THE FOURIER TRANSFORM IN WEIGHTED REARRANGEMENT INVARIANT SPACES

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Abstract. It is shown that if the Fourier transform is a bounded map on a rearrangement-invariant space of functions on $\mathbb{R}^n$, modified by a weight, then the weight is bounded above and below and the space is equivalent to L^2. Also, if it is bounded from L^p to L^q, each modified by the same weight, then the weight is bounded above and below and $1 \le p = q' \le 2$. Applications prove the non-boundedness on these spaces of an operator related to the Schrödinger equation.

1. Introduction

Plancherel's theorem and the Hausdorff-Young inequality show that if $1 \le p \le 2$ and $1/p + 1/q = 1$, then there exists a C such that for all $f \in L^1 \cap L^p$,

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}. \tag{1.1}$$

It is well known that these are the only (L^p, L^q) pairs between which the Fourier transform is a bounded map. In addition, Theorem 1(ii) of [2] shows that if X is a rearrangement invariant space of functions, then the Fourier transform is bounded on X if and only if $X = L^2$, with equivalent norms.

In this paper we modify these spaces by introducing a weight function and show that the Fourier transform is bounded only if the weights are bounded above and below, reducing both problems to their respective unweighted cases. This provides a much larger class of spaces on which the Hausdorff-Young inequalities in (1.1) are effectively the only possible Fourier norm inequalities. See Theorems 2.5 and 2.6.

Following Lemma 8 of [2] we apply these results to identify a large class of spaces on which the Schrödinger multiplier $\exp(-4\pi^2 it|y|^2)$ does not give rise to a bounded convolution operator. See Corollaries 3.1 and 3.2.

If $f\colon \mathbb{R}^n \to \mathbb{C}$ is integrable, then its Fourier transform,

$$\widehat{f}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot t} f(t)\,dt,$$

is a bounded continuous function. We restrict our attention throughout to integrable functions and avoid unnecessary extensions of the Fourier transform.

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Modulation and translation operators ε_z and τ_z are defined by setting

$$\varepsilon_z f(x) = e^{-2\pi i x\cdot z} f(x) \quad \text{and} \quad \tau_z f(x) = f(x+z).$$

Observe that if f is integrable, then $\tau_z \widehat{f} = (\varepsilon_z f)\widehat{\ }$ and $(\tau_z f)\widehat{\ } = \varepsilon_{-z}\widehat{f}$, for $z \in \mathbb{R}^n$.

For each $r > 0$, let $Q_r = (-r/2, r/2)^n$ denote the cube in $\mathbb{R}^n$ with centre zero and side length r. Let $q_r = \chi_{Q_r}$ denote its characteristic function. Notice that if $y = (y_1, \ldots, y_n) \in Q_{1/(2r)}$, then for each j, $-\pi/4 \le \pi r y_j \le \pi/4$ so

$$\Big|\frac{\sin(\pi r y_j)}{\pi r y_j}\Big| \ge \frac{\sin(\pi/4)}{\pi/4} \ge \frac{1}{2}.$$

Thus, for all $y, z \in \mathbb{R}^n$,

$$|(\tau_z q_r)\widehat{\ }(y)| = \Big|e^{2\pi i y\cdot z}\prod_{j=1}^{n}\frac{\sin(\pi r y_j)}{\pi y_j}\Big| \ge \frac{r^n}{2^n} q_{1/(2r)}(y). \tag{1.2}$$

Let X be a rearrangement invariant space of complex valued functions on $\mathbb{R}^n$. For the definition, we refer the reader to [1] and only recall the properties of X that we need here. The space X is a Banach space, equipped with a norm $\|\cdot\|_X$, and satisfying the following:

(a) Characteristic functions of sets of finite measure are in X.
(b) If $g \in X$ and $|f| \le |g|$ almost everywhere, then $f \in X$ and $\|f\|_X \le \|g\|_X$.
(c) If $0 \le f_k \in X$ for each k and f_k increases pointwise almost everywhere to f as $k \to \infty$, then $f \in X$ and $\|f_k\|_X \to \|f\|_X$ as $k \to \infty$.
(d) If $g \in X$ and f and g are equimeasurable, that is, for all $\alpha > 0$, the sets $\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}$ and $\{x \in \mathbb{R}^n \colon |g(x)| > \alpha\}$ have the same (Lebesgue) measure, then $f \in X$ and $\|f\|_X \le \|g\|_X$.
(e) The associate space X' is defined to be the set of measurable g such that

