FROM RESTRICTED TYPE TO STRONG TYPE ESTIMATES ON QUASI-BANACH REARRANGEMENT INVARIANT SPACES

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Abstract. Let X be a quasi-Banach rearrangement invariant space and let T be a (ε, δ)-atomic operator for which a restricted type estimate of the form $\|T\chi_E\|_X \le D(|E|)$ for some positive function D and every measurable set E is known. Then, we show that this estimate can be extended to the set of all positive functions $f \in L^1$ such that $\|f\|_\infty \le 1$, in the sense that $\|Tf\|_X \le D(\|f\|_1)$. This inequality allows us to obtain strong type estimates for T on several classes of spaces as soon as some information about the Galb of the space X is known. This paper will consider the case of weighted Lorentz spaces $X = \Lambda^q(w)$ and their weak version.

1. Introduction

It is well known and we refer to the papers [1], [3], [4] and [14] that, for many interesting operators only a restricted estimate on characteristic functions is known, and it is of a general interest to show what kind of strong type estimate can be obtained from it. This is, for example, the principle of the weak type extrapolation theory where we have an operator satisfying

$$\|T\chi_E\|_{L^{p,\infty}} \le \frac{1}{p-1}|E|^{1/p},$$

for every $1 < p \le p_0$, and it is an open question to see if this implies that T is bounded from the Orlicz space $L\log L$ into $L^{1,\infty}$. A positive solution to this question will give us, when applied to the Carleson operator

$$Sf(x) = \sup_n |S_n f(x)|,$$

where $S_n f(x) = (D_n * f)(x)$, D_n is the Dirichlet kernel on $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ and $f \in L^1(\mathbb{T})$, the almost everywhere convergence of the Fourier series of a function in $L\log L(\mathbb{T})$.

2000 *Mathematics Subject Classification.* 47A30, 47A63.

Key words and phrases. Rearrangement inequality, Extrapolation theory, Restricted weak type estimates, (ε, δ)-atomic operator, Galb.

This work has been partially supported by MTM2004-02299 and by CURE 2001SGR 00069 and by funding from IMUB and NSERC.

In a recent paper [4], it was proved that if the operator T is (ε, δ)-atomic approximable (see Definition 2.2), then an estimate of the form

$$(T\chi_E)^*(t) \leq h(t, |E|),$$

for every measurable set, can be extended to every function f bounded by 1 and, from it, some strong type estimates on logarithmic type spaces were proved. In particular, if $h(t,s) \leq R(t)D(s)$, the above inequality is equivalent to $\|T\chi_E\|_X \leq D(|E|)$, where X is a weak weighted Lorentz space (see definition below).

The first purpose of this paper consists in proving, in Section 2, that a slight modification of the main theorem in [4] shows that if T is (ε, δ)-atomic approximable, and $\|T\chi_E\|_X \leq D(|E|)$ for some positive function D and every measurable set E where X is any quasi-Banach r.i. space, then $\|Tf\|_X \leq D(\|f\|_1)$, for every $f \in L^1$ such that $\|f\|_\infty \leq 1$.

Our second step will be to obtain, from this inequality, a strong type estimate, for which we need to have some information on the Galb(X), which is defined (see [22]) by

$$\mathrm{Galb(X)} = \Big\{ (\mathrm{c_n})_\mathrm{n};\ \sum_\mathrm{n} \mathrm{c_n f_n} \in \mathrm{X}, \mathrm{whenever}\ \|\mathrm{f_n}\|_\mathrm{X} \leq 1 \Big\},$$

endowed with the norm $\|c\|_{\mathrm{Galb(X)}} = \sup_{\|f_n\|_X \leq 1} \Big\| \sum_n c_n f_n \Big\|_X$. In particular, we study, in Section 3, this Galb for the case of the weighted Lorentz spaces $X = \Lambda^q(w)$, for $0 < q < \infty$, and also, of the weak spaces $\Lambda^{q,\infty}(w)$. To this end, we use the following formula for the decreasing rearrangement of a sum of functions (see [8]): if $f = \sum_n c_n f_n$, then

$$f^*(3t) \leq \sum_n c_n \left(f_n^*(t) + \frac{1}{t} \int_{a_n t}^t f_n^*(s) ds \right),$$

where $\{a_n\}_n$ are positive numbers such that $\sum_n a_n = 1$, and we need to solve the problem of computing, for $q \geq 1$,

$$\sup \frac{\int_0^\infty \left(\frac{1}{t} \int_{at}^t f(s)\, ds \right)^q w(t)\, dt}{\int_0^\infty f(t)^q w(t)\, dt},$$

where the supremum extends over the set of decreasing functions f. This problem will be solved in Section 5. Finally, in Section 4 we present some concrete examples and applications.

