} f.$$

For almost every such t (the Lebesgue points of f for which K' exists) we may take the limit as $\varepsilon \to 0+$ to get

$$K'(t) \leq f(t) \quad \text{and} \quad K'(t) \geq f(t).$$

This completes the proof.

3 The main result

One of fundamental tasks of interpolation theory is that of describing all interpolation spaces for a given couple. The uniform Calderón couples are important because they have the remarkable property that their interpolation spaces are completely described by the $\mathcal{K}$-method of interpolation. This property is a consequence of the Brudnyĭ-Krugljak K-divisibility theorem for Banach couples (see [2].) In the case of rearrangement invariant spaces and general Banach function spaces, further deep results on Calderón couples may be found in [7], [4], and [5] and references cited there.

We recall that a Banach couple (X_0, X_1) is said to be a uniform Calderón couple with constant γ if the following holds. Whenever $f, g \in X_0 + X_1$ satisfy

$$K(t, g; X_0, X_1) \leq K(t, f; X_0, X_1), \quad t > 0,$$

then there exists a linear operator $S : X_0 + X_1 \to X_0 + X_1$ with $Sf = g$ and

$$\|S\|_{X_0 \to X_0} \leq \gamma \quad \text{and} \quad \|S\|_{X_1 \to X_1} \leq \gamma.$$

In this section we show that (L^1, D^∞) is a uniform Calderón couple with constant $\gamma = 1$ by explicitly constructing the operator S in three stages,

$$f \mapsto f^o \mapsto g^o \mapsto g,$$

given in Theorems 3.5, 3.7 and 3.6 respectively. Here f^o and g^o denote the derivatives,

$$f^o(t) = \frac{d}{dt} K(t, f; L^1, D^\infty) \quad \text{and} \quad g^o(t) = \frac{d}{dt} K(t, g; L^1, D^\infty),$$

which exist almost everywhere on $(0, \infty)$ as non-negative, non-increasing functions.

To begin we introduce some averaging operators that will serve as building blocks. Suppose g is a non-negative measurable function and $\mathcal{I}$ is a countable collection of disjoint open subintervals of $(0,\infty)$ such that $\int_I g < \infty$ for each $I \in \mathcal{I}$. Define the operator $A_{g,\mathcal{I}}$ on locally integrable functions by

$$A_{g,\mathcal{I}}h(x) = \begin{cases} g \int_I h / \int_I g, & x \in I \in \mathcal{I} \\ h(x), & x \notin \cup_{I\in\mathcal{I}} I. \end{cases}$$

If $\mathcal{I} = \{I\}$ we naturally write $A_{g,I}$ for $A_{g,\mathcal{I}}$ and if $g \equiv 1$ we omit it and write $A_{\mathcal{I}}$ or A_I.

Observe that the operator $A_{g,\mathcal{I}}$ behaves like a projection, that is, $A_{g,\mathcal{I}}A_{g,\mathcal{I}} = A_{g,\mathcal{I}}$. If $g \geq 0$ then $A_{g,\mathcal{I}}$ is positive. Also, if we assume that each interval of $\mathcal{I}$ is of finite measure as well as satisfying $\int_I g < \infty$, then it is a simple matter to check that

$$(A_{g,\mathcal{I}}A_{\mathcal{I}})g = g. \tag{4}$$

Lemma 3.1 *Suppose that for each interval $(a,b) \in \mathcal{I}$ and each $x \in (a,b)$, g satisfies*

$$\frac{1}{x-a}\int_a^x g \leq \frac{1}{b-a}\int_a^b g.$$

Then the operator $A_{g,\mathcal{I}}$ is a contraction on both L^1 and D^∞.

Proof. Suppose $J \subset \mathbb{R}$ is a set that contains every interval of $\mathcal{I}$ that it intersects. Set

$$\mathcal{I}_J = \{I \in \mathcal{I} : I \subset J\}.$$

Then

$$\begin{aligned}\int_J |A_{g,\mathcal{I}}f| &= \int_{J\setminus\cup_{I\in\mathcal{I}} I} |f| + \sum_{I\in\mathcal{I}_J}\int_I |A_{g,\mathcal{I}}f| \\ &\leq \int_{J\setminus\cup_{I\in\mathcal{I}} I} |f| + \sum_{I\in\mathcal{I}_J}\int_I |f| \\ &= \int_J |f|.\end{aligned}$$

In particular if $J = (0,\infty)$, we get

$$\|A_{g,\mathcal{I}}f\|_{L^1} \leq \|f\|_{L^1}$$

so $A_{g,\mathcal{I}}$ is a contraction on L^1.

If $J = (0,x)$ for some $x \notin \cup_{I\in\mathcal{I}} I$ then J contains every interval of $\mathcal{I}$ that it intersects and thus

$$\frac{1}{x}\int_0^x |A_{g,\mathcal{I}}f| \leq \frac{1}{x}\int_0^x |f| \leq \|f\|_{D^\infty}.$$

If $x \in (a,b) \in \mathcal{I}$ then $(0,a)$ contains every interval of $\mathcal{I}$ that it intersects. We have

$$\begin{aligned}
\int_0^x |A_{g,\mathcal{I}} f| &= \int_0^a |A_{g,\mathcal{I}} f| + \int_a^x |A_{g,\mathcal{I}} f| \\
&\le \int_0^a |f| + \frac{\int_a^b |f|}{\int_a^b g} \int_a^x g \\
&\le \int_0^a |f| + \frac{x-a}{b-a} \int_a^b |f| \\
&= \frac{b-x}{b-a} \int_0^a |f| + \frac{x-a}{b-a} \int_0^b |f| \\
&\le \left(\frac{b-x}{b-a} a + \frac{x-a}{b-a} b \right) \|f\|_{D^\infty} \\
&= x \|f\|_{D^\infty}.
\end{aligned}$$

Therefore, we have

$$\frac{1}{x} \int_0^x |A_{g,\mathcal{I}} f| \le \|f\|_{D^\infty}$$

for this x as well. Taking the supremum over all $x > 0$ yields

$$\|A_{g,\mathcal{I}} f\|_{D^\infty} \le \|f\|_{D^\infty}$$

and completes the proof.

In the case $g \equiv 1$ the hypothesis of Lemma 3.1 is automatically satisfied.

Corollary 3.2 *The operator $A_{\mathcal{I}}$ is a positive contraction on both L^1 and D^∞.*

The next two lemmas provide a method of handling averages over intervals of infinite measure.

Lemma 3.3 *If $f \in L^1 + D^\infty$ and*

$$|\gamma| \le \limsup_{b \to \infty} \frac{1}{b} \int_0^b |f|$$

then there is a linear functional $\Psi : L^1 + D^\infty \to \mathbb{R}$, of norm at most one, such that $L^1 \subset \ker(\Psi)$, $\Psi(f) = \gamma$, and if $\gamma, f \ge 0$ then Ψ is positive.

Proof. Note that if $f \in L^1$ then $\gamma = 0$. Since $f \in L^1 + D^\infty$, we can write $f = f_0 + f_1$ with $f_0 \in L^1$ and $f_1 \in D^\infty$ to get

$$\int_0^b |f| \le \int_0^b |f_0| + \int_0^b |f_1| \le \|f_0\|_{L^1} + b\|f_1\|_{D^\infty}$$

for any $b > 0$. Now

$$|\gamma| \leq \limsup_{b\to\infty} \frac{1}{b}\|f_0\|_{L^1} + \|f_1\|_{D^\infty} = \|f_1\|_{D^\infty}.$$

Let $V = L^1 + \mathbb{R}f$, considered as a subspace of $L^1 + D^\infty$, and define $\Psi : V \to \mathbb{R}$ by

$$\Psi(h + \alpha f) = \alpha\gamma$$

for $h \in L^1$ and $\alpha \in \mathbb{R}$. This is well defined because if $h + \alpha f = \bar{h} + \bar{\alpha} f$ with h and $\bar{h}$ in L^1 then either $\alpha = \bar{\alpha}$ or else $f = (h - \bar{h})/(\alpha - \bar{\alpha}) \in L^1$ so that $\gamma = 0$.

The norm of this linear functional is at most one because it is zero if $\alpha = 0$ and if $\alpha \neq 0$ then whenever $h + \alpha f = f_0 + f_1$ with $f_0 \in L^1$ and $f_1 \in D^\infty$ we have $f = (f_0 - h)/\alpha + f_1/\alpha$ with $(f_0 - h)/\alpha \in L^1$ and $f_1/\alpha \in D^\infty$ so

$$|\Psi(h + \alpha f)| = |\alpha\gamma| \leq |\alpha|\|f_1/\alpha\|_{D^\infty} = \|f_1\|_{D^\infty} \leq \|f_0\|_{L^1} + \|f_1\|_{D^\infty}.$$

Taking the infimum over all such decompositions of $h + \alpha f$ we have

$$|\Psi(h + \alpha f)| \leq \|h + \alpha f\|_{L^1 + D^\infty}.$$

To see that Ψ is positive when $\gamma, f \geq 0$, suppose that $h + \alpha f \geq 0$. If $\alpha \geq 0$ then $\Psi(h + \alpha f) = \alpha\gamma \geq 0$. If $\alpha < 0$ then $0 \leq (-\alpha)f \leq h$ so $f \in L^1$. It follows that $\gamma = 0$ and again $\Psi(h + \alpha f) = \alpha\gamma \geq 0$.

By the Hahn-Banach Theorem the functional Ψ extends to all of $L^1 + D^\infty$ with no increase in norm. The Hahn-Banach Theorem for positive functionals in Banach lattices (see [15]) shows that if Ψ is positive then there is a positive extension. This completes the proof.