$$\|g\|_{X'} = \sup\Big\{\int_{\mathbb{R}} |fg| \colon \|f\|_X \le 1\Big\}$$

is finite. It is also a rearrangement invariant space, and $(X')' = X$.
(f) If T is a sublinear operator on $L^1 + L^\infty$ that maps L^1 to L^1 and L^∞ to L^∞ such that $\|Tf\|_{L^1} \le \|f\|_{L^1}$ for all $f \in L^1$ and $\|Tf\|_{L^\infty} \le \|f\|_{L^\infty}$ for all $f \in L^\infty$, then T maps X to X and $\|Tf\|_X \le \|f\|_X$ for all $f \in X$.
(g) If $f \in X$ and $z \in \mathbb{R}^n$, then $\tau_z f \in X$ and $\|\tau_z f\|_X = \|f\|_X$.
(h) If $r > 0$, then $r^n = \|q_r\|_X \|q_r\|_{X'}$.

The last property deserves some justification. Since $q_r = q_r^2$ and r^n is the integral of q_r, the definition of X' gives "$\le$". With $Tf = q_r \frac{1}{r^n}\int_{Q_r}|f|$, we have $\|Tf\|_{L^1} \le \|f\|_{L^1}$ for $f \in L^1$ and $\|Tf\|_{L^\infty} \le \|f\|_{L^\infty}$ for $f \in L^\infty$ so $\|Tf\|_{X'} \le \|f\|_{X'}$ for all $f \in X'$. If $\|f\|_{X'} \le 1$, then $\|q_r\|_{X'}\int_{\mathbb{R}^n}|fq_r|\,dx = r^n\|Tf\|_{X'} \le r^n$. Taking the supremum over all such f gives "$\ge$".

2. Main results

Let $\mathcal{A} = \{\alpha \colon \mathbb{R}^n \to (0,\infty) | \int_{\mathbb{R}^n}\alpha = 1\}$. These are the functions we will use to smooth weight functions by convolution. We begin with a general duality result, which we only need for rearrangement invariant spaces.

Lemma 2.1. *Let P and Q be rearrangement invariant spaces of complex valued functions on $\mathbb{R}^n$, U and V be strictly positive measurable functions and $C > 0$. Suppose that if f is integrable and $Uf \in P$, then $V\widehat{f} \in Q'$ and*

$$\|V\widehat{f}\|_{Q'} \leq C\|Uf\|_P.$$

Then, for all integrable g such that $g/V \in Q$, we have $\widehat{g}/U \in P'$ and

$$\|\widehat{g}/U\|_{P'} \leq C\|g/V\|_Q.$$

Proof. Suppose g is integrable and $g/V \in Q$. Choose $h \in P$ with $\|h\|_P \leq 1$. For each positive integer k, set $h_k(x) = q_k(x)|h(x)||\widehat{g}(x)|/\widehat{g}(x)$ when $h(x) \leq kU(x)$ and $\widehat{g}(x) \neq 0$. Set $h_k(x) = 0$ otherwise. Evidently, $|h_k| \leq |h|$ so $\|h_k\|_P \leq 1$. Also, h_k/U is integrable, so

$$\int_{\mathbb{R}^n} \frac{|\widehat{g}|}{U}|h_k| = \int_{\mathbb{R}^n} \widehat{g}\frac{h_k}{U} = \int_{\mathbb{R}^n} g\Big(\frac{h_k}{U}\Big)\widehat{\ } \leq \|g/V\|_Q\|V(h_k/U)\widehat{\ }\|_{Q'} \leq C\|g/V\|_Q.$$

Letting $k \to \infty$, the monotone convergence theorem implies

$$\int_{\mathbb{R}^n} \frac{|\widehat{g}|}{U}|h| \leq C\|g/V\|_Q < \infty.$$

Therefore $\widehat{g}/U \in P'$ and $\|\widehat{g}/U\|_{P'} \leq C\|g/V\|_Q$. □

Next we show that if a weighted Fourier inequality holds, it also holds with a smoothed weight in the codomain.

