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RESEARCH ARTICLE

A Formula for the Norm of an Averaging Operator on Weighted Lebesgue Space

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Members of a family of averaging operators associated with systems of partial differential operators are studied as maps on a class of weighted Lebesgue spaces. Those spaces on which the operators are bounded maps are determined and the operator norms are given precisely.

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1. Introduction

The operator $\mathcal{R}_\alpha$ is defined by

$$\mathcal{R}_\alpha f(r,x) = \frac{\alpha}{\pi}\int_{-1}^{1}\int_{-1}^{1} f(r|s|\sqrt{1-t^2}, x+tr)(1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1}\,dt\,ds$$

for $\alpha > 0$ and by

$$\mathcal{R}_0 f(r,x) = \frac{1}{\pi}\int_{-1}^{1} f(r\sqrt{1-t^2}, x+tr)(1-t^2)^{-1/2}\,dt$$

when $\alpha = 0$. These operators have been extensively studied in [1] and [6], as well as in a more general form in [3]. They arise in connection with the system

$$\Delta_1 = \frac{\partial}{\partial x}, \quad \Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}$$

of partial differential operators. The boundedness of integral operators associated with a related differential operator is studied in [4].

The Lebesgue spaces L^p with weights of the form $|x|^\alpha$ are a natural collection to consider when boundedness of integral operators is concerned, see [7, 8, 10]. This is particularly true when studying integral operators connected with differential systems. Knowing the range of the parameter α and the index p for which an operator is bounded on weighted Lebesgue space gives quantitative information about the rate of growth of the transformed functions, about the operator itself,

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and about the original differential system. For some research in this direction, see [2, 5, 9, 11, 12].

In this paper we consider the boundedness of the operators $\mathcal{R}_\alpha$ on the weighted Lebesgue spaces $L^p((0,\infty)\times(-\infty,\infty), r^\beta drdx)$. For convenience we refer to this space as L^p_β and denote its norm by

$$\|f\|_{p,\beta} = \left(\int_{-\infty}^{\infty}\int_0^\infty |f(r,x)|^p r^\beta \,dr\,dx\right)^{1/p}.$$

Here β can be any real number. The space $L^\infty_\beta \equiv L^\infty$ does not depend on β, and

$$\|f\|_{\infty,\beta} \equiv \|f\|_\infty = \operatorname{ess\,sup}\{|f(r,x)| : (r,x)\in(0,\infty)\times(-\infty,\infty)\}.$$

Our object is to investigate whether or not $\mathcal{R}_\alpha$ is a bounded operator on L^p_β, that is, whether or not there exists a constant C such that the inequality

$$\|R_\alpha f\|_{p,\beta} \le C\|f\|_{p,\beta}$$

holds for all $f\in L^p_\beta$. This question is completely answered. In addition, we provide a formula for the least possible constant C for which the inequality holds. This is called the *operator norm* of R_α on L^p_β.

As usual, for $p\in[1,\infty]$ we define p' by $1/p+1/p'=1$.

2. Determining the Operator Norm

Making the substitution $t=\sin\theta$ and observing that the integral defining $\mathcal{R}_\alpha$ is symmetric in s, places the operators $\mathcal{R}_\alpha$ and $\mathcal{R}_0$ in the form that we will use most often:

$$\mathcal{R}_\alpha f(r,x) = \frac{2\alpha}{\pi}\int_0^1\int_{-\pi/2}^{\pi/2} f(rs\cos\theta, x+r\sin\theta)(\cos\theta)^{2\alpha}(1-s^2)^{\alpha-1}\,d\theta\,ds$$

for $\alpha>0$ and

$$\mathcal{R}_0 f(r,x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(r\cos\theta, x+r\sin\theta)\,d\theta.$$

It is clear from this form that $\mathcal{R}_0 f(r,x)$ is the average of f over the right semicircle of radius r, centred at $(0,x)$.