We shall denote by $L^0(\mathbb{R}^n)$ the class of Lebesgue measurable functions that are finite a.e. and $g^*(t) = \inf\{s : \lambda_g(s) \leq t\}$ is the decreasing rearrangement of g, where $\lambda_g(y) = |\{x \in \mathbb{R} : |g(x)| > y\}|$ is the distribution function of g with respect to Lebesgue measure. We refer the reader to [2] for further information about distribution functions, decreasing rearrangements and rearrangement invariant (r.i.) spaces.

If, in the definition of a norm, the triangle inequality is weakened to the requirement that for some constant c, $\|x + y\| \leq c(\|x\| + \|y\|)$ holds for all x and y, then we have a quasi-norm. A complete quasi-normed space is called a quasi-Banach space. It is well known that spaces ℓ^p for $0 < p < 1$ are quasi-Banach spaces. Observe that if X is a quasi-Banach r.i. space of measurable functions on $\mathbb{R}^n$ then there is a r.i. quasi-Banach space X^* of measurable functions on $\mathbb{R}$ such that $\|f\|_X = \|f^*\|_{X^*}$ for all $f \in X$. One simply defines $\|g\|_{X^*} = \|G\|_X$ where $G(x) = \omega g(|x|)|x|^{n-1}$ with ω chosen so that g and G are equimeasurable. It is a simple matter to verify that X^* is a quasi-Banach space.

For a measurable set E, χ_E denotes the characteristic function of E, $|E|$ is the Lebesgue measure of E and, for simplicity in our arguments, we say that an operator T is sublinear if $T(\lambda f) = \lambda T f$ and

$$\left| T\left(\sum_{n\in\mathbb{N}} f_n \right) \right| \leq \sum_{n\in\mathbb{N}} |T f_n|.$$

If we only have that $|T(f+g)| \leq |Tf| + |Tg|$, then we need to assume some extra boundedness condition on our operator T such as

$$T : L^1 + L^\infty \longrightarrow L^0$$

is bounded or to use some standard density argument to obtain our conclusions.

2. From restricted weak type to strong type

We shall work in $\mathbb{R}^n$ and Q will represent a cube with sides parallel to the axes. The results can be extended in the natural way to $\mathbb{T}^N$ (identifying $\mathbb{T}^N$ with $[0,1)^N$). In [4], the following definitions were introduced:

Definition 2.1. *Given $\delta > 0$, a function $a \in L^1(\mathbb{R}^n)$ is called a δ-atom if it satisfies the following properties:*
(i) $\int_{\mathbb{R}^n} a(x)\,dx = 0$, and
(ii) there exists a cube Q such that $|Q| \leq \delta$ and supp $a \subset Q$.

Definition 2.2. *(a) A sublinear operator T, defined on $L^1 + L^\infty$ and taking values in L^0, is (ε, δ)-atomic if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying*

$$\|Ta\|_{L^1+L^\infty} \leq \varepsilon \|a\|_1, \tag{1}$$

for every δ-atom a.
(b) A sublinear operator T is (ε, δ)-atomic approximable if there exists a sequence $(T_n)_n$ of (ε, δ)-atomic operators such that, for every measurable

set E, $|T_n\chi_E| \leq |T\chi_E|$ and, for every $f \in L^1$ such that $\|f\|_\infty \leq 1$, and every $t > 0$,

$$(Tf)^*(t) \leq \liminf_n (T_n f)^*(t)\cdot$$

In particular, any maximal operator of the form $\sup_j |K_j * f|$, where $K_j \in L^{p_j}$ for some $1 \leq p_j < \infty$ is (ε, δ)-atomic approximable (see [4] for more examples of this kind of operators). Also, as we shall see in this paper, any operator bounded from L^p into L^p with $0 < p < 1$ is not (ε, δ)-atomic approximable.