Lemma 3.4 *If $a \geq 0$, $f \in L^1 + D^\infty$, and $g \in L^1 + D^\infty$ satisfies*

$$\frac{1}{x-a}\int_a^x |g| \leq \limsup_{b\to\infty} \frac{1}{b}\int_0^b |f|, \quad x > a,$$

then there exists an operator $B_{a,f,g}$ defined on $L^1 + D^\infty$ such that

(i) $B_{a,f,g}$ is a contraction on both L^1 and D^∞,

(ii) for all $h \in L^1 + D^\infty$, $B_{a,f,g}h = h$ on $(0, a]$ and $B_{a,f,g}h$ is a constant multiple of g on (a, ∞),

(iii) $B_{a,f,g}f = g$ on (a, ∞), and

(iv) if $f, g \geq 0$ then $B_{a,f,g}$ is positive.

Proof. Set

$$\gamma = \limsup_{b\to\infty} \frac{1}{b}\int_0^b |f|$$

and observe that if $\gamma = 0$ then $g \equiv 0$ on (a, ∞). Let Ψ be the functional of Lemma 3.3 and define $B_{a,f,g}$ by

$$B_{a,f,g}h(x) = \begin{cases} (g/\gamma)\Psi(h), & x > a \\ h(x), & x \le a. \end{cases}$$

Evidently, properties (ii), (iii), and (iv) are satisfied. If $h \in L^1$ then $B_{a,f,g}h = h\chi_{(0,a)}$ so $B_{a,f,g}$ is clearly a contraction on L^1. If $h \in D^\infty$ then

$$|\Psi(h)| \le \|h\|_{L^1+D^\infty} \le \|h\|_{D^\infty}$$

so for any $b \le a$ we have

$$\int_0^b |B_{a,f,g}h| = \int_0^b |h| \le b\|h\|_{D^\infty}$$

and for any $b > a$,

$$\int_0^b |B_{a,f,g}h| = \int_0^a |h| + \left(\frac{1}{\gamma}\int_a^b |g|\right) |\Psi(h)| \le a\|h\|_{D^\infty} + (b-a)\|h\|_{D^\infty} = b\|h\|_{D^\infty}.$$

Dividing by b and taking the supremum yields

$$\|B_{a,f,g}h\|_{D^\infty} \le \|h\|_{D^\infty}$$

and completes the proof.

In the next two theorems we construct the maps that take $f \mapsto f^o$ and $g^o \mapsto g$.

Theorem 3.5 *If $f \in L^1 + D^\infty$ then there is a bounded linear map on $L^1 + D^\infty$ that is a contraction on both L^1 and D^∞, takes f to f^o, and is positive if $f \ge 0$.*

Proof. First observe that the map $h \mapsto (|f|/f)h$ is a contraction on both L^1 and D^∞ and takes f to $|f|$. Since $f^o = |f|^o$, we may assume henceforth that $f \ge 0$.

Let $\mathcal{I} = \mathcal{I}_f$ and $a = a_f \in [0, \infty]$ be those given by Theorem 2.3. On $(0, a]$, $f^o = A_{\mathcal{I}}f$, and on (a, ∞) f^o takes the value,

$$\gamma = \limsup_{b\to\infty} \frac{1}{b}\int_0^b f.$$

The constant function $g = \gamma$ clearly satisfies the hypotheses of Lemma 3.4 so the positive operator $B_{a,f,\gamma}$ is a contraction on both L^1 and D^∞ and

$$B_{a,f,\gamma}f(x) = \begin{cases} f(x), & x \leq a \\ f^o(x), & x > a. \end{cases}$$

By Corollary 3.2, the positive operator $A_{\mathcal{I}}$ is a contraction on both L^1 and D^∞ and $A_{\mathcal{I}}B_{a,f,\gamma}f = f^o$. This completes the proof.

Theorem 3.6 *If $g \in L^1 + D^\infty$ then there is a bounded linear map on $L^1 + D^\infty$ that is a contraction on both L^1 and D^∞, takes g^o to g, and is positive if $g \geq 0$.*

Proof. First observe that the map $h \mapsto (g/|g|)h$ is a contraction on both L^1 and D^∞ and takes $|g|$ to g. Since $g^o = |g|^o$, we may assume henceforth that $g \geq 0$.

Let $\mathcal{I} = \mathcal{I}_g$ and $a = a_g \in [0, \infty]$ be those given by Theorem 2.3. Then $g^o = A_{\mathcal{I}}g$ on $(0, a]$ and g^o is constant on (a, ∞), taking the value

$$\gamma = \limsup_{b \to \infty} \frac{1}{b} \int_0^b g.$$

Theorem 2.3 also shows that for every $b > a$,

$$\frac{1}{b-a} \int_a^b g \leq \gamma.$$

As in the proof of Theorem 3.5, both $A_{\mathcal{I}}$ and $B_{a,g,\gamma}$ are positive and

$$A_{\mathcal{I}}B_{a,g,\gamma}g = g^o.$$

Set $\bar{g} = B_{a,g,\gamma}g$ so that $A_{\mathcal{I}}\bar{g} = g^o$. It follows from the construction of $B_{a,g,\gamma}$ that $\bar{g} = g$ on $(0, a)$ and $\limsup_{b\to\infty} \frac{1}{b}\int_0^b \bar{g} = \gamma$ so we may apply Lemma 3.4 with $f = \bar{g}$ to get the positive operator $B_{a,\bar{g},g}$, a contraction on both L^1 and D^∞, that satisfies

$$B_{a,\bar{g},g}\bar{g} = g.$$

Putting these together with (4) applied to $\bar{g}$ we have

$$B_{a,\bar{g},g}A_{\bar{g},\mathcal{I}}g^o = B_{a,\bar{g},g}A_{\bar{g},\mathcal{I}}A_{\mathcal{I}}\bar{g} = B_{a,\bar{g},g}\bar{g} = g.$$

To show that the map $B_{a,\bar{g},g}A_{\bar{g},\mathcal{I}}$ has all the desired properties, it remains to observe that $A_{\bar{g},\mathcal{I}}$ is a positive contraction on both L^1 and D^∞. Theorem 2.3 shows that the function g satisfies the hypothesis of Lemma 3.1. Since this condition depends only on the values of g on $(0, a)$ and $g = \bar{g}$ on that interval, the function $\bar{g}$ also satisfies the hypothesis of Lemma 3.1. Thus the map $A_{\bar{g},\mathcal{I}}$ is a contraction on both L^1 and D^∞. It is clear from the definition that $A_{\bar{g},\mathcal{I}}$ is also a positive map. This completes the proof.

To complete the construction of a map from f to g, we need the step $f^o \mapsto g^o$. The next theorem provides this step because both f^o and g^o are non-increasing functions.

Theorem 3.7 *Let $f, g \in L^1 + D^\infty$ be non-negative and non-increasing, set $F(x) = \int_0^x f$ and $G(x) = \int_0^x g$ for all $x > 0$, and suppose that $G \leq F$. Then there exists a bounded positive operator on $L^1 + D^\infty$ that is a contraction on both L^1 and D^∞ and maps f to g.*

Proof. Let $q_1, q_2, \dots$ be an enumeration of the positive rationals and for each n define

$$\ell_n(x) = g(q_n)(x - q_n) + G(q_n), \quad x > 0.$$

Then ℓ_n is a tangent line at q_n to the concave function G, that is, $G \leq \ell_n$ and $G(q_n) = \ell_n(q_n)$. Define

$$F_n = \min\{F, \ell_1, \ell_2, \dots, \ell_n\}$$

and observe that F_n is a concave function, $G \leq F_n$ and $F_n(q_k) = G(q_k)$ for $1 \leq k \leq n$. Finally, define

$$I_n = \{x > 0 : F_n(x) < F_{n-1}(x)\}$$

and notice that I_n is an open interval, possible empty, such that $q_k \notin I_n$ for all $k < n$. Also observe that $F_n = \ell_n$ on I_n so that

$$F_n'(x) = \begin{cases} F_{n-1}'(x), & x \notin I_n \\ g(q_n), & x \in I_n. \end{cases}$$

Now we define a sequence of positive operators C_n satisfying $F_n' = C_n F_{n-1}'$ such that each C_n is a contraction on both L^1 and D^∞. If I_n is empty then $F_n = F_{n-1}$ so we may take C_n to be the identity operator.

If $I_n = (a, b)$ for $0 \leq a < b < \infty$ then

$$g(q_n) = \frac{F_n(b) - F_n(a)}{b - a} = \frac{F_{n-1}(b) - F_{n-1}(a)}{b - a} = \frac{1}{b - a} \int_a^b F_{n-1}'$$

so $C_n = A_{I_n}$ satisfies $F_n' = C_n F_{n-1}'$. By Corollary 3.2, C_n is a positive contraction on both L^1 and D^∞.

In the remaining case, $I_n = (a, \infty)$ for some $a \geq 0$ and for each $b > a$,

$$\begin{aligned} \int_a^b F_{n-1}' &= F_{n-1}(b) - F_{n-1}(a) \\ &> F_n(b) - F_{n-1}(a) \\ &= g(q_n)(b - q_n) + G(q_n) - F_{n-1}(a). \end{aligned}$$

It follows that

$$\begin{aligned}\limsup_{b\to\infty}\frac{1}{b}\int_0^b F'_{n-1} &= \limsup_{b\to\infty}\frac{1}{b-a}\int_a^b F'_{n-1}\\ &\geq \limsup_{b\to\infty}\frac{g(q_n)(b-q_n)+G(q_n)-F_{n-1}(a)}{b-a}\\ &= g(q_n).\end{aligned}$$

so we can let $C_n = B_{a,F'_{n-1},g(q_n)}$, the operator constructed in Lemma 3.4. It is positive because both F'_{n-1} and $g(q_n)$ are non-negative.

Define the operators $D_n = C_n \dots C_2C_1$ for each n and note that each D_n is positive and is a contraction on both L^1 and D^∞.