To see that $\mathcal{R}_\alpha$ is also an averaging operator for $\alpha>0$ we make the change of variable

$$u = rs\cos\theta, \quad v = x + r\sin\theta.$$

The Jacobian determinant is $|\partial(u,v)/\partial(\theta,s)| = r^2(\cos\theta)^2$ and the map $(\theta,s)\to(u,v)$ takes the open rectangle $(-\pi/2,\pi/2)\times(0,1)$ one-to-one and onto the open half disc $D_r^+(0,x) = \{(u,v) : u>0, u^2+(v-x)^2<r^2\}$. We have

$$\mathcal{R}_\alpha f(r,x) = \frac{2\alpha}{r^{2\alpha}\pi}\iint_{D_r^+(0,x)} f(u,v)(r^2-u^2-(v-x)^2)^{\alpha-1}\,dudv.$$

An easy calculation gives

$$\frac{2\alpha}{r^{2\alpha}\pi}\iint_{D_r^+(0,x)}(r^2-u^2-(v-x)^2)^{\alpha-1}\,dudv=1$$

showing that $\mathcal{R}_\alpha$ is also an averaging operator. Indeed, $\mathcal{R}_\alpha f(r,x)$ is the weighted average of f over $D_r^+(0,x)$, where the weight $(r^2-u^2-(v-x)^2)^{\alpha-1}$ is a power of the distance to the boundary of the disc.

Our first result is an immediate consequence of these observations.

Theorem 2.1 *If $\alpha\geq 0$ then $\mathcal{R}_\alpha$ is a bounded operator on L^∞ with operator norm equal to 1. That is, the inequality $\|\mathcal{R}_\alpha f\|_\infty\leq\|f\|_\infty$ holds for all bounded f and when $C<1$ the inequality $\|\mathcal{R}_\alpha f\|_\infty\leq C\|f\|_\infty$ fails to hold for some bounded f.*

Proof. Let f be a bounded function and view $\|f\|_\infty$ as a constant function. Certainly, $f\leq\|f\|_\infty$ almost everywhere. Since $\mathcal{R}_\alpha$ is an averaging operator, for each (r,x) we have,

$$|\mathcal{R}_\alpha f(r,x)|\leq\mathcal{R}_\alpha(\|f\|_\infty)(r,x)=\|f\|_\infty.$$

Taking the essential supremum over all (r,x) proves that $\|\mathcal{R}_\alpha f\|_\infty\leq\|f\|_\infty$ and shows that the operator norm of $\mathcal{R}_\alpha$ is at most 1. Taking f to be a non-zero constant function reduces the above inequality to equality, showing that $\|\mathcal{R}_\alpha f\|_\infty\leq C\|f\|_\infty$ fails when $C<1$ and proving that the operator norm is at least 1. This completes the proof.

The case $p=1$ is also treated separately both for technical reasons and because the operator norm is achieved for all non-negative functions f.

Theorem 2.2 *If $\alpha\geq 0$ and $\beta<0$ then $\mathcal{R}_\alpha$ is a bounded operator on L^1_β with operator norm equal to*

$$\frac{\Gamma(\alpha+1)\Gamma(-\frac{\beta}{2})}{\Gamma(\alpha+\frac{1}{2}-\frac{\beta}{2})\Gamma(\frac{1}{2})}.$$

If $\beta\geq 0$, then $\mathcal{R}_\alpha$ is not a bounded operator on L^1_β.

Proof. Suppose $f\in L^1_\beta$. Then

$$\begin{aligned}\|\mathcal{R}_0 f\|_{1,\beta}&=\int_{-\infty}^{\infty}\int_0^{\infty}|\mathcal{R}_0 f(r,x)|r^\beta\,dr\,dx\\&\leq\int_{-\infty}^{\infty}\int_0^{\infty}\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}|f(r\cos\theta,x+r\sin\theta)|\,d\theta\,r^\beta\,dr\,dx.\end{aligned}$$

Interchanging the order of integration and making the change of variable $t=r\cos\theta$

and $y = x + r\sin\theta$ yields

$$
\begin{aligned}
\|\mathcal{R}_0 f\|_{1,\beta} &\le \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty}\int_0^{\infty} |f(r\cos\theta, x + r\sin\theta)| r^{\beta}\, dr\, dx\, d\theta \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty}\int_0^{\infty} |f(t,y)| t^{\beta}\, dt\, dy\, (\cos\theta)^{-\beta-1}\, d\theta \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (\cos\theta)^{-\beta-1}\, d\theta \|f\|_{1,\beta} \\
&= \frac{\Gamma(-\frac{\beta}{2})}{\Gamma(\frac{1}{2}-\frac{\beta}{2})\Gamma(\frac{1}{2})}\|f\|_{1,\beta},
\end{aligned}
$$

provided $\beta < 0$. The last integral above diverges when $\beta \ge 0$.