Definition 2.3. *Given an operator T and a quasi-Banach r.i. space X, we define the fundamental function of T with respect to X, by*

$$\varphi_{X,T}(r) = \sup_{|E| \leq r} ||T\chi_E||_X.$$

Observe that if T is the identity operator, $\varphi_{X,T}$ is nothing but φ_X, the usual fundamental function of X.

Definition 2.4. *Given $\delta > 0$, we say that $\mathcal{F}_\delta$ is a δ-net if it is a collection of open cubes of the following form:*

$$\mathcal{F}_\delta = \{Q_j; |Q_j| = \delta, Q_j \text{ are pairwise disjoint}, \cup \overline{Q_j} = \mathbb{R}^n\}.$$

Theorem 2.1. *Let X be a quasi-Banach r.i. space and T a sublinear (ε, δ)-atomic approximable operator. Then, for every positive function $f \in L^1$ such that $\|f\|_\infty \leq 1$,*

$$\|Tf\|_X \leq \varphi_{X,T}(\|f\|_1).$$

Proof: In view of Definition 2.2, it is enough to prove the result for an (ε, δ)-atomic operator T.

Given X, let X^* be the space of measurable functions on $(0, \infty)$ such that $\|f\|_X = \|f^*\|_{X^*}$. Let $f \in L^1$ be a positive function such that $\|f\|_\infty \leq 1$ and, given $\varepsilon > 0$, let us consider a δ-net $\mathcal{F}_\delta$ where δ is the number associated to ε by the property that T is (ε, δ)-atomic.

Given $Q_i \in \mathcal{F}_\delta$, let $f_i = f\chi_{Q_i}$. Then,

$$\int_{\mathbb{R}^n} f_i(x)dx \leq |Q_i|,$$

and hence, we can take a cube $\tilde{Q}_i \subset Q_i$ and satisfying:

$$|\tilde{Q}_i| = \int_{\mathbb{R}^n} f_i(x)dx = \int_{Q_i} f(x)dx.$$

Then, it is clear that the function $g_i = f_i - \chi_{\tilde{Q}_i}$ is a δ-atom and

$$\|g_i\|_1 \leq \int_{Q_i} |f(x)|dx + |\tilde{Q}_i| = 2\int_{Q_i} |f(x)|dx.$$

Now, $f = \sum_i f_i = \sum_i g_i + \chi_E$, where $E = \cup \tilde{Q}_i$. Then, by sublinearity

$$|Tf| \leq \sum_i |Tg_i| + |T\chi_E| \equiv G + |T\chi_E|.$$

For fixed $n > 1$, we have

$$\begin{aligned}(Tf)^*(x)\chi_{(1/n,n)}(x) &\leq G^*((1/n^2)x)\chi_{(1/n,n)}(x)\\ &+ (T\chi_E)^*((1-1/n^2)x)\chi_{(1/n,n)}(x)\\ &\equiv Q_n(x) + R_n(x).\end{aligned}$$

For $x \in (1/n, n)$, we have $0 \leq x - 1/n \leq (1 - 1/n^2)x$, and hence

$$R_n(x) \leq (T\chi_E)^*(x - 1/n)\chi_{(1/n,n)}(x)$$

and it follows that $R_n^* \leq (T\chi_E)^*$. On the other hand,

$$Q_n(x) \leq G^*(1/n^3)\chi_{(1/n,n)}(x)$$

and

$$\begin{aligned}G^*(1/n^3) &= \left(\sum_i |Tg_i|\right)^*(1/n^3) \leq n^3 \int_0^{1/n^3} \left(\sum_i |Tg_i|\right)^*\\ &\leq \sum_i n^3 \int_0^{1/n^3} (Tg_i)^* \leq \sum_i n^3 \int_0^1 (Tg_i)^*\\ &\leq n^3 \sum_i \|Tg_i\|_{L^1+L^\infty} \leq n^3\varepsilon \sum_i \|g_i\|_1 \leq 2n^3\varepsilon\|f\|_1.\end{aligned}$$