Suppose $h \in L^1 + D^\infty$. If $n > k$ then $q_k \notin I_n$ so $I_n \subset (q_k, \infty)$ or $I_n \subset (0, q_k)$. In the former case the operator C_n does not change the function on $(0, q_k)$ and in the latter case the operation of C_n is to average the function over the interval $I_n \subset (0, q_k)$. In either case

$$\int_0^{q_k} D_n h = \int_0^{q_k} C_n(D_{n-1}h) = \int_0^{q_k} D_{n-1}h.$$

It follows that for each each k, the sequence $\int_0^{q_k} D_n h$ is constant for $n \geq k$.

Define $H : \mathbb{Q} \cap (0, \infty) \to \mathbb{R}$ by

$$H(q_k) = \lim_{n\to\infty}\int_0^{q_k} D_n h.$$

Claim: For each $h \in L^1 + D^\infty$ the function H extends uniquely to a continuous function on $[0, \infty)$. The extension is absolutely continuous on $[0, y]$ for each $y > 0$ and is non-decreasing if $h \geq 0$.

Proof: Since H is densely defined, uniqueness of the continuous extension is immediate once we show it exists. Moreover, if $h \geq 0$ then $D_n h \geq 0$ for each n. It follows that H is non-decreasing and so is any continuous extension of H. It remains to show that H extends to an absolutely continuous function on $[0, y]$ for each $y > 0$.

Fix $y > 0$ and choose m so that $q_m \in (y, \infty)$. Let $h_n = (D_n h)\chi_{[0,q_m]}$ for each n. Since $h \in L^1 + D^\infty$ and D_n is a contraction on both L^1 and D^∞, $D_n h \in L^1 + D^\infty$ for each n. It follows that $h_n \in L^1([0, q_m])$ and so is its rearrangement, h_n^*.

For each $n > m$, either $I_n \subset (q_m, \infty)$ or $I_n \subset (0, q_m)$. In the former case we have $h_n = h_{n-1}$ and in the latter case we have

$$h_n(x) = \begin{cases} \frac{1}{|I|} \int_I h_{n-1}, & x \in I \\ h_{n-1}(x), & x \notin I. \end{cases}$$

Proposition 3.7 of [1] shows that

$$\int_0^x h_n^* \leq \int_0^x h_{n-1}^*$$

for all $x \in [0, q_m]$ and induction yields

$$\int_0^x h_n^* \leq \int_0^x h_m^*$$

for all $n \geq m$.

Fix $\varepsilon > 0$. Since each h_n^* is integrable on $(0, q_m]$ we can choose δ so that

$$\int_0^\delta h_n^* < \varepsilon$$

for all $n \leq m$ and hence for all n.

If $x \in [0, y] \subset [0, q_m]$ and $r_1, r_2, r_3 \ldots$ is a sequence of rational numbers that converges to x then we can choose J so that $|r_j - r_k| < \delta$ whenever $j, k \geq J$. Therefore, whenever $j, k \geq J$ we have

$$|H(r_j) - H(r_k)| = \lim_{n\to\infty} \left| \int_{r_j}^{r_k} h_n \right| \leq \lim_{n\to\infty} \int_0^\delta h_n^* < \varepsilon.$$

This shows that the sequence $H(r_1), H(r_2), H(r_3) \ldots$ is Cauchy and hence converges. If $r_1', r_2', r_3' \ldots$ is another sequence of rationals converging to x then, by considering the interleaved sequence $r_1, r_1', r_2, r_2', \ldots$ we easily see that $H(r_1'), H(r_2'), H(r_3') \ldots$ converges to the same limit. We denote the limit by $H(x)$. Clearly if x is rational this agrees with the original function H.

To see that H is absolutely continuous on $[0, y]$ we take $\varepsilon > 0$ and δ as above. If $(x_1, x_1'), (x_2, x_2'), \ldots, (x_J, x_J')$ is a finite sequence of non-empty, non-overlapping subintervals of $[0, y]$ satisfying

$$\sum_{j=1}^J x_j' - x_j < \delta$$

then we may choose sequences of rationals $r_{j,1}, r_{j,2}, \ldots$ and $r_{j,1}', r_{j,2}', \ldots$ such that $r_{j,k} \to x_j$ and $r_{j,k}' \to x_j'$ as $k \to \infty$. For k sufficiently large we have $r_{j,k}' > r_{j,k}$ for each j and

$$\sum_{j=1}^J r_{j,k}' - r_{j,k} < \delta.$$

Therefore, $E_k = \cup_{j=1}^J (r_{j,k}, r'_{j,k})$ has measure less than δ so

$$\begin{aligned}
\sum_{j=1}^J |H(x'_j) - H(x_j)| &= \lim_{k\to\infty} \sum_{j=1}^J |H(r'_{j,k}) - H(r_{j,k})| \\
&\le \lim_{k\to\infty} \sum_{j=1}^J \lim_{n\to\infty} \int_{r_{j,k}}^{r'_{j,k}} |h_n| \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \int_{E_k} |h_n| \\
&\le \lim_{k\to\infty} \lim_{n\to\infty} \int_0^\delta h_n^* < \varepsilon.
\end{aligned}$$

This completes the proof of the claim.

On any interval $[0, y]$ the absolutely continuous function H is differentiable almost everywhere and is the integral of its derivative. Therefore, setting

$$Dh = H'$$

yields

$$\int_0^x Dh = H(x)$$

for all $x \ge 0$. Since D_n is a linear operator for each n and

$$\int_0^q Dh = H(q) = \lim_{n\to\infty} \int_0^q D_n h$$

for each rational q it follows readily that D is linear.

Also, for each rational q we have

$$\int_0^q g = G(q) = \lim_{n\to\infty} F_n(q) = \lim_{n\to\infty} \int_0^q F'_n = \lim_{n\to\infty} \int_0^q D_n f = \int_0^q Df.$$

Thus $Df = g$ as required.

Recall that if $h \ge 0$ then H is non-decreasing and so $Dh = H' \ge 0$. Thus D is a positive operator and it follows that $|Dh| \le D(|h|)$.

Now each D_n is a contraction on L^1 so

$$\begin{aligned}
\int_0^\infty |Dh| \le \sup_{0\le q\in\mathbb{Q}} \int_0^q D(|h|) &= \sup_{0\le q\in\mathbb{Q}} \lim_{n\to\infty} \int_0^q D_n(|h|) \\
&\le \lim_{n\to\infty} \int_0^\infty D_n(|h|) \le \int_0^\infty |h|.
\end{aligned}$$

Thus D is a contraction on L^1.

Also, each D_n is a contraction on D^∞ so

$$\sup_{x>0} \frac{1}{x} \int_0^x |Dh| \le \sup_{x>0} \frac{1}{x} \int_0^x D(|h|) = \sup_{0<q\in\mathbb{Q}} \frac{1}{q} \int_0^q D(|h|)$$
$$= \sup_{0<q\in\mathbb{Q}} \lim_{n\to\infty} \frac{1}{q} \int_0^q D_n(|h|) \le \lim_{n\to\infty} \sup_{x>0} \frac{1}{x} \int_0^x D_n(|h|) \le \sup_{x>0} \frac{1}{x} \int_0^x |h|.$$

Thus D is a contraction on D^∞.

Since D is a contraction on both L^1 and D^∞ it is clearly bounded on $L^1 + D^\infty$. This completes the proof.

Theorem 3.8 *If f and g are functions in $L^1 + D^\infty$ such that*

$$K(t, g; L^1, D^\infty) \le K(t, f; L^1, D^\infty), \quad t > 0,$$

then there is an operator on $L^1 + D^\infty$ that is a contraction on both L^1 and D^∞, maps f to g, and is positive if $f, g \ge 0$. In particular, (L^1, D^∞) is a uniform Calderón couple.

Proof. Suppose that $K(t, g; L^1, D^\infty) \le K(t, f; L^1, D^\infty)$. Since simple functions are dense in L^1 and contained in $L^1 \cap D^\infty$, [1, Proposition 1.15] shows that

$$K(0+, g; L^1, D^\infty) = K(0+, f; L^1, D^\infty) = 0.$$

Therefore, g^o and f^o are non-increasing functions satisfying

$$\int_0^t g^o = K(t, g; L^1, D^\infty) \le K(t, f; L^1, D^\infty) = \int_0^t f^o.$$

By Theorem 3.7 there is a bounded positive linear operator on $L^1 + D^\infty$ that is a contraction on both L^1 and D^∞ and maps f^o to g^o. Combining this with the results of Theorems 3.5 and 3.6 completes the proof.

One of the main results in the theory of real interpolation is the K-divisibility theorem [2, Theorem 3.2.7] of Brudnyĭ and Krugljak. In the next corollary we show that the constant of K-divisibility of the couple (L^1, D^∞) equals one. For the definition of the K-divisibility constant of a couple see [2, page 325].

Corollary 3.9 *Suppose $f \in L^1 + D^\infty$ and $\{\varphi_n\}$ is a sequence of positive, concave functions such that $\sum_{n=1}^\infty \varphi_n(1) < \infty$. If*

$$K(t, f; L^1, D^\infty) \le \sum_n \varphi_n(t),$$

for all $t > 0$, then there exists a sequence $\{f_n\}$ of functions in $L^1 + D^\infty$ such *that*

$$f = \sum_{n=1}^{\infty} f_n \quad (\textit{convergence in } L^1 + D^\infty)$$

and

$$K(t, f_n; L^1, D^\infty) \le \varphi_n(t),$$

for all $t > 0$ and for each positive integer n. Moreover, if $f \ge 0$ then the functions f_n may be taken to be non-negative.