Since this inequality reduces to equality when f is non-negative, the constant is best possible. In particular, when $\beta \ge 0$ the best constant is infinite so $\mathcal{R}_0$ is not a bounded operator on L^1_β. This completes the proof in the case $\alpha = 0$.

When $\alpha > 0$ we proceed similarly,

$$
\begin{aligned}
\|\mathcal{R}_\alpha f\|_{1,\beta} &= \int_{-\infty}^{\infty}\int_0^{\infty} |\mathcal{R}_\alpha f(r,x)| r^{\beta}\, dr\, dx \\
&\le \int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\alpha}{\pi}\int_0^1\int_{-\pi/2}^{\pi/2} |f(rs\cos\theta, x + r\sin\theta)| \times \\
&\qquad (\cos\theta)^{2\alpha}(1-s^2)^{\alpha-1}\, d\theta\, ds\, r^{\beta}\, dr\, dx.
\end{aligned}
$$

Interchanging again and making the change of variable, $t = rs\cos\theta$ and $y = x + r\sin\theta$, yields

$$
\begin{aligned}
\|\mathcal{R}_\alpha f\|_{1,\beta} &\le \frac{2\alpha}{\pi}\int_0^1\int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty}\int_0^{\infty} |f(rs\cos\theta, x + r\sin\theta)| \times \\
&\qquad r^{\beta}\, dr\, dx (\cos\theta)^{2\alpha}(1-s^2)^{\alpha-1}\, d\theta\, ds \\
&= \frac{2\alpha}{\pi}\int_0^1\int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty}\int_0^{\infty} |f(t,y)| t^{\beta}\, dt\, dy \times \\
&\qquad (\cos\theta)^{2\alpha-\beta-1} s^{-\beta-1}(1-s^2)^{\alpha-1}\, d\theta\, ds \\
&= \frac{2\alpha}{\pi}\int_0^1 s^{-\beta-1}(1-s^2)^{\alpha-1}\, ds \int_{-\pi/2}^{\pi/2} (\cos\theta)^{2\alpha-\beta-1}\, d\theta \|f\|_{1,\beta} \\
&= \frac{\Gamma(\alpha+1)\Gamma(-\frac{\beta}{2})}{\Gamma(\alpha+\frac{1}{2}-\frac{\beta}{2})\Gamma(\frac{1}{2})}\|f\|_{1,\beta},
\end{aligned}
$$

provided $\beta < 0$. The "ds" integral above diverges when $\beta \ge 0$.

Since this inequality also reduces to equality when f is non-negative, the constant is best possible. In particular, when $\beta \ge 0$ the best constant is infinite so $\mathcal{R}_\alpha$ is not a bounded operator on L^1_β. This completes the proof.

The next two theorems determine the values of β for which R_α is a bounded operator on L^p_β when $1 < p < \infty$. We begin by looking at the case $\alpha = 0$.

Theorem 2.3 *Suppose $1 < p < \infty$. Then $\mathcal{R}_0$ is a bounded operator on L^p_β if and only if $\beta < p-1$. Moreover, if $\beta < p-1$ then the operator norm is*

$$\frac{\Gamma(\frac{1}{2}-\frac{\beta+1}{2p})}{\Gamma(1-\frac{\beta+1}{2p})\Gamma(\frac{1}{2})}.$$

Proof. Suppose first that $\beta > p-1$ and define the function f by setting $f(t,y) = 1/t$ when $(t,y) \in (0,1)\times(-2,2)$ and $f(t,y) = 0$ otherwise. Then

$$\|f\|^p_{p,\beta} = \int_{-\infty}^{\infty}\int_0^{\infty} |f(t,y)|^p t^\beta \, dt\, dy = 4\int_0^1 t^{\beta-p}\, dt < \infty$$

so $f \in L^p_\beta$. On the other hand, if $0 < r < 1$ and $-1 < x < 1$, then

$$\mathcal{R}_0 f(r,x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{r\cos\theta}\, d\theta = \infty$$

so $\mathcal{R}_0 f \notin L^p_\beta$. Thus R_0 is not a map from L^p_β to L^p_β.