Theorem 2.2. *Let X and Y be rearrangement invariant spaces of complex valued functions on $\mathbb{R}^n$ and let $\alpha \in \mathcal{A}$. Let u and v be non-negative, measurable functions on $\mathbb{R}^n$ such that v is not almost everywhere zero. Suppose there exists a $C > 0$ such that, if f is integrable and $uf \in X$, then $v\widehat{f} \in Y$ and*

$$\|v\widehat{f}\|_Y \leq C\|uf\|_X.$$

Then $u > 0$ almost everywhere; $v * q_1$ is bounded above; $0 < v * \alpha < \infty$ almost everywhere; and, if f is integrable and $uf \in X$, then $(v * \alpha)\widehat{f} \in Y$ and

$$\|(v * \alpha)\widehat{f}\|_Y \leq C\|uf\|_X.$$

Proof. Suppose f is integrable, $uf \in X$, $g \in Y'$ and $\|g\|_{Y'} \leq 1$. Interchanging the order of integration and replacing y by $y + z$ produces the equation

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} v(y-z)\alpha(z)\,dz|\widehat{f}(y)g(y)|\,dy = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |v(y)(\tau_z\widehat{f})(y)(\tau_z g)(y)|\,dy\,\alpha(z)\,dz.$$

Since Y' is rearrangement invariant, $\|\tau_z g\|_{Y'} \leq 1$. Also, $\tau_z\widehat{f} = (\varepsilon_z f)\widehat{\ }$, so we have

$$\int_{\mathbb{R}^n} v * \alpha(y)|\widehat{f}(y)g(y)|\,dy \leq \int_{\mathbb{R}^n} \|v\tau_z\widehat{f}\|_Y\alpha(z)\,dz = \int_{\mathbb{R}^n} \|v(\varepsilon_z f)\widehat{\ }\|_Y\alpha(z)\,dz.$$

But $|\varepsilon_z f| = |f|$, so

$$\int_{\mathbb{R}^n} v * \alpha(y)|\widehat{f}(y)g(y)|\,dy \leq \int_{\mathbb{R}^n} C\|uf\|_X\alpha(z)\,dz = C\|uf\|_X < \infty.$$

Therefore, $(v * \alpha)\widehat{f} \in Y$ and $\|(\alpha * v)\widehat{f}\|_Y \leq C\|uf\|_X$.

Choose a set F of finite, positive measure such that u is bounded on F. If u is zero on a set of positive measure, choose the set F so that u is zero on F. Set $f = \chi_F$. Then f is bounded, integrable and not almost everywhere zero so its

Fourier transform is continuous and not identically zero. Choose $a \in \mathbb{R}^n$ and $\delta > 0$ so that if $a - y \in Q_\delta$, then $|\widehat{f}(y)| \geq \delta$. Then, for all $y, z \in \mathbb{R}^n$,

$$\delta q_\delta(z-y) \leq |\widehat{f}(a+y-z)| q_\delta(z-y) = |(\varepsilon_{a-z} f)\widehat{\ }(y)|\, q_\delta(z-y).$$

The choice of F ensures that $u\varepsilon_{a-z} f$ is bounded and integrable so $u\varepsilon_{a-z} f \in X$. Therefore,

$$\frac{\delta}{\|q_\delta\|_{Y'}} v * q_\delta(z) \leq \int_{\mathbb{R}^n} v(y) |(\varepsilon_{a-z} f)\widehat{\ }(y)| \frac{q_\delta(z-y)}{\|q_\delta\|_{Y'}}\, dy \leq \|v(\varepsilon_{a-z} f)\widehat{\ }\|_Y \leq C\|uf\|_X.$$

If u were zero on a set of positive measure, the choice of F would make the right-hand side zero and force v to be almost-everywhere zero, contrary to hypothesis. Therefore $u > 0$ almost everywhere. The strict positivity of α ensures that $v * \alpha > 0$ almost everywhere.

Since $C\|uf\|_X$ finite and independent of z, $v * q_\delta$ is bounded above. It is a simple matter to cover Q_1 by finitely many translates of Q_δ to get

$$q_1 \leq \sum_{j=1}^{N} \tau_{z_j} q_\delta,$$

for some finite sequence $z_1, \ldots, z_N$, and hence

$$v * q_1 \leq \sum_{j=1}^{N} v * (\tau_{z_j} q_\delta) = \sum_{j=1}^{N} \tau_{z_j} (v * q_\delta)$$

which is bounded above. It follows that

$$(v * \alpha) * q_1 = (v * q_1) * \alpha$$

is bounded above, which implies that $v * \alpha$ is finite almost everywhere. □

Combining duality with smoothing of the codomain weight permits smoothing of both weights.