Proof. To start, observe that it follows from Lemma 2.1 that if $h \in L^1 + D^\infty$ is a non-negative, non-increasing function then for $t > 0$,

$$K(t, h; L^1, D^\infty) = \int_0^t h.$$

Set $K(t) = K(t, f; L^1, D^\infty)$ and note that $K(0+) = 0$. For each n, let g_n be the derivative (which exists almost everywhere) of the non-negative, concave function $\min\{K, \varphi_n\}$ so that

$$K(t) \le \sum_{n=1}^{\infty} \min\{K(t), \varphi_n(t)\} = \sum_{n=1}^{\infty} \int_0^t g_n = \int_0^t \sum_{n=1}^{\infty} g_n$$

for all $t > 0$.

Since each g_n is non-negative, non-increasing, and $\int_0^1 g_n \le \varphi_n(1) < \infty$ it follows that $g_n \in L^1 + L^\infty \subset L^1 + D^\infty$ for each n. Thus,

$$\sum_{n=1}^{\infty} \|g_n\|_{L^1+D^\infty} = \sum_{n=1}^{\infty} K(1, g_n; L^1, D^\infty) = \sum_{n=1}^{\infty} \int_0^1 g_n \le \sum_{n=1}^{\infty} \varphi_n(1) < \infty.$$

This implies that the series $\sum g_n$ converges in the Banach space $L^1 + D^\infty$. Consequently, for all $t > 0$

$$K(t, f; L^1, D^\infty) \le \int_0^t \sum_{n=1}^{\infty} g_n = K\Big(t, \sum_{n=1}^{\infty} g_n; L^1, D^\infty\Big).$$

By Theorem 3.8 there exists a linear operator $S : L^1 + D^\infty \to L^1 + D^\infty$ mapping $\sum_{n=1}^{\infty} g_n$ to f that is a contraction on both L^1 and D^∞. Hence

$$f = \sum_{n=1}^{\infty} f_n \quad (\text{convergence in } L^1 + D^\infty)$$

where $f_n = Sg_n$ for each n. Also we have

$$K(t, f_n; L^1, D^\infty) \le K(t, g_n; L^1, D^\infty) = \min\{K(t), \varphi_n(t)\} \le \varphi_n(t).$$

for all $t > 0$ and each n.

Each $g_n \ge 0$ so if $f \ge 0$ then the operator S is positive and hence each $f_n \ge 0$. This completes the proof.

4 The case of general measures

Suppose that λ is a measure on the Borel subsets of $\mathbb{R}$ satisfying $\Lambda(x) \equiv \lambda(-\infty, x] < \infty$ for all $x \in \mathbb{R}$. In this section we show that $(L^1_\lambda, D^\infty_\lambda)$ is a uniform Calderón couple.

Let m denote the Lebesgue measure on the half-line $(0, \infty)$. To construct an order-preserving, measurable transformation from $(\mathbb{R}, \lambda)$ into a subspace of $((0, \infty), m)$, let $\Omega = \{t > 0 : t \leq \Lambda(y) \text{ for some } y \in \mathbb{R}\}$ and define $\varphi : \Omega \to \mathbb{R}$ by

$$\varphi(t) = \inf\{y : t \leq \Lambda(y)\}.$$

The transformation φ induces a map of functions by composition. If f is a λ-measurable function on $\mathbb{R}$ define the map T by

$$Tf = (f \circ \varphi)\chi_\Omega.$$

Clearly Tf is a Lebesgue measurable function on $(0, \infty)$.

Since Λ is right continuous it is easy to see that for all $x \in \mathbb{R}$ and $t \in \Omega$ we have

$$\varphi(t) \leq x \quad \text{if and only if} \quad t \leq \Lambda(x). \tag{5}$$

A similar observation for $\Lambda(x-)$ is also needed: If $t < \Lambda(x-)$ then $\varphi(t) < x$ and if $\varphi(t) < x$ then $t \leq \Lambda(x-)$. Consequently, for all $x \in \mathbb{R}$ and all $t \in \Omega \setminus \{\Lambda(x-)\}$,

$$\varphi(t) < x \quad \text{if and only if} \quad t < \Lambda(x-). \tag{6}$$

Standard measure theory arguments give properties of φ in the next two lemmas.

Lemma 4.1 *For λ-almost every $x \in \mathbb{R}$, $\varphi(\Lambda(x)) = x$.*

Proof. Since Λ is right continuous, $\Lambda(\varphi(t)) \geq t$ for each $t \in \Omega$. If x is in the set $\Lambda^{-1}(t)$ then $\varphi(t) \leq x$ and

$$0 \leq \lambda(\varphi(t), x] = \Lambda(x) - \Lambda(\varphi(t)) \leq t - t = 0.$$

If follows that $\lambda(\Lambda^{-1}(t) \setminus \{\varphi(t)\}) = 0$. The non-empty sets among $\Lambda^{-1}(t) \setminus \{\varphi(t)\}$, $t \in \Omega$, are a collection of disjoint intervals so there are necessarily at most countably many of them. Therefore the set

$$E = \cup_{t \in \Omega} \Lambda^{-1}(t) \setminus \{\varphi(t)\}$$

is of λ-measure zero. If $\varphi(\Lambda(x)) \neq x$ then $x \in \Lambda^{-1}(\Lambda(x)) \setminus \{\varphi(\Lambda(x))\} \subset E$ so $\varphi(\Lambda(x)) = x$ holds λ-almost everywhere.

Lemma 4.2 *For any non-negative measurable function f on $\mathbb{R}$,*

$$\int_\Omega f \circ \varphi = \int_{\mathbb{R}} f \, d\lambda.$$

Proof. The set Ω is an interval and φ is non-decreasing. Therefore φ is a measurable point mapping from the Borel subsets of Ω to the Borel subsets of $\mathbb{R}$. The change of variable formula in [14, Proposition 15.1] shows that for any non-negative measurable function f,

$$\int_\Omega f \circ \varphi = \int_{\mathbb{R}} f \, d\mu,$$

where the measure μ is defined by $\mu(A) = m(\varphi^{-1}(A))$. To show that $\mu = \lambda$ it is enough to show that these two σ-finite Borel measures agree on sets of the form $(-\infty, x]$, for $x \in \mathbb{R}$. By (5) we have

$$\mu(-\infty, x] = m\{t \in \Omega : \varphi(t) \le x\} = m\{t \in \Omega : t \le \Lambda(x)\} = \Lambda(x) = \lambda(-\infty, x].$$

This completes the proof.

Lemma 4.3 *The map T is a positive, isometric embedding of L^1_λ into L^1 and also of D^∞_λ into D^∞.*

Proof. The map T is clearly positive. If $f \in L^1_\lambda$ then by Lemma 4.2

$$\|Tf\|_{L^1} = \int_0^\infty |Tf| = \int_\Omega |f| \circ \varphi = \int_{\mathbb{R}} |f| \, d\lambda = \|f\|_{L^1_\lambda}.$$

Thus T is an isometric embedding of L^1_λ into L^1.

Now suppose $f \in D^\infty_\lambda$. Fix $t \in \Omega$ and set $b = \Lambda(\varphi(t))$ and $a = \Lambda(\varphi(t)-)$. Note that $t \in [a, b]$ and that φ is constant on $(a, b]$. Also by (5), $\varphi(s) \le \varphi(t)$ if and only if $s \le \Lambda(\varphi(t)) = b$ so

$$\chi_{(-\infty,\varphi(t)]} \circ \varphi = \chi_{(0,b]}.$$

Therefore by Lemma 4.2,

$$\begin{aligned}
\int_0^b |f| \circ \varphi &= \int_\Omega (|f| \circ \varphi)(\chi_{(-\infty,\varphi(t)]} \circ \varphi) \\
&= \int_\Omega (|f| \chi_{(-\infty,\varphi(t)]}) \circ \varphi \\
&= \int_{\mathbb{R}} |f| \chi_{(-\infty,\varphi(t)]} \, d\lambda \\
&= \int_{(-\infty,\varphi(t)]} |f| \, d\lambda.
\end{aligned}$$

If $b = t$ then
$$\int_0^t |f| \circ \varphi = \int_{(-\infty,\varphi(t)]} |f| \, d\lambda \le t\|f\|_{D^\infty_\lambda}.$$
If $b \ne t$ then $b - a = \lambda\{\varphi(t)\} > 0$ so

$$\begin{aligned}
\int_0^t |f| \circ \varphi &= \int_0^b |f| \circ \varphi - (b-t)|f(\varphi(t))| \\
&= \int_{(-\infty,\varphi(t)]} |f| \, d\lambda - \frac{b-t}{b-a}|f(\varphi(t))|\lambda\{\varphi(t)\} \\
&= \frac{b-t}{b-a}\int_{(-\infty,\varphi(t))} |f| \, d\lambda + \frac{t-a}{b-a}\int_{(-\infty,\varphi(t)]} |f| \, d\lambda \\
&\le \left(\frac{b-t}{b-a}\Lambda(\varphi(t)-) + \frac{t-a}{b-a}\Lambda(\varphi(t))\right)\|f\|_{D^\infty_\lambda} \\
&= t\|f\|_{D^\infty_\lambda}.
\end{aligned}$$

Therefore
$$\|Tf\|_{D^\infty} = \sup_{t\ge 0}\frac{1}{t}\int_0^t |Tf| = \sup_{t\in\Omega}\frac{1}{t}\int_0^t |f| \circ \varphi \le \|f\|_{D^\infty_\lambda}.$$

For the reverse inequality we use Lemma 4.1 to see that for λ-almost every $x \in \mathbb{R}$, $\varphi(\Lambda(x)) = x$ so, setting $t = \Lambda(x)$ in the above argument puts us in the case $b = \Lambda(\varphi(\Lambda(x))) = \Lambda(x) = t$ and we have
$$\int_{(-\infty,x]} |f| \, d\lambda = \int_{(-\infty,\varphi(t)]} |f| \, d\lambda = \int_0^t |f| \circ \varphi \le t\|Tf\|_{D^\infty}.$$
Dividing by $\Lambda(x) = t$ and taking the supremum over such x shows that
$$\|f\|_{D^\infty_\lambda} \le \|Tf\|_{D^\infty}$$
to complete the proof.