When $\beta = p-1$ a similar argument shows R_0 does not map L^p_β to L^p_β. This time, let $f(t,y) = 1/(t(1-\log t))$ when $(t,y) \in (0,1)\times(-2,2)$ and $f(t,y) = 0$ otherwise. We have

$$\|f\|^p_{p,\beta} = \int_{-\infty}^{\infty}\int_0^{\infty} |f(t,y)|^p t^\beta \, dt\, dy = 4\int_0^1 \frac{1}{t(1-\log t)^p}\, dt < \infty$$

so $f \in L^p_\beta$. But, if $0 < r < 1$ and $-1 < x < 1$, then

$$\mathcal{R}_0 f(r,x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{r\cos\theta(1-\log(r\cos\theta))}\, d\theta = \infty$$

so $\mathcal{R}_0 f \notin L^p_\beta$.

Now suppose that $\beta < p-1$ and fix $f \in L^p_\beta$. Let γ be a real constant to be determined later. Then, by Hölder's inequality,

$$\begin{aligned}|\mathcal{R}_0 f(r,x)| &\le \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} |f(r\cos\theta, x + r\sin\theta)|\, d\theta \\ &\le \frac{C_1^{1/p'}}{\pi}\left(\int_{-\pi/2}^{\pi/2} |f(r\cos\theta, x+r\sin\theta)|^p (\cos\theta)^{p\gamma}\, d\theta\right)^{1/p},\end{aligned}$$

where

$$C_1 = \int_{-\pi/2}^{\pi/2} (\cos\theta)^{-p'\gamma}\, d\theta.$$

Using this estimate we get,

$$\|\mathcal{R}_0 f\|_{p,\beta}^p \le \frac{C_1^{p/p'}}{\pi^p} \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\pi/2}^{\pi/2} |f(r\cos\theta, x + r\sin\theta)|^p \times (\cos\theta)^{p\gamma}\, d\theta\, r^\beta\, dr\, dx$$
$$= \frac{C_1^{p/p'}}{\pi^p} \int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty}\int_0^{\infty} |f(r\cos\theta, x + r\sin\theta)|^p \times r^\beta\, dr\, dx (\cos\theta)^{p\gamma}\, d\theta.$$

The change of variable $t = r\cos\theta$ and $y = x + r\sin\theta$ yields

$$\|\mathcal{R}_0 f\|_{p,\beta}^p \le \frac{C_1^{p/p'}}{\pi^p} \int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty}\int_0^{\infty} |f(t,y)|^p t^\beta\, dt\, dy (\cos\theta)^{p\gamma-\beta-1}\, d\theta$$
$$= \frac{C_1^{p/p'} C_2}{\pi^p} \|f\|_{p,\beta}^p,$$

where

$$C_2 = \int_{-\pi/2}^{\pi/2} (\cos\theta)^{p\gamma-\beta-1}\, d\theta.$$

If there exists a γ that makes both the integrals C_1 and C_2 finite, then $\mathcal{R}_0$ is a bounded operator on L^p_β. For C_1 to be finite requires that $-p'\gamma > -1$ and for C_2 to be finite requires that $p\gamma - \beta - 1 > -1$. These reduce to the requirement that $\gamma \in (\beta/p, 1/p')$, which is a non-empty interval because we have assumed that $\beta < p - 1$.

To obtain a specific upper bound for the operator norm let $\gamma = (\beta + 1)/(pp')$ and verify that it lies in the above interval. The upper bound obtained is,

$$\frac{C_1^{1/p'} C_2^{1/p}}{\pi} = \frac{\Gamma(\frac{1}{2} - \frac{\beta+1}{2p})}{\Gamma(1 - \frac{\beta+1}{2p})\Gamma(\frac{1}{2})}.$$