Corollary 2.3. *Under the hypotheses of Theorem 2.2,*

(i) If g is integrable and $g/(v * \alpha) \in Y'$, then $\widehat{g}/u \in X'$ and

$$\|\widehat{g}/u\|_{X'} \leq C\|g/(v * \alpha)\|_{Y'}.$$

(ii) $(1/u) * q_1$ is bounded above, $0 < (1/u) * \alpha < \infty$ almost everywhere and, if g is integrable and $g/(v * \alpha) \in Y'$, then $((1/u) * \alpha)\widehat{g} \in X'$ and

$$\|((1/u) * \alpha)\widehat{g}\|_{X'} \leq C\|g/(v * \alpha)\|_{Y'}.$$

(iii) If f is integrable and $f/((1/u) * \alpha) \in X$, then $(v * \alpha)\widehat{f} \in Y$ and

$$\|(v * \alpha)\widehat{f}\|_Y \leq C\|f/((1/u) * \alpha)\|_X.$$

Proof. For (i), apply Lemma 2.1 to the result of Theorem 2.2. For (ii), apply Theorem 2.2 to (i). For (iii), apply Lemma 2.1 to (ii). □

Next we select a sequence of elements of $\mathcal{A}$ that can be used as an approximate identity on the original weights.

Lemma 2.4. *There exist $\alpha_1, \alpha_2, \ldots$ in $\mathcal{A}$ such that if $w \geq 0$ and $w * q_1$ is bounded above, then $w * \alpha_k$ is continuous for each k and as $k \to \infty$, $w * \alpha_k \to w$ almost everywhere on $\mathbb{R}^n$.*

Proof. Fix $\alpha_0 \in \mathcal{A}$. For each positive integer k, set

$$\alpha_k = \frac{1}{k} q_1 * \alpha_0 + \frac{k-1}{k} k^n q_{1/k}.$$

We readily verify that $\alpha_k \in \mathcal{A}$ for each k.

Since $w * q_1$ is bounded above, $w * q_1 * \alpha_0$ is bounded above and w is locally integrable. Lebesgue's differentiation theorem shows that for almost every $z \in \mathbb{R}^n$, $k^n w * q_{1/k}(z) \to w(z)$ as $k \to \infty$. It follows that, as $k \to \infty$, $w * \alpha_k \to w$ pointwise almost everywhere.

Now fix k and $z \in \mathbb{R}^n$. Let B be an upper bound for $w * q_1$. Then for $h \in \mathbb{R}^n$,

$$|w * q_1 * \alpha_0(z+h) - w * q_1 * \alpha_0(z)| \le B \int_{\mathbb{R}^n} |\alpha_0(z+h-y) - \alpha_0(z-y)|\, dy \to 0$$

as $h \to 0$ because translation is continuous in L^1. So $w * q_1 * \alpha_0$ is continuous at z.

If $h \in Q_1$ and $y \notin Q_2$, then $y + h \notin Q_1$ and we have $q_{1/k}(y+h) = q_{1/k}(y) = 0$. So for sufficiently small h,

$$|w * q_{1/k}(z+h) - w * q_{1/k}(z)| \le \int_{Q_2} w(z-y)|q_{1/k}(y+h) - q_{1/k}(y)|\, dy.$$

Since w is locally integrable and $|q_{1/k}(y+h) - q_{1/k}(y)| \to 0$ almost everywhere as $h \to 0$, the dominated convergence theorem shows $w * q_{1/k}$ is continuous at z.

These combine to show that $w * \alpha_k$ is continuous on $\mathbb{R}^n$. □

Now we are ready to prove our main result: The Fourier transform is bounded on a non-trivial weighted rearrangement invariant space only if the weight is equivalent to a constant function.

Theorem 2.5. *Let X be a rearrangement invariant space of complex valued functions on $\mathbb{R}^n$ and let w be a non-negative measurable function on $\mathbb{R}^n$ that is not almost everywhere zero. Suppose there exists a $C > 0$ such that, if f is integrable and $wf \in X$, then $w\widehat{f} \in X$ and*

$$\|w\widehat{f}\|_X \le C\|wf\|_X.$$

Then there exist positive real numbers m and M such that $m \le w(x) \le M$ for almost every $x \in \mathbb{R}^n$. Moreover, $X = L^2$ with equivalent norms.