Let $\mathcal{I}_\lambda$ be the collection of non-empty intervals of the form $(\Lambda(x-), \Lambda(x)]$ for $x \in \mathbb{R}$ and let A_λ be the positive operator
$$A_\lambda h(x) = A_{\mathcal{I}_\lambda}(h\chi_\Omega).$$
Clearly $h \mapsto h\chi_\Omega$ is a positive contraction on both L^1 and D^∞ so by Corollary 3.2, A_λ is also a positive contraction on both L^1 and D^∞.

Lemma 4.4 *The images of the operators T and A_λ coincide. Specifically, $A_\lambda T = T$, $T(L^1_\lambda) = A_\lambda(L^1)$ and $T(D^\infty_\lambda) = A_\lambda(D^\infty)$. Consequently, the maps $T : L^1_\lambda \to A_\lambda(L^1)$, $T : D^\infty_\lambda \to A_\lambda(D^\infty)$, and $T : L^1_\lambda + D^\infty_\lambda \to A_\lambda(L^1 + D^\infty)$ are all positive, isometric isomorphisms with positive inverses.*

Proof. Suppose f is a measurable function on $\mathbb{R}$. Since φ is constant on each interval in $\mathcal{I}_\lambda$, so is Tf. Also Tf vanishes off Ω. Thus $A_\lambda(Tf) = Tf$. It follows that $T(L^1_\lambda) \subset A_\lambda(L^1)$ and $T(D^\infty_\lambda) \subset A_\lambda(D^\infty)$.

Suppose h is a measurable function on $[0, \infty)$. Then $A_\lambda h$ is constant on each interval of $\mathcal{I}_\lambda$ and vanishes off Ω. In particular $A_\lambda h(\Lambda(\varphi(t)) = A_\lambda h(t)$ for each $t \in \Omega$. Therefore,

$$T((A_\lambda h) \circ \Lambda) = ((A_\lambda h) \circ \Lambda \circ \varphi)\chi_\Omega = (A_\lambda h)\chi_\Omega = A_\lambda h$$

and we have $A_\lambda(L^1) \subset T(L^1_\lambda)$ and $A_\lambda(D^\infty) \subset T(D^\infty_\lambda)$, proving the first statement of the theorem. The second follows from Lemma 4.3 and the observation that if $A_\lambda h \geq 0$ then so is its preimage under T, $(A_\lambda h) \circ \Lambda$.

This isomorphism enables us to express the K-functional for the pair $(L^1_\lambda, D^\infty_\lambda)$ in terms of the K-functional for (L^1, D^∞).

Lemma 4.5 *If $f \in L^1_\lambda + D^\infty_\lambda$ and $t > 0$, $K(t, f; L^1_\lambda, D^\infty_\lambda) = K(t, Tf; L^1, D^\infty)$.*

Proof. Fix $f \in L^1_\lambda + D^\infty_\lambda$ and $t > 0$. If $f = f_0 + f_1$ then $Tf = Tf_0 + Tf_1$ so by Lemma 4.3

$$\|f_0\|_{L^1_\lambda} + t\|f_1\|_{D^\infty_\lambda} = \|Tf_0\|_{L^1} + t\|Tf_1\|_{D^\infty} \geq K(t, Tf; L^1, D^\infty).$$

Taking the infimum over all possible decompositions $f = f_0 + f_1$ yields $K(t, f; L^1_\lambda, D^\infty_\lambda) \geq K(t, Tf; L^1, D^\infty)$.

If $Tf = g_0 + g_1$ then by Lemma 4.4 there exist f_0 and f_1 such that $A_\lambda g_0 = Tf_0$ and $A_\lambda g_1 = Tf_1$. Moreover,

$$Tf = A_\lambda(Tf) = A_\lambda(g_0 + g_1) = T(f_0 + f_1)$$

and, since T is an isometry, $f = f_0 + f_1$. Thus

$$\begin{aligned}
K(t, f; L^1_\lambda, D^\infty_\lambda) &\leq \|f_0\|_{L^1_\lambda} + t\|f_1\|_{D^\infty_\lambda} \\
&= \|Tf_0\|_{L^1} + t\|Tf_1\|_{D^\infty} \\
&= \|A_\lambda g_0\|_{L^1} + t\|A_\lambda g_1\|_{D^\infty} \\
&\leq \|g_0\|_{L^1} + t\|g_1\|_{D^\infty}.
\end{aligned}$$

Taking the infimum over all possible decompositions $Tf = g_0 + g_1$ yields $K(t, f; L^1_\lambda, D^\infty_\lambda) \leq K(t, Tf; L^1, D^\infty)$.

Theorem 4.6 *If $f, g \in L^1_\lambda + D^\infty_\lambda$ satisfy*

$$K(t, g; L^1_\lambda, D^\infty_\lambda) \leq K(t, f; L^1_\lambda, D^\infty_\lambda), \quad t > 0,$$

then there exists a bounded operator that is a contraction on both L^1_λ and D^∞_λ, maps f to g, and is positive if $f, g \geq 0$. In particular, $(L^1_\lambda, D^\infty_\lambda)$ is a uniform Calderón couple.

Proof. For such a pair f and g, Lemma 4.5 gives us

$$K(t, Tg; L^1, D^\infty) \leq K(t, Tf; L^1, D^\infty)$$

and Theorem 3.8 provides an operator D on $L^1 + D^\infty$, that is a contraction on both L^1 and D^∞, such that $DTf = Tg$. By Lemma 4.4, $A_\lambda T = T$ and T^{-1} maps $A_\lambda(L^1 + D^\infty)$ isometrically onto $L^1_\lambda + D^\infty_\lambda$. Therefore, the operator $T^{-1}A_\lambda DT$ is bounded on $L^1_\lambda + D^\infty_\lambda$, is a contraction on both L^1_λ and D^∞_λ, and satisfies

$$T^{-1}A_\lambda DTf = T^{-1}A_\lambda Tg = T^{-1}Tg = g.$$

The operators T, A_λ, and T^{-1} are positive and if $f, g \geq 0$ then $Tf, Tg \geq 0$ so the operator D is also positive. This completes the proof.

Using the same approach as in Theorem 4.6 and the result of Corollary 3.9 we may deduce the following.

Corollary 4.7 *The statement of Corollary 3.9 holds with L^1 replaced by L^1_λ and D^∞ replaced by D^∞_λ.*

5 Connections with the level function

Here we introduce the level function construction with respect to a general measure on $\mathbb{R}$ and describe its connection with the K-functional for the pair $(L^1_\lambda, D^\infty_\lambda)$.

As in the last section, we let λ be a measure on the Borel subsets of $\mathbb{R}$ that satisfies $\Lambda(x) = \lambda(-\infty, x] < \infty$ for $x \in \mathbb{R}$. We say that a non-negative function F is λ-concave on $\mathbb{R}$ provided

$$(\Lambda(b) - \Lambda(x))(F(x) - F(a)) \geq (F(b) - F(x))(\Lambda(x) - \Lambda(a))$$

whenever $a \leq x \leq b$.

In the special case that λ is the Lebesgue measure on $(0, \infty)$, λ-concavity reduces to the usual definition of concavity and the level function of f reduces to the function f^o introduced in Section 3. There, the function $f^o(t)$ was the derivative of the least concave majorant of $\int_0^t |f|$. For a general measure λ the construction of f^o is analogous but uses the Radon-Nikodym derivative and the notion of a least λ-concave majorant. The general construction implies the following results, presented in [22, Lemma 2.2 and Theorem 2.3].

Proposition 5.1 *To each $f \in L^1_\lambda + D^\infty_\lambda$ there corresponds a non-negative, non-increasing function f^o, called the level function of f with respect to λ, such that $\int_{(-\infty,x]} f^o\, d\lambda$ is the least λ-concave majorant of $\int_{(-\infty,x]} |f|\, d\lambda$. For a non-negative, non-increasing g,*

$$\int f^o g\, d\lambda = \sup \int |f|\bar{g}\, d\lambda$$

where the supremum is taken over all non-negative, non-increasing $\bar{g}$ such that

$$\int_{(-\infty,x]} \bar{g}\, d\lambda \le \int_{(-\infty,x]} g\, d\lambda \text{ for all } x \in \mathbb{R}.$$

The next lemma shows how the isometry introduced in Section 4 makes the connection between concavity and λ-concavity. Recall the definitions of Ω and φ given at the beginning of Section 4.

Lemma 5.2 *For any $t \in \Omega$, $\Lambda(\varphi(t)-) \le t \le \Lambda(\varphi(t))$ and, if $\theta_t \in [0, 1]$ is chosen so that $t = (1 - \theta_t)\Lambda(\varphi(t)-) + \theta_t\Lambda(\varphi(t))$, then*

$$\int_0^t f \circ \varphi = (1 - \theta_t) \int_{(-\infty,\varphi(t))} f\, d\lambda + \theta_t \int_{(-\infty,\varphi(t)]} f\, d\lambda.$$

Proof. Since Λ is right continuous

$$\Lambda(\varphi(t)) = \Lambda(\inf\{y : t \le \Lambda(y)\}) = \inf\{\Lambda(y) : t \le \Lambda(y)\} \ge t.$$

On the other hand, if $x < \varphi(t)$ then $t > \Lambda(x)$ so

$$\Lambda(\varphi(t)-) = \lim_{x\to\varphi(t)^-} \Lambda(x) \le t.$$

Thus $\Lambda(\varphi(t)-) \le t \le \Lambda(\varphi(t))$ and we can choose θ_t as above.