To get a lower bound for the operator norm we fix $\eta > 0$ and $M > 1$, and define f by setting $f(t,y) = t^{(\eta-\beta-1)/p}$ when $(t,y) \in (0,1) \times (-M, M)$ and $f(t,y) = 0$ otherwise. The norm of f in L^p_β is

$$\|f\|_{p,\beta} = \left(\int_{-M}^{M}\int_0^1 \left(t^{(\eta-\beta-1)/p}\right)^p t^\beta\, dt\, dy\right)^{1/p} = (2M/\eta)^{1/p}.$$

On the other hand, if $0 < r < 1$ and $1 - M < x < M - 1$ then for any $\theta \in (-\pi/2, \pi/2)$, we have $0 < r\cos\theta < 1$ and $-M < x + r\sin\theta < M$ so

$$\mathcal{R}_0 f(r,x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (r\cos\theta)^{(\eta-\beta-1)/p}\, d\theta = r^{(\eta-\beta-1)/p} \frac{\Gamma(\frac{1}{2} + \frac{\eta-\beta-1}{2p})}{\Gamma(1 + \frac{\eta-\beta-1}{2p})\Gamma(\frac{1}{2})}.$$

It follows that

$$\|\mathcal{R}_0 f\|_{p,\beta} \ge \frac{\Gamma(\frac{1}{2}+\frac{\eta-\beta-1}{2p})}{\Gamma(1+\frac{\eta-\beta-1}{2p})\Gamma(\frac{1}{2})}\left(\int_{1-M}^{M-1}\int_0^1 \left(r^{(\eta-\beta-1)/p}\right)^p r^\beta\,dr\,dx\right)^{1/p}$$
$$= \frac{\Gamma(\frac{1}{2}+\frac{\eta-\beta-1}{2p})}{\Gamma(1+\frac{\eta-\beta-1}{2p})\Gamma(\frac{1}{2})}\,(2(M-1)/\eta)^{1/p}.$$

For each η and M the ratio $\|\mathcal{R}_0 f\|_{p,\beta}/\|f\|_{p,\beta}$ is a lower bound for the operator norm of $\mathcal{R}_0$. Letting $M\to\infty$ first and then letting $\eta\to 0$ gives the lower bound,

$$\frac{\Gamma(\frac{1}{2}-\frac{\beta+1}{2p})}{\Gamma(1-\frac{\beta+1}{2p})\Gamma(\frac{1}{2})}$$

as required. The upper and lower bounds coincide so the operator norm is determined.

THEOREM 2.4 *Suppose $1<p<\infty$. Then $\mathcal{R}_\alpha$ is a bounded operator on L^p_β if and only if $\beta<p-1$. Moreover, if $\beta<p-1$ then the operator norm is*

$$\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-\frac{\beta+1}{2p})}{\Gamma(\alpha+1-\frac{\beta+1}{2p})\Gamma(\frac{1}{2})}.$$

Proof. Suppose that $\beta>p-1$ and define the function f by setting $f(t,y)=1/t$ when $(t,y)\in(0,1)\times(-2,2)$ and $f(t,y)=0$ otherwise. Then

$$\|f\|_{p,\beta}^p = \int_{-\infty}^{\infty}\int_0^{\infty}|f(t,y)|^p t^\beta\,dt\,dy = 4\int_0^1 t^{\beta-p}\,dt<\infty$$

so $f\in L^p_\beta$. On the other hand, if $0<r<1$ and $-1<x<1$, then

$$\mathcal{R}_\alpha f(r,x) = \frac{2\alpha}{\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{r\cos\theta}(\cos\theta)^{2\alpha}\,d\theta\int_0^1(1-s^2)^{\alpha-1}\frac{ds}{s}=\infty$$

so $\mathcal{R}_\alpha f\notin L^p_\beta$. Thus R_α is not a map from L^p_β to L^p_β.

When $\beta=p-1$ a similar argument shows R_α does not map L^p_β to L^p_β. This time, let $f(t,y)=1/(t(1-\log t))$ when $(t,y)\in(0,1)\times(-2,2)$ and $f(t,y)=0$ otherwise. We have

$$\|f\|_{p,\beta}^p = \int_{-\infty}^{\infty}\int_0^{\infty}|f(t,y)|^p t^\beta\,dt\,dy = 4\int_0^1\frac{1}{t(1-\log t)^p}\,dt<\infty$$

so $f\in L^p_\beta$. But, if $0<r<1$ and $-1<x<1$, then

$$\mathcal{R}_\alpha f(r,x) = \frac{2\alpha}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^1\frac{(\cos\theta)^{2\alpha}(1-s^2)^{\alpha-1}}{rs\cos\theta(1-\log(rs\cos\theta))}\,ds\,d\theta=\infty$$

so $\mathcal{R}_0 f\notin L^p_\beta$.