Using these estimates for R_n and Q_n we have

$$\|(Tf)^*\chi_{(1/n,n)}\|_{X^*} \leq 2n^2\varepsilon\|f\|_1\|\chi_{(1/n,n)}\|_{X^*} + \|T\chi_E\|_X.$$

First we let $\varepsilon \to 0$ and then we let $n \to \infty$ to get

$$\|Tf\|_X \leq \|T\chi_E\|_X.$$

Since $|E| = \sum_i |\tilde{Q}_i| = \sum_i \int_{Q_i} f = \|f\|_1$, the result follows. □

Also, as a consequence of the previous remark we obtain the following:

Proposition 2.1. *Let X be a quasi-Banach r.i. space and let T be a non-zero (ε, δ)-atomic approximable operator. Then $\varphi_{X,T}$ is quasi-concave.*

Proof: Clearly $\varphi_{X,T}(r)$ is non-decreasing. Suppose $s > r$. If $|E| \leq s$ then $\|(r/s)\chi_E\|_\infty \leq 1$ so

$$(1/s)\|T\chi_E\|_X \leq (1/r)\varphi_{X,T}(\|(r/s)\chi_E\|_1) = (1/r)\varphi_{X,T}(r).$$

Since this holds for all such E, $(1/r)\varphi_{X,T}(r)$ is non-increasing. □

Every quasi-concave function is equivalent to a concave function so we shall assume from now on that D is a concave function with $\varphi_{X,T} \preceq D$.

Remark 2.1. *From the above Proposition, we obtain that if X is any quasi-Banach r.i. space and $0 < p < 1$, any operator T mapping L^p to X is not (ε, δ)-atomic approximable. In particular, convolution operators on L^p with discrete measures with coefficients in ℓ^p are not (ε, δ)-atomic approximable.*

Definition 2.5. *Given a sequence space $\int \subseteq \ell^1$ and a concave function D, we shall denote by $D(\int)$, the set of all measurable functions f such that*

$$||f||_{D(\int)} = \inf \left\{ \left\| \Big(c_n D(||f_n||_1) \Big)_n \right\|_{\int} \right\}$$

is finite. Here the infimum extends over the set of all possible decompositions of $f = \sum_n c_n f_n$, a.e. such that $||f_n||_\infty \leq 1$.

It is an exercise to prove the following.

Theorem 2.2. *If D is concave then $L^1 \cap L^\infty \subseteq D(\int)$. If, in addition, $s \preceq D(s)$, then $D(\int) \subseteq L^1$.*

Let us now give some concrete examples which will be useful in what follows:

Examples:

a) If $\int = \ell^p$ with $0 < p \leq 1$, then taking the following decomposition of a function f

$$f = \sum_{n \in \mathbb{Z}} 2^n f_n,$$

where $f_n = 2^{-n} f \chi_{\{2^{n-1} \leq |f| < 2^n\}}$, we have that

$$\begin{aligned} ||f||_{D(\int)} &\preceq \left(\sum_{n \in \mathbb{Z}} 2^{np} D^p(\lambda_f(2^n)) \right)^{1/p} \\ &\preceq \left(\int_0^\infty y^{p-1} D^p(\lambda_f(y)) dy \right)^{1/p} \\ &\sim \left(\int_0^\infty f^*(t)^p dD^p(t) \right)^{1/p} = ||f||_{\Lambda^p(w)}, \end{aligned}$$

where λ_f is the distribution function of f and $\Lambda^p(w)$ is the weighted Lorentz space with weight $w(t) = dD^p(t)$, and hence we have proved that

$$\Lambda^p(dD^p) \subseteq D(\ell^p).$$

Therefore, using the previous theorem, we have that

$$\Lambda^p(dD^p) + L^1 \cap L^\infty \subseteq D(\ell^p).$$

At this point, and since $0 < p \leq 1$, it will be good to know when this second space $\Lambda^p(dD^p) + L^1 \cap L^\infty$ is strictly bigger than $\Lambda^p(dD^p)$. Obviously, these two spaces coincide if and only if $L^1 \cap L^\infty \subseteq \Lambda^p(dD^p)$. It follows from Theorem 3.3 in [18], that