Set $x = \varphi(t)$ and observe that φ is constant on $(\Lambda(x-), \Lambda(x)]$ because if $\Lambda(x-) < s \le \Lambda(x)$, then $\{y : s \le \Lambda(y)\} = [x, \infty)$ and hence $\varphi(s) = x$. Now,

$$\int_{\Lambda(x-)}^t f \circ \varphi = (t - \Lambda(x-))f(x) = \theta_t(\Lambda(x) - \Lambda(x-))f(x) = \theta_t \int_{\Lambda(x-)}^{\Lambda(x)} f \circ \varphi.$$

Therefore,

$$\int_0^t f \circ \varphi = (1 - \theta_t) \int_0^{\Lambda(x-)} f \circ \varphi + \theta_t \int_0^{\Lambda(x)} f \circ \varphi.$$

By (6) and (5) we have

$$\chi_{(0,\Lambda(x-))} = \chi_{(-\infty,x)} \circ \varphi \quad \text{and} \quad \chi_{(0,\Lambda(x)]} = \chi_{(-\infty,x]} \circ \varphi$$

so we may rewrite the last expression as

$$\int_0^t f \circ \varphi = (1-\theta_t) \int_\Omega (f\chi_{(-\infty,x)}) \circ \varphi + \theta_t \int_\Omega (f\chi_{(-\infty,x]}) \circ \varphi.$$

Applying Lemma 4.2 twice yields

$$\begin{aligned}\int_0^t f \circ \varphi &= (1-\theta_t) \int_{\mathbb{R}} f\chi_{(-\infty,x)}\, d\lambda + \theta_t \int_{\mathbb{R}} f\chi_{(-\infty,x]}\, d\lambda \\ &= (1-\theta_t) \int_{(-\infty,x)} f\, d\lambda + \theta_t \int_{(-\infty,x]} f\, d\lambda\end{aligned}$$

and completes the proof.

Theorem 5.3 *The least concave majorant of* $\int_0^t f \circ \varphi$ *is* $\int_0^t f^o \circ \varphi$.

Proof. Recall that $\int_{(-\infty,x]} f^o\, d\lambda$ is the least λ-concave majorant of $\int_{(-\infty,x]} f\, d\lambda$. In particular,

$$\int_{(-\infty,x]} f\, d\lambda \leq \int_{(-\infty,x]} f^o\, d\lambda$$

for each $x \in \mathbb{R}$, and consequently,

$$\int_{(-\infty,x)} f\, d\lambda \leq \int_{(-\infty,x)} f^o\, d\lambda$$

for each $x \in \mathbb{R}$ as well.

Since φ is non-decreasing and f^o is non-increasing, $\int_0^t f^o \circ \varphi$ is concave. To see that it majorizes $\int_0^t f \circ \varphi$ we set $x = \varphi(t)$ and apply the last lemma to get

$$\begin{aligned}\int_0^t f \circ \varphi &= (1-\theta_t) \int_{(-\infty,x)} f\, d\lambda + \theta_t \int_{(-\infty,x]} f\, d\lambda \\ &\leq (1-\theta_t) \int_{(-\infty,x)} f^o\, d\lambda + \theta_t \int_{(-\infty,x]} f^o\, d\lambda \\ &= \int_0^t f^o \circ \varphi.\end{aligned}$$

It remains to show that $\int_0^t f \circ \varphi$ has no smaller concave majorant. If H is any concave majorant, then for each $x \in \mathbb{R}$ the last lemma, with $t = \Lambda(x)$, yields

$$\int_{(-\infty,x]} f\, d\lambda = \int_0^{\Lambda(x)} f \circ \varphi \leq H(\Lambda(x)).$$

It is a simple matter to check that $H \circ \Lambda$ is λ-concave and conclude that

$$\int_{(-\infty,x]} f^o\, d\lambda \leq H(\Lambda(x))$$

for each $x \in \mathbb{R}$. Since H is concave, it is continuous on $(0, \infty)$ so this implies

$$\int_{(-\infty,x)} f^o \, d\lambda \leq H(\Lambda(x-))$$

as well for each $x \in \mathbb{R}$. We apply the last lemma once more to complete the proof. For $t \in \Omega$ and $x = \varphi(t)$,

$$\begin{aligned}\int_0^t f^o \circ \varphi &= (1-\theta_t) \int_{(-\infty,x)} f^o \, d\lambda + \theta_t \int_{(-\infty,x]} f^o \, d\lambda \\ &\leq (1-\theta_t) H(\Lambda(x-)) + \theta_t H(\Lambda(x)) \\ &\leq H((1-\theta_t)\Lambda(x-) + \theta_t \Lambda(x)) \\ &= H(t).\end{aligned}$$

Theorem 5.4 *If* $f \in L^1_\lambda + D^\infty_\lambda$ *then*

$$K(t, f; L^1_\lambda, D^\infty_\lambda) = \int_0^t (f^o)^* = K(t, f^o; L^1_\lambda, L^\infty_\lambda).$$

Proof. The second equality is a standard result so we prove only the first. Lemmas 4.5 and 2.1 show that $K(t, f; L^1_\lambda, D^\infty_\lambda)$ is the least concave majorant of $\int_0^t Tf$ and by the last lemma this is just $\int_0^t f^o \circ \varphi$. We complete the proof by showing that $f^o \circ \varphi = (f^o)^*$ almost everywhere. Since $f^o \circ \varphi$ is non-increasing it is enough to show that it is equimeasurable with f^o. For any $\alpha > 0$, Lemma 4.2 shows that

$$m\{t : f \circ \varphi(t) > \alpha\} = \int_\Omega \chi_{(\alpha,\infty)} \circ f \circ \varphi = \int_{\mathbb{R}} \chi_{(\alpha,\infty)} \circ f \, d\lambda = \lambda\{t : f(t) > \alpha\}.$$

This completes the proof.

6 Exact Interpolation Spaces

In [3], Calderón gave a complete description of the exact interpolation spaces between L^1_λ and L^∞_λ in terms of the K-functional. Couples whose K-functionals satisfy this property became known as *Calderón couples.* Brudnyĭ and Krugljak later showed that all exact interpolation spaces for any uniform Calderón couple can be generated by the $\mathcal{K}$-method of interpolation. A careful analysis of the proof of their general result in the special case of the couple $(L^1_\lambda, L^\infty_\lambda)$, combined with the fact that the constant of K-divisibility for this couple equals one, leads to a beautiful complement to Calderón's description; a method of generating the norms of all the spaces in $\mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$ using only the K-functional. We formulate this known result in Proposition 6.1 to facilitate comparison with Theorem 6.2 in which we give an analogous description of all the interpolation spaces between the down spaces L^1_λ and D^∞_λ.

Also in this section, we show the down space construction maps $\mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$ into $\mathrm{Int}(L^1_\lambda, D^\infty_\lambda)$ and its image is exactly the spaces having the Fatou property.

A Banach function space Φ of Lebesgue measurable functions on $(0,\infty)$ is called a *parameter of the $\mathcal{K}$-method* provided $\min\{1,t\} \in \Phi$.

Recall that the norm in the $\mathcal{K}$-method, for the couple $(L^1_\lambda, D^\infty_\lambda)$, is given by

$$\|f\|_{K_\Phi(L^1_\lambda, D^\infty_\lambda)} = \|K(\cdot, f; L^1_\lambda, D^\infty_\lambda)\|_\Phi.$$

Proposition 6.1 *Let λ be a σ-finite measure and $X \subset L^1_\lambda + L^\infty_\lambda$ a Banach space. The following are equivalent.*

(i) $X \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$.

(ii) For some parameter Φ of the $\mathcal{K}$-method, $X = K_\Phi(L^1_\lambda, L^\infty_\lambda)$ with equality of norms.

(iii) If $g \in X$ and

$$\int_0^t f^* \le \int_0^t g^*$$

for all $t > 0$ then $f \in X$ and $\|f\|_X \le \|g\|_X$.

Next we present a direct analogue of this description for the exact interpolation spaces between L^1_λ and D^∞_λ provided λ is a measure on Borel subsets of $\mathbb{R}$ such that $\Lambda(x) \equiv \int_{(-\infty,x]} d\lambda < \infty$ for all $x \in \mathbb{R}$. It is possible to establish the next result using the general methods of [2]. However, in keeping with our self-contained approach, we provide a direct proof.

Theorem 6.2 *Let $Y \subset L^1_\lambda + D^\infty_\lambda$ be a Banach space. The following are equivalent.*

(i) $Y \in \mathrm{Int}(L^1_\lambda, D^\infty_\lambda)$.

(ii) For some parameter Φ of the $\mathcal{K}$-method, $Y = K_\Phi(L^1_\lambda, D^\infty_\lambda)$ with equality of norms.

(iii) If $g \in Y$ and

$$\int_{(-\infty,x]} f^o \, d\lambda \le \int_{(-\infty,x]} g^o \, d\lambda \text{ for all } x \in \mathbb{R}. \tag{7}$$

then $f \in Y$ and $\|f\|_Y \le \|g\|_Y$. Here f^o is the level function of f with respect to λ, introduced in Proposition 5.1.

Proof. We begin by observing that (7) is equivalent to

$$K(t,f;L^1_\lambda,D^\infty_\lambda) \le K(t,g;L^1_\lambda,D^\infty_\lambda) \quad \text{for all } t>0. \tag{8}$$

The equivalence follows readily from Lemma 5.2 and Theorem 5.4. Now suppose that (ii) holds, $g \in Y$, and f satisfies (7). Then

$$\|K(\cdot,f;L^1_\lambda,D^\infty_\lambda)\|_\Phi \le \|K(\cdot,g;L^1_\lambda,D^\infty_\lambda)\|_\Phi = \|g\|_Y < \infty$$

so $f \in K_\Phi(L^1_\lambda,D^\infty_\lambda) = Y$ and $\|f\|_Y \le \|g\|_Y$. This shows that (ii) implies (iii).