Proof. By Theorem 2.2 and Corollary 2.3 we get: $w > 0$ almost everywhere; for each $\alpha \in \mathcal{A}$, $0 < w * \alpha < \infty$ and $0 < (1/w) * \alpha < \infty$ almost everywhere; $w * q_1$ and $(1/w) * q_1$ are bounded above; if g is integrable and $g/(w * \alpha) \in X'$, then $((1/w) * \alpha)\widehat{g} \in X'$ and

$$\|((1/w) * \alpha)\widehat{g}\|_{X'} \le C\|g/(w * \alpha)\|_{X'};$$

and if f is integrable and $f/((1/w) * \alpha) \in X$, then $(w * \alpha)\widehat{f} \in X$ and

$$\|(w * \alpha)\widehat{f}\|_X \le C\|f/((1/w) * \alpha)\|_X.$$

Since $s + 1/s \ge 2$ for $s > 0$, we see that, for all $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big(\frac{w(x-z)}{w(x-y)} + \frac{w(x-y)}{w(x-z)}\Big)\alpha(y)\alpha(z)\, dy\, dz \ge \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 2\alpha(y)\alpha(z)\, dy\, dz = 2,$$

which implies $1 \le w * \alpha(x)(1/w) * \alpha(x)$. Therefore,

$$\frac{1}{(2r)^n} \le \int_{Q_{1/(2r)}} w * \alpha(x)(1/w) * \alpha(x)\, dx \le \|(w * \alpha)q_{1/(2r)}\|_X \|((1/w) * \alpha)q_{1/(2r)}\|_{X'}.$$

But for any $y, z \in \mathbb{R}^n$, inequality (1.2) implies

$$\|(w * \alpha)q_{1/(2r)}\|_X \leq \frac{2^n}{r^n}\|(w * \alpha)(\tau_z q_r)\widehat{\ }\|_X \leq \frac{2^n C}{r^n}\|\tau_z q_r/((1/w) * \alpha)\|_X$$

and

$$\|((1/w) * \alpha)q_{1/(2r)}\|_{X'} \leq \frac{2^n}{r^n}\|((1/w) * \alpha)(\tau_y q_r)\widehat{\ }\|_{X'} \leq \frac{2^n C}{r^n}\|\tau_y q_r/(w * \alpha)\|_{X'}.$$

These, together with properties (h) and (g) above, yield

$$1 \leq 2^{3n} C^2 \frac{\|\tau_z q_r/((1/w) * \alpha)\|_X}{\|\tau_z q_r\|_X} \frac{\|\tau_y q_r/(w * \alpha)\|_{X'}}{\|\tau_y q_r\|_{X'}}.$$

This inequality holds for all $\alpha \in \mathcal{A}$ so it holds with α replaced by each α_k from the sequence given in Lemma 2.4. The lemma applies to both w and $1/w$. Using the continuity of $(1/w) * \alpha_k$ and $w * \alpha_k$, and letting $r \to 0$, we get

$$1 \leq 2^{3n} C^2 \frac{1}{(1/w) * \alpha_k(z)} \frac{1}{w * \alpha_k(y)}.$$

Now we let $k \to \infty$ to see that for almost every z and almost every y we have

$$1 \leq 2^{3n} C^2 \frac{w(z)}{w(y)}.$$

Choose y_0 such that the inequality holds for almost every z and choose z_0 such that the inequality holds for almost every y. Then set $m = 2^{-3n} C^{-2} w(y_0)$ and $M = 2^{3n} C^2 w(z_0)$ to get $m \leq w \leq M$ almost everywhere.

With this inequality in hand, the hypothesis of the theorem implies that if f is integrable and $f \in X$, then $\widehat{f} \in X$ and $\|\widehat{f}\|_X \leq (MC/m)\|f\|_X$. Since X is rearrangement invariant, the Fourier transform extends to be bounded on all of X and Theorem 1(ii) of [2] implies that $X = L^2$, with equivalent norms. □

Now we turn our attention to weighted Lebesgue spaces and the Hausdorff-Young inequality. Note that if $1 \leq p \leq \infty$, L^p is rearrangement invariant space and $(L^p)' = L^{p'}$ where $1/p + 1/p' = 1$.