Now suppose that $\beta<p-1$ and fix $f\in L^p_\beta$. Let γ, δ, and ε be real constants to be determined later. Clearly, $|\mathcal{R}_\alpha f(r,x)|\le\mathcal{R}_\alpha|f|(r,x)$ and by Hölder's inequality

this is no greater than

$$\frac{2\alpha C_1^{1/p'}}{\pi}\left(\int_0^1\int_{-\pi/2}^{\pi/2}|f(rs\cos\theta,x+r\sin\theta)|^p(\cos\theta)^{\gamma p}s^{\delta p}(1-s^2)^{\varepsilon p}\,d\theta\,ds\right)^{1/p},$$

where

$$C_1=\int_{-\pi/2}^{\pi/2}(\cos\theta)^{(2\alpha-\gamma)p'}\,d\theta\int_0^1 s^{-\delta p'}(1-s^2)^{(\alpha-1-\varepsilon)p'}\,ds.$$

Using this estimate we get,

$$\begin{aligned}\|\mathcal{R}_\alpha f\|_{p,\beta}^p &\le \frac{(2\alpha)^pC_1^{p/p'}}{\pi^p}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^1\int_{-\pi/2}^{\pi/2}|f(rs\cos\theta,x+r\sin\theta)|^p\times\\ &\quad(\cos\theta)^{\gamma p}s^{\delta p}(1-s^2)^{\varepsilon p}\,d\theta\,ds\,r^\beta\,dr\,dx\\ &=\frac{(2\alpha)^pC_1^{p/p'}}{\pi^p}\int_0^1\int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty}\int_0^{\infty}|f(rs\cos\theta,x+r\sin\theta)|^p\times\\ &\quad r^\beta\,dr\,dx(\cos\theta)^{\gamma p}s^{\delta p}(1-s^2)^{\varepsilon p}\,d\theta\,ds.\end{aligned}$$

The change of variable $t=rs\cos\theta$ and $y=x+r\sin\theta$ yields

$$\begin{aligned}\|\mathcal{R}_\alpha f\|_{p,\beta}^p &\le \frac{(2\alpha)^pC_1^{p/p'}}{\pi^p}\int_0^1\int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty}\int_0^{\infty}|f(t,y)|^pt^\beta\,dt\,dy\times\\ &\quad(\cos\theta)^{\gamma p-\beta-1}s^{\delta p-\beta-1}(1-s^2)^{\varepsilon p}\,d\theta\,ds\\ &=\frac{(2\alpha)^pC_1^{p/p'}C_2}{\pi^p}\|f\|_{p,\beta}^p,\end{aligned}$$

where

$$C_2=\int_{-\pi/2}^{\pi/2}(\cos\theta)^{\gamma p-\beta-1}\,d\theta\int_0^1 s^{\delta p-\beta-1}(1-s^2)^{\varepsilon p}\,ds.$$

If there exist γ, δ, and ε that make the four integrals in C_1 and C_2 all finite, then $\mathcal{R}_\alpha$ is a bounded operator on L^p_β. The requirements are that

$$(2\alpha-\gamma)p',\ -\delta p',\ (\alpha-1-\varepsilon)p',\ \gamma p-\beta-1,\ \delta p-\beta-1,\ \varepsilon p$$

all be greater than -1. These conditions reduce to

$$\gamma\in(\beta/p,2\alpha+1/p'),\quad \delta\in(\beta/p,1/p'),\quad \varepsilon\in(-1/p,\alpha-1/p).$$

Since $\alpha>0$ and $\beta<p-1$ all three intervals are non-empty so it is possible to choose γ, δ, and ε that make C_1 and C_2 finite. Thus $\mathcal{R}_\alpha$ is a bounded operator on L^p_β.