Next suppose that (iii) holds and S is a bounded linear operator on $L^1_\lambda + D^\infty_\lambda$ that is a contraction on both L^1_λ and D^∞_λ. If $g \in Y$ then for each $t > 0$,

$$K(t,Sg;L^1_\lambda,D^\infty_\lambda) \le K(t,g;L^1_\lambda,D^\infty_\lambda)$$

which is equivalent to (7) with $f = Sg$. It follows that $Sg \in Y$ and $\|Sg\|_Y \le \|g\|_Y$. Thus S is a contraction on Y and $Y \in \mathrm{Int}(L^1_\lambda,D^\infty_\lambda)$. This proves that (iii) implies (i).

To see that (i) implies (ii) suppose that $Y \in \mathrm{Int}(L^1_\lambda,D^\infty_\lambda)$. If φ is a Lebesgue measurable function on $(0,\infty)$ let $\tilde\varphi$ denote the least concave majorant of $|\varphi|$ if it exists and set $\tilde\varphi = \infty$ otherwise. For any $h \in L^1_\lambda + D^\infty_\lambda$ set

$$\rho(h) = \begin{cases} \|h\|_Y, & h \in Y \\ \infty, & h \notin Y \end{cases}$$

and define

$$\|\varphi\|_\Phi = \sup\{\rho(h) : h \in L^1_\lambda + D^\infty_\lambda \text{ and } K(t,h;L^1_\lambda,D^\infty_\lambda) \le \tilde\varphi(t) \text{ for all } t > 0\}.$$

Let Φ be the collection of those functions φ for which $\|\varphi\|_\Phi < \infty$.

Clearly, $\|\varphi\|_\Phi \ge 0$ for all φ with equality when $\varphi = 0$ almost everywhere. The homogeneity of $\|\cdot\|_\Phi$ is also easy to check, as is the property that if $\psi \in \Phi$ and $|\varphi| \le |\psi|$ almost everywhere then $\varphi \in \Phi$ and $\|\varphi\|_\Phi \le \|\psi\|_\Phi$.

To show that Φ is a Banach function space it remains to check that only the zero function has zero norm, that the triangle inequality holds, and that the space is complete.

Suppose $\|\varphi\|_\Phi = 0$ and fix $x \in \mathbb{R}$ such that $\Lambda(x) > 0$. (We ignore the trivial case when λ is the zero measure.) Let R be any real number satisfying $0 \le R \le \tilde\varphi(\Lambda(x))/\Lambda(x)$ and set $h = R\chi_{(-\infty,x]}$. The simple function h is non-increasing so we have $h = h^o$ and therefore $(h^o)^* = R\chi_{(0,\Lambda(x))}$. By the concavity of the non-negative function $\tilde\varphi$,

$$K(t,h;L^1_\lambda,D^\infty_\lambda) = \int_0^t (h^o)^* = R\min\{\Lambda(x),t\} \le \tilde\varphi(t)$$

for all $t > 0$. Now $h \in L^1_\lambda \cap D^\infty_\lambda \subset Y$ so

$$\|h\|_Y \le \|\varphi\|_\Phi = 0.$$

Since Y is embedded in $L^1_\lambda + D^\infty_\lambda$ we have

$$0 = \|h\|_{L^1_\lambda + D^\infty_\lambda} = R \min\{\Lambda(x), 1\}$$

and we conclude that $R = 0$ and hence $\tilde{\varphi}(\Lambda(x)) = 0$. Since $\tilde{\varphi}$ is non-negative and concave it must be identically zero and therefore φ is zero almost everywhere.

Let $\sum \varphi_n$ be an absolutely convergent series in Φ. Since $\|\tilde{\psi}\|_\Phi = \|\psi\|_\Phi$ for each $\psi \in \Phi$, $\sum \tilde{\varphi}_n$ is also absolutely convergent in Φ. Set $\varphi(t) = \sum_{n=1}^\infty \tilde{\varphi}_n(t)$ for each $t > 0$. A standard argument shows that this series converges everywhere. If $h \in L^1_\lambda + D^\infty_\lambda$ with $K(t, h; L^1_\lambda, D^\infty_\lambda) \le \tilde{\varphi}(t)$ for all $t > 0$ then

$$K(t, h; L^1_\lambda, D^\infty_\lambda) \le \sum_{n=1}^\infty \tilde{\varphi}_n(t)$$

for all $t > 0$. By Corollary 4.7 there exist functions h_n such that $h = \sum_{n=1}^\infty h_n$ (convergence in $L^1_\lambda + D^\infty_\lambda$) and, for each n, $K(t, h_n; L^1_\lambda, D^\infty_\lambda) \le \tilde{\varphi}_n(t)$ for all $t > 0$. Since

$$\sum_{n=1}^\infty \|h_n\|_Y \le \sum_{n=1}^\infty \|\varphi_n\|_\Phi < \infty,$$

the series $\sum h_n$ converges in Y. By the continuous inclusion of Y in $L^1_\lambda + D^\infty_\lambda$ the limit equals h and

$$\|h\|_Y \le \sum_{n=1}^\infty \|\varphi_n\|_\Phi.$$

Taking the supremum over all such h yields

$$\left\| \sum_{n=1}^\infty \varphi_n \right\|_\Phi \le \left\| \sum_{n=1}^\infty |\varphi_n| \right\|_\Phi \le \sum_{n=1}^\infty \|\varphi_n\|_\Phi < \infty.$$

Restricting this argument to just two terms proves the triangle inequality in Φ so Φ is a normed space. The unrestricted argument proves completeness.

Suppose now that $f \in L^1_\lambda + D^\infty_\lambda$ and

$$\|K(\cdot, f; L^1_\lambda, D^\infty_\lambda)\|_\Phi < \infty.$$

Clearly $f \in Y$ and we have

$$\|f\|_Y \le \|K(\cdot, f; L^1_\lambda, D^\infty_\lambda)\|_\Phi.$$

On the other hand, if $f \in Y$ and $h \in L^1_\lambda + D^\infty_\lambda$ satisfies

$$K(t, h; L^1_\lambda, D^\infty_\lambda) \le K(t, f; L^1_\lambda, D^\infty_\lambda)$$

for all $t > 0$ then by Theorem 4.6, there is an operator S on $L^1_\lambda + D^\infty_\lambda$ that is a contraction on both L^1_λ and D^∞_λ such that $Sf = h$. Since $Y \in \mathrm{Int}(L^1_\lambda, D^\infty_\lambda)$ we have $h \in Y$ and

$$\|h\|_Y = \|Sf\|_Y \leq \|f\|_Y.$$

Taking the supremum over all such h yields

$$\|K(\cdot, f; L^1_\lambda, D^\infty_\lambda)\|_\Phi \leq \|f\|_Y.$$

In particular, since $\chi_{(-\infty,x]} \in L^1_\lambda \cap D^\infty_\lambda \subset Y$ for all $x \in \mathbb{R}$, and

$$\begin{aligned}\min\{1,t\} &\leq \max\{1, 1/\Lambda(x)\} \min\{\Lambda(x), t\} \\ &= \max\{1, 1/\Lambda(x)\} K(t, \chi_{(-\infty,x]}; L^1_\lambda, D^\infty_\lambda)\end{aligned}$$

we see that $\min\{1,t\}$ is in Φ. Thus Φ is a parameter of the $\mathcal{K}$-method. We conclude that $Y = K_\Phi(L^1_\lambda, D^\infty_\lambda)$ with equality of norms. This completes the proof.

Given a Banach function space X, the norms in X' and X'' are given by

$$\|g\|_{X'} = \sup_{0 \leq f} \frac{\int f|g|\, d\lambda}{\|f\|_X},$$

and

$$\|f\|_{X''} = \sup_{0 \leq g} \frac{\int |f| g\, d\lambda}{\|g\|_{X'}}. \tag{9}$$

Comparing (1) and (9) we find that $\|f\|_{X^\downarrow} \leq \|f\|_{X''}$ for each $f \in X''$. It follows that $X \subset X'' \subset X^\downarrow$.

Lemma 6.3 *Let X be a Banach function space of λ-measurable functions. Then*

(i) $X^\downarrow = (X'')^\downarrow$ with equality of norms,

(ii) if $f \in L^1_\lambda + D^\infty_\lambda$ and $f^o \in X$ then $f \in X^\downarrow$ and $\|f\|_{X^\downarrow} \leq \|f^o\|_{X''}$,

(iii) if $X \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$ and $f \in X$ then $f^o \in X$ and $\|f^o\|_X \leq \|f\|_X$,

(iv) if $X \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$ and $f \in X^\downarrow$ then $f^o \in X''$ and $\|f^o\|_{X''} = \|f\|_{X^\downarrow}$.

Proof. (i). The definition of the norm in the down spaces, together with the fact that $X' = X'''$ with equality of norms ([23, Theorem 68.2b]) yields $X^\downarrow = (X'')^\downarrow$ with equality of norms.

(ii). Fix $f \in L^1_\lambda + D^\infty_\lambda$. If g is non-increasing and $\|g\|_{X'} \leq 1$ then by Proposition 5.1

$$\int |f| g\, d\lambda \leq \int f^o g\, d\lambda \leq \|f^o\|_{X''}.$$

Taking the supremum over all such g yields $\|f\|_{X^\downarrow} \le \|f^o\|_{X''}$.