Proposition 2.2. *$L^1 \cap L^\infty \subseteq \Lambda^p(dD^p)$ if and only if*

$$\int_0^\infty \left(\frac{\max(1,y)}{D^p(y)}\right)^{p/(p-1)} dD^p(y) < \infty.$$

b) If $\int = \ell(\log \ell)$, and $s \preceq D(s)$, then, taking the decomposition

$$f = \underline{f} + \sum_{n\geq 1} 2^n f_n,$$

where $\underline{f} = f\chi_{\{|f|\leq 1\}}$ and f_n as before, we get that

$$||f||_{D(\int)} \preceq D(||\underline{f}||_1) + \left(\int_1^\infty (\log^+ \log^+ y)\, D(\lambda_f(y))dy\right).$$

From this, it follows using homogeneity that

$$L \log\log L(D) \subseteq D(\textstyle\int),$$

where

$$||f||_{L\log\log L(D)} = \int_0^\infty f^*(t)(1 + \log^+ \log^+ f^*(t))dD(t).$$

In particular, if $D(s) = s\left(1 + \log^+ \frac{1}{s}\right)$, then

$$L \log\log L(D) = L \log L \log\log L.$$

Now, in this concrete case, it was proved in [7], taking the ideas of [1], that we can improve the above result by taking the decomposition of f

$$f = f_0 + \sum_{n\geq 1} 2^{2^n} f_n,$$

where $f_0 = f\chi_{\{|f|\leq 2\}}$ and $f_n = 2^{-2^n} f\chi_{\{2^{2^{n-1}} \leq |f| < 2^{2^n}\}}$. Using this decomposition, it can be proved that

$$L \log L \log\log\log L \subseteq D(\textstyle\int),$$

and, in fact, it was proved in [7], that if $D(s) \geq s$ and $D(s^2) \leq sD(s)$, then

$$L \log\log\log L(D) \subseteq D(\textstyle\int).$$

For our next purpose, we need the following concept which was introduced in [22].

Definition 2.6. *The Galb of a quasi-Banach space X is defined by*

$$\mathrm{Galb(X)} = \left\{(\mathrm{c_n})_\mathrm{n};\ \sum_\mathrm{n} \mathrm{c_n f_n} \in \mathrm{X}, \text{whenever } \|\mathrm{f_n}\|_\mathrm{X} \leq 1\right\},$$

endowed with the "norm" $\|c\|_{\mathrm{Galb(X)}} = \sup_{\|f_n\|_X \le 1} \Big\| \sum_n c_n f_n \Big\|_X$.

Now, since the motivation of our work is to obtain certain type of estimates for an operator T for which a restricted estimates is known, on many occasions, it will be enough to have a weak type estimate for the operator T or even to have that $Tf(x) < \infty$ a.e. x, for every $f \in X$ in order to apply some continuity Banach principle. To this end, it will be enough to identify certain sets containing the Galb(X).

Definition 2.7. *The Weak Galb of a quasi-Banach space X is defined by*

$$\mathrm{WGalb(X)} = \Big\{(\mathrm{c_n})_\mathrm{n};\ \sum_\mathrm{n} \mathrm{c_n f_n} \in \mathrm{M_X}, \text{whenever } \|\mathrm{f_n}\|_\mathrm{X} \le 1\Big\},$$

endowed with the norm $\|c\|_{\mathrm{WGalb(X)}} = \sup_{\|f_n\|_X \le 1} \Big\| \sum_n c_n f_n \Big\|_{M_X}$, *where M_X is the maximal Marcinkiewicz space defined by*

$$M_X = \{f; \|f\|_{M_X} = \sup_{t>0} f^*(t)\varphi_X(t) < \infty\}.$$

The Finite Galb of X is defined by

$$\mathrm{FGalb(X)} = \Big\{(\mathrm{c_n})_\mathrm{n};\ \sum_\mathrm{n} \mathrm{c_n f_n} \text{ are finite a.e., whenever } \|\mathrm{f_n}\|_\mathrm{X} \le 1\Big\}.$$