To obtain a specific upper bound for the operator norm let $\gamma=(2\alpha p'+\beta+$

$1)/(pp')$, $\delta = (\beta+1)/(pp')$ and $\varepsilon = (\alpha-1)/p$. The upper bound obtained is,

$$\frac{2\alpha C_1^{1/p'} C_2^{1/p}}{\pi} = \frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-\frac{\beta+1}{2p})}{\Gamma(\alpha+1-\frac{\beta+1}{2p})\Gamma(\frac{1}{2})}.$$

To get a lower bound for the operator norm we fix $\eta > 0$ and $M > 1$, and define f by setting $f(t,y) = t^{(\eta-\beta-1)/p}$ when $(t,y) \in (0,1) \times (-M,M)$ and $f(t,y) = 0$ otherwise. The norm of f in L^p_β is

$$\|f\|_{p,\beta} = \left(\int_{-M}^{M}\int_0^1 \left(t^{(\eta-\beta-1)/p}\right)^p t^\beta \,dt\,dy\right)^{1/p} = (2M/\eta)^{1/p}.$$

On the other hand, if $0 < r < 1$ and $1-M < x < M-1$ then for any $s \in (0,1)$ and $\theta \in (-\pi/2, \pi/2)$, we have $0 < rs\cos\theta < 1$ and $-M < x + r\sin\theta < M$ so

$$\begin{aligned}\mathcal{R}_\alpha f(r,x) &= \frac{2\alpha}{\pi}\int_0^1\int_{-\pi/2}^{\pi/2} (rs\cos\theta)^{(\eta-\beta-1)/p}(\cos\theta)^{2\alpha}(1-s^2)^{\alpha-1}\,d\theta\,ds \\ &= r^{(\eta-\beta-1)/p}\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}+\frac{\eta-\beta-1}{2p})}{\Gamma(\alpha+1+\frac{\eta-\beta-1}{2p})\Gamma(\frac{1}{2})}.\end{aligned}$$

It follows that

$$\begin{aligned}\|\mathcal{R}_\alpha f\|_{p,\beta} &\geq \frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}+\frac{\eta-\beta-1}{2p})}{\Gamma(\alpha+1+\frac{\eta-\beta-1}{2p})\Gamma(\frac{1}{2})}\left(\int_{1-M}^{M-1}\int_0^1 \left(r^{(\eta-\beta-1)/p}\right)^p r^\beta\,dr\,dx\right)^{1/p} \\ &= \frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}+\frac{\eta-\beta-1}{2p})}{\Gamma(\alpha+1+\frac{\eta-\beta-1}{2p})\Gamma(\frac{1}{2})}\,(2(M-1)/\eta)^{1/p}.\end{aligned}$$

For each η and M the ratio $\|\mathcal{R}_\alpha f\|_{p,\beta}/\|f\|_{p,\beta}$ is a lower bound for the operator norm of $\mathcal{R}_\alpha$. Letting $M \to \infty$ first and then letting $\eta \to 0$ gives the lower bound,

$$\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-\frac{\beta+1}{2p})}{\Gamma(\alpha+1-\frac{\beta+1}{2p})\Gamma(\frac{1}{2})}$$

as required.

In is important to point out the operator norms calculated in the previous four theorems are all related. We do this in the following summary.

THEOREM 2.5 *Suppose $1 \leq p \leq \infty$ and $\alpha \geq 0$. The operator $\mathcal{R}_\alpha$ is a bounded map on L^p_β if and only if $\beta < p-1$. In this case the operator norm is*

$$\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-\frac{\beta+1}{2p})}{\Gamma(\alpha+1-\frac{\beta+1}{2p})\Gamma(\frac{1}{2})}.$$

Earlier work suggests that $L^p_{2\alpha+1}$ is a natural space for the operator $\mathcal{R}_\alpha$. In a final corollary we restrict our attention to the case $\beta = 2\alpha+1$.

Corollary 2.6 *Suppose $1 \leq p \leq \infty$ and $\alpha \geq 0$. The operator $\mathcal{R}_\alpha$ is a bounded map on $L^p_{2\alpha+1}$ if and only if $\alpha < (p/2) - 1$. In this case the operator norm is*

$$\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2} - \frac{\alpha+1}{p})}{\Gamma(\frac{\alpha+1}{p'})\Gamma(\frac{1}{2})}.$$

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