(iii). Suppose now that X is an exact interpolation space between L^1_λ and L^∞_λ. The norm in D^∞_λ is smaller than the norm in L^∞_λ so

$$K(t, f; L^1_\lambda, D^\infty_\lambda) \le K(t, f; L^1_\lambda, L^\infty_\lambda)$$

for all $f \in L^1_\lambda + D^\infty_\lambda$ and all $t > 0$. If $\bar{f}$ is a non-negative, non-increasing function satisfying

$$K(t, \bar{f}; L^1_\lambda, L^\infty_\lambda) \le K(t, f^o; L^1_\lambda, L^\infty_\lambda)$$

for all $t > 0$ then, combining these two inequalities with Theorem 5.4 yields

$$K(t, \bar{f}; L^1_\lambda, L^\infty_\lambda) \le K(t, f; L^1_\lambda, L^\infty_\lambda)$$

for $t > 0$. Since $X \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$, we may apply Calderón's celebrated result to conclude that $\bar{f} \in X$ and

$$\|\bar{f}\|_X \le \|f\|_X.$$

In particular, taking $\bar{f} = f^o$ yields the required result.

In the proof of (iv) we require a corresponding result for the space X'. Let $g \in L^1_\lambda + D^\infty_\lambda$ and $\bar{g}$ be any non-negative, non-increasing function that satisfies

$$K(t, \bar{g}; L^1_\lambda, L^\infty_\lambda) \le K(t, g^o; L^1_\lambda, L^\infty_\lambda)$$

for all $t > 0$. For any $f \in X$, Proposition 5.1 shows that

$$\int g^o f^o \, d\lambda = \sup\left\{ \int |g| \bar{f} \, d\lambda : 0 \le \bar{f} \downarrow, K(\cdot, \bar{f}; L^1_\lambda, L^\infty_\lambda) \le K(\cdot, f^o; L^1_\lambda, L^\infty_\lambda) \right\}.$$

Here we have used the equivalence of (7) and (8), applied to the functions $\bar{f}$ and f^o. Proposition 5.1 yields

$$\int \bar{g}|f| \, d\lambda \le \int \bar{g} f^o \, d\lambda \le \int g^o f^o \, d\lambda$$

and our inequality for $\bar{f}$ in the proof of (iii) above shows that

$$\int |g| \bar{f} \, d\lambda \le \|g\|_{X'} \|\bar{f}\|_X \le \|g\|_{X'} \|f\|_X.$$

Combining these gives the estimate

$$\int \bar{g}|f| \, d\lambda \le \|g\|_{X'} \|f\|_X.$$

Taking the supremum over all such f gives

$$\|\bar{g}\|_{X'} \le \|g\|_{X'}.$$

(iv). Let $f \in X^\downarrow$. For each $g \in X'$, Proposition 5.1 (using the equivalence of (7) and (8) applied to the functions $\bar g$ and g^o) shows that

$$\begin{aligned}
&\int f^o|g|\,d\lambda \le \int f^o g^o\,d\lambda \\
&= \sup\left\{\int |f|\bar g\,d\lambda : 0 \le \bar g \downarrow, K(\cdot,\bar g; L^1_\lambda, L^\infty_\lambda) \le K(\cdot, g^o; L^1_\lambda, L^\infty_\lambda)\right\} \\
&\le \sup\left\{\|f\|_{X^\downarrow}\|\bar g\|_{X'} : 0 \le \bar g \downarrow, K(\cdot,\bar g; L^1_\lambda, L^\infty_\lambda) \le K(\cdot, g^o; L^1_\lambda, L^\infty_\lambda)\right\} \\
&\le \|f\|_{X^\downarrow}\|g\|_{X'}.
\end{aligned}$$

We conclude that $f^o \in X''$ and $\|f^o\|_{X''} \le \|f\|_{X^\downarrow}$ as required.

It is well known that X has the Fatou property if and only if $X = X''$ isometrically. The last lemma simplifies somewhat in this case.

Our final result exposes the close connection between the rearrangement invariant spaces $(\mathrm{Int}(L^1_\lambda, L^\infty_\lambda))$, the level function, and the down space construction. It extends and strengthens Corollary 2.4 of [19].

Theorem 6.4 *Suppose $Y \subset L^1_\lambda + D^\infty_\lambda$. Then $Y \in \mathrm{Int}(L^1_\lambda, D^\infty_\lambda)$ if and only if*

$$\|f\|_Y = \|f^o\|_X \text{ for all } f \in Y \text{ and } \quad Y = \{f \in L^1_\lambda + D^\infty_\lambda : f^o \in X\} \qquad (10)$$

for some $X \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$. Also, $Y = X^\downarrow$, with equality of norms, for some $X \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$ if and only if $Y \in \mathrm{Int}(L^1_\lambda, D^\infty_\lambda)$ and Y has the Fatou property.

Proof. If $Y \in \mathrm{Int}(L^1_\lambda, D^\infty_\lambda)$ then Theorem 6.2 shows that $Y = K_\Phi(L^1_\lambda, D^\infty_\lambda)$, with equality of norms, for some parameter Φ of the $\mathcal{K}$-method. Let $X = K_\Phi(L^1_\lambda, L^\infty_\lambda)$. Then $X \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$ and, by Theorem 5.4, if $f \in Y$ then

$$\|f\|_Y = \|K(\cdot, f; L^1_\lambda, D^\infty_\lambda)\|_\Phi = \|K(\cdot, f^o; L^1_\lambda, L^\infty_\lambda)\|_\Phi = \|f^o\|_X.$$

Also,

$$\begin{aligned}
Y &= \{f \in L^1_\lambda + D^\infty_\lambda : K(\cdot, f; L^1_\lambda, D^\infty_\lambda) \in \Phi\} \\
&= \{f \in L^1_\lambda + D^\infty_\lambda : K(\cdot, f^o; L^1_\lambda, L^\infty_\lambda) \in \Phi\} \\
&= \{f \in L^1_\lambda + D^\infty_\lambda : f^o \in X\}.
\end{aligned}$$

Conversely, if (10) holds for some $X \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$ then, by Proposition 6.1, $X = K_\Phi(L^1_\lambda, L^\infty_\lambda)$, with equality of norms, for some parameter Φ of the $\mathcal{K}$-method. Thus

$$\|f\|_Y = \|f^o\|_X = \|K(\cdot, f^o; L^1_\lambda, L^\infty_\lambda)\|_\Phi = \|K(\cdot, f; L^1_\lambda, D^\infty_\lambda)\|_\Phi,$$

for $f \in Y$, and

$$
\begin{aligned}
Y &= \{f \in L^1_\lambda + D^\infty_\lambda : f^o \in X\} \\
&= \{f \in L^1_\lambda + D^\infty_\lambda : K(\cdot, f^o; L^1_\lambda, L^\infty_\lambda) \in \Phi\} \\
&= \{f \in L^1_\lambda + D^\infty_\lambda : K(\cdot, f; L^1_\lambda, D^\infty_\lambda) \in \Phi\} \\
&= K_\Phi(L^1_\lambda, D^\infty_\lambda).
\end{aligned}
$$

Therefore $Y = K_\Phi(L^1_\lambda, D^\infty_\lambda)$ with equality of norms and so $Y \in \mathrm{Int}(L^1_\lambda, D^\infty_\lambda)$. This proves the first statement of the theorem.

Now suppose that $Y = X^\downarrow$ with equality of norms for some $X \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$. As we mentioned in the introduction, $X^\downarrow$ has the Fatou property. It is a consequence of [23, Theorem 71.2] that any contraction on X is a contraction on X'' so we also have $X'' \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$. Lemma 6.3(iv) shows for every $f \in X^\downarrow$,

$$\|f\|_Y = \|f\|_{X^\downarrow} = \|f^o\|_{X''}.$$

The spaces $X^\downarrow$ and X'' are defined in terms of their norms so

$$
\begin{aligned}
Y = X^\downarrow &= \{f \in L^1_\lambda + D^\infty_\lambda : \|f\|_{X^\downarrow} < \infty\} \\
&= \{f \in L^1_\lambda + D^\infty_\lambda : \|f^o\|_{X''} < \infty\} \\
&= \{f \in L^1_\lambda + D^\infty_\lambda : f^o \in X''\}.
\end{aligned}
$$

Thus (10) holds with X replaced by X'' and we may apply the first statement of the theorem to conclude that $Y \in \mathrm{Int}(L^1_\lambda, D^\infty_\lambda)$.

For the converse, we suppose that $Y \in \mathrm{Int}(L^1_\lambda, D^\infty_\lambda)$ has the Fatou property. The first part of the theorem provides an $X \in \mathrm{Int}(L^1_\lambda, L^\infty_\lambda)$ such that (10) holds. To complete the proof we show that $\|f^o\|_X = \|f\|_{X^\downarrow}$ for all $f \in X^\downarrow$. In view of Lemma 6.3(iv) it is enough to show that $\|f^o\|_X = \|f^o\|_{X''}$ for all $f \in X^\downarrow$. The inequality $\|f^o\|_X \geq \|f^o\|_{X''}$ is immediate.

According to [23, Theorem 71.2],

$$\|f^o\|_{X''} = \inf \lim_{n\to\infty} \|f_n\|_X$$

where the infimum is taken over all those non-negative sequences $\{f_n\}$ of λ-measurable functions such that $f_n \uparrow f^o$ λ-almost everywhere. If $\{f_n\}$ is such a sequence, then (10), the Fatou property in Y, Lemma 6.3(iii), and the observation that $f^o = (f^o)^o$, show that

$$\|f^o\|_X = \|(f^o)^o\|_X = \|f^o\|_Y = \lim_{n\to\infty} \|f_n\|_Y = \lim_{n\to\infty} \|f_n^o\|_X \leq \lim_{n\to\infty} \|f_n\|_X.$$

Taking the infimum yields $\|f^o\|_X \leq \|f^o\|_{X''}$ and completes the proof.

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