Theorem 2.6. *Suppose $p, q \in [1, \infty]$, w is a positive, measurable function on $\mathbb{R}^n$ and there exists a C such that $\|w\widehat{f}\|_{L^q} \leq C\|wf\|_{L^p}$ whenever f is integrable and $fw \in L^p$. Then $1 \leq p \leq 2$, $q = p'$ and there exist positive real numbers m and M such that $m \leq w(x) \leq M$ for almost every $x \in \mathbb{R}^n$.*

Proof. By Theorem 2.2 and Corollary 2.3 we get: $0 < w$ almost everywhere; for each $\alpha \in \mathcal{A}$, $0 < w * \alpha < \infty$ and $0 < (1/w) * \alpha < \infty$ almost everywhere; $w * q_1$ and $(1/w) * q_1$ are bounded above; if g is integrable and $g/(w * \alpha) \in L^{q'}$, then $((1/w) * \alpha)\widehat{g} \in L^{p'}$ and

$$\|((1/w) * \alpha)\widehat{g}\|_{L^{p'}} \leq C\|g/(w * \alpha)\|_{L^{q'}};$$

and if f is integrable and $f/((1/w) * \alpha) \in L^p$, then $(w * \alpha)\widehat{f} \in L^q$ and

$$\|(w * \alpha)\widehat{f}\|_{L^q} \leq C\|f/((1/w) * \alpha)\|_{L^p}.$$

As in the proof of Theorem 2.5, we see that $1 \leq w * \alpha(x)(1/w) * \alpha(x)$ for all $x \in \mathbb{R}^n$. If both p' and q are finite, this implies

$$\left(\frac{1}{(2r)^n}\right)^{\frac{1}{p'}+\frac{1}{q}} \leq \left(\int_{Q_{1/(2r)}} \left(w * \alpha(x)^q\right)^{p'/(p'+q)} \left((1/w) * \alpha(x)^{p'}\right)^{q/(p'+q)} dx\right)^{\frac{1}{p'}+\frac{1}{q}}.$$

Applying Hölder's inequality with indices $(p'+q)/p'$ and $(p'+q)/q$ we get

$$(2r)^{-\frac{n}{p'}-\frac{n}{q}} \le \|(w*\alpha)q_{1/(2r)}\|_{L^q}\|((1/w)*\alpha)q_{1/(2r)}\|_{L^{p'}}.$$

It is easy to verify that this inequality remains valid when one or both of p' and q is infinite.

But for any $y, z \in \mathbb{R}^n$, inequality (1.2) implies

$$\|(w*\alpha)q_{1/(2r)}\|_{L^q} \le \frac{2^n}{r^n}\|(w*\alpha)(\tau_z q_r)\widehat{\ }\|_{L^q} \le \frac{2^n C}{r^n}\|\tau_z q_r/((1/w)*\alpha)\|_{L^p}$$

and

$$\|((1/w)*\alpha)q_{1/(2r)}\|_{L^{p'}} \le \frac{2^n}{r^n}\|((1/w)*\alpha)(\tau_y q_r)\widehat{\ }\|_{L^{p'}} \le \frac{2^n C}{r^n}\|\tau_y q_r/(w*\alpha)\|_{L^{q'}}.$$

Since $\|\tau_z q_r\|_{L^p} = r^{n/p}$ and $\|\tau_y q_r\|_{L^{q'}} = r^{n/q'}$, the above inequalities combine to show that

$$1 \le 2^{n(2+\frac{1}{p'}+\frac{1}{q})}C^2 \frac{\|\tau_z q_r/((1/w)*\alpha)\|_{L^p}}{\|\tau_z q_r\|_{L^p}} \frac{\|\tau_y q_r/(w*\alpha)\|_{L^{q'}}}{\|\tau_y q_r\|_{L^{q'}}}.$$

As in the proof of Theorem 2.5, we replace α by α_k, let $r \to 0$ to get

$$1 \le 2^{n(2+\frac{1}{p'}+\frac{1}{q})}C^2 \frac{1}{(1/w)*\alpha_k(z)} \frac{1}{w*\alpha_k(y)},$$

and let $k \to \infty$ to get

$$1 \le 2^{n(2+\frac{1}{p'}+\frac{1}{q})}C^2 \frac{w(z)}{w(y)}.$$

It follows as above that there exist positive real numbers m and M such that $m \le w(x) \le M$ for almost every $x \in \mathbb{R}^n$. The hypothesis of the theorem now implies that if f is integrable and $f \in L^p$, then $\widehat{f} \in L^q$ and

$$\|\widehat{f}\|_{L^q} \le (MC/m)\|\widehat{f}\|_{L^p}.$$

This can only happen when $1 \le p = q' \le 2$.

To see this well known fact we may use (1.2) with $z = 0$, taking $f = q_r$ to see that $q = p'$ is a necessary condition for (1.1). Also, Theorem 1(i) of [2] shows that $L^1 + L^2$ is the largest rearrangement invariant space which the Fourier Transform maps into a space of locally integrable functions. If $p > 2$ then L^p is not a subset of $L^1 + L^2$, making $p \le 2$ also a necessary condition for (1.1). □

3. The Schrödinger Multiplier

The Fourier transform separates variables in the Schrödinger equation

$$\partial_t u(t,x) = i\Delta_x u(t,x). \tag{3.1}$$

The resulting multiplier is $\exp(-4\pi^2 it|y|^2)$. That is, if $u(0,x) = h(x)$, with h integrable, the solution to (3.1) is $u(t,x) = S_t h(x)$, where the operator S_t is defined by

$$\widehat{S_t h}(y) = \exp(-4\pi^2 it|y|^2)\widehat{h}(y).$$

The operator S_t can be extended (along with the Fourier transform) to spaces other than L^1 in order to solve the Schrödinger equation for non-integrable initial data. Concrete extensions of S_t to other spaces of functions follow from boundedness of the operator on the integrable functions in the space. So it is natural to investigate the spaces on which S_t is bounded. Lemma 8 of [2] showed that S_t is not bounded on any rearrangement invariant space of functions unless the space is L^2. Here we modify rearrangement invariant spaces with a weight function and show that a similar negative result holds on this larger class.

Corollary 3.1. *Let X be a rearrangement invariant space of complex valued functions on $\mathbb{R}^n$ and let $t > 0$. Fix a non-negative measurable function w and set $w_t(x) = w(x/(4\pi t))$ for $x \in \mathbb{R}^n$. Suppose there exists a $C > 0$ such that if h is integrable and $wh \in X$, then $w_t S_t h \in X$ and*

$$\|w_t S_t h\|_X \leq C\|wh\|_X.$$

Then w is bounded above and below, and $X = L^2$ with equivalent norms.

Proof. Since X is rearrangement invariant, the dilation map $f(x) \to f(x/(4\pi t))$ is bounded on X. (It is trivially bounded on L^∞ and a simple change of variable shows that it is bounded on L^1.) Let M be a bound for this map.

A Fourier transform calculation shows that the operator S_t can be written as the convolution

$$S_t h(y) = (4\pi i t)^{-n/2} \int_{\mathbb{R}^n} \exp\left(i\frac{|y-x|^2}{4t}\right) h(x)\, dx$$

and the simplification in Lemma 8 of [2], taking $\sigma(x) = \exp(i|x|^2/(4t))$, gives

$$S_t h(y) = (4\pi i t)^{-n/2} \sigma(y) \widehat{\sigma h}(y/(4\pi t)).$$

Suppose f is integrable and $wf \in X$. Set $h = f/\sigma$. Since $|\sigma(x)| = 1$ for all x, h is integrable and $wh \in X$. Thus $\|w_t S_t h\|_X \leq C\|wh\|_X$. The calculation above shows that

$$w(y/(4\pi t))|\widehat{f}(y/(4\pi t))| = (4\pi t)^{n/2} w_t(y)|S_t h(y)|.$$

Since $w_t S_t h \in X$ and X is closed under dilations, $w\widehat{f} \in X$. Moreover,

$$\|w\widehat{f}\|_X \leq M(4\pi t)^{n/2}\|w_t S_t h\|_X \leq M(4\pi t)^{n/2} C\|wh\|_X = M(4\pi t)^{n/2} C\|wf\|_X.$$

Now we apply Theorem 2.5 to see that w is bounded above and below, and $X = L^2$ with equivalent norms. □

We state the next corollary without proof, as it follows from Theorem 2.6 in the same way as Corollary 3.1 follows from Theorem 2.5.

Corollary 3.2. *Let $p, q \in [1, \infty]$ and let $t > 0$. Fix a non-negative weight function w and set $w_t(x) = w(x/(4\pi t))$ for all $x \in \mathbb{R}^n$. Suppose there exists a C such that if h is integrable and $wh \in L^p$, then $w_t S_t h \in L^q$ and*

$$\|w_t S_t h\|_{L^q} \leq C\|wh\|_{L^p}.$$

Then w is bounded above and below, $1 \leq p \leq 2$ and $q = p'$.

References

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