It is trivial that

$$\mathrm{Galb(M_X)} = \mathrm{WGalb(M_X)}$$

and

$$\mathrm{Galb(X)} \subseteq \mathrm{WGalb(X)} \subseteq \mathrm{FGalb(X)}.$$

We shall see in Proposition 4.1 that the three concepts are different. Note that the advantage of the Finite Galb is the fact that if two quasi-Banach spaces X and Y satisfy that $X \subseteq Y$ continuously, then

$$\mathrm{FGalb(Y)} \subseteq \mathrm{FGalb(X)}.$$

A first general and important fact is the following:

Theorem 2.3. *Let X be a quasi-Banach r.i. space. Then,*

$$\mathrm{Galb(X)} \subseteq \mathrm{WGalb(X)} \subseteq \mathrm{FGalb(X)} \subseteq \ell^1 \cap L_{\varphi_X^{-1}}, \tag{2}$$

where

$$L_{\varphi_X^{-1}} = \Big\{(c_n)_n;\ \sum_n \varphi_X^{-1}(|c_n|) < \infty\Big\}.$$

Proof: The embedding in ℓ^1 is immediate. To show that $\mathrm{FGalb(X)} \subseteq L_{\varphi_X^{-1}}$ we suppose that $\sum_n \varphi_X^{-1}(|c_n|)$ diverges. It is a standard argument to select sets A_n of measure $\varphi_X^{-1}(|c_n|)$ such that $\sum_n \chi_{A_n} = \infty$ on a set of positive measure. With $f_n = (1/c_n)\chi_{A_n}$, $\|f_n\|_X = 1$ and so $(c_n)_n \notin \mathrm{FGalb(X)}$. □

Remark 2.2. *Obviously* $\mathrm{Galb(X)} = \ell^1$ *if and only if* X *is a Banach space. If this is not the case, we shall study conditions on our spaces to have that* $\mathrm{Galb(X)} = L_{\varphi_X^{-1}} \cap \ell^1$.

Our second main result can now be formulated in the following way:

Theorem 2.4. *Let* T *be a sublinear* (ε, δ)*-atomic approximable operator and let* X *be a quasi-Banach r.i. space. Then, if* $\varphi_{X,T}(\mathrm{Galb(X)})$ *is as in Definition 2.5,*
a)

$$T : \varphi_{X,T}(\mathrm{Galb(X)}) \longrightarrow X$$

is bounded.
b) Similarly

$$T : \varphi_{X,T}(\mathrm{WGalb(X)}) \longrightarrow M_X$$

is bounded.
c) For every $f \in \varphi_{X,T}(\mathrm{FGalb(X)})$, $Tf(x) < +\infty$ *almost everywhere.*

Proof: We shall only prove a), since the proof of b) and c) are completely similar.

If $f = \sum_n c_n f_n$ then by sublinearity

$$\|Tf\|_X \leq \|c_n T f_n\|_{\mathrm{Galb(X)}}.$$

If we suppose that $\|f_n\|_\infty \leq 1$ for each n then by Theorem 2.1,

$$\|Tf_n\|_X \leq \varphi_{X,T}(\|f_n\|_1)$$

and

$$\|Tf\|_X \leq \|f\|_{\varphi_{X,T}(\mathrm{Galb(X)})}.$$

follows by taking the infimum over all such representations of f. $\square$

In particular, if T is a sublinear (ε, δ)-atomic approximable operator, the following corollaries follow, from the examples given above.

Corollary 2.1. *If* X *is a Banach space, then*

$$T : \Lambda^1(d\varphi_{X,T}) \longrightarrow X$$

is bounded.

Corollary 2.2. *If* $\mathrm{GalbX} = \ell^{\mathrm{p}}$ *with* $0 < p < 1$, *then*

$$T : \Lambda^p(d\varphi^p_{X,T}) + L^1 \cap L^\infty \longrightarrow X$$

is bounded.

Corollary 2.3. *If* $\mathrm{GalbX} \subseteq \ell(\log \ell)^\alpha$ *and* $s \preceq \varphi_{X,T}(s)$, *then*

$$T : L(\log\log L)^\alpha(d\varphi_{X,T}) \longrightarrow X$$

is bounded. And if, in addition, $\varphi_{X,T}(s^2) \preceq s\varphi_{X,T}(s)$*, then*

$$T : L(\log\log\log L)^\alpha(d\varphi_{X,T}) \longrightarrow X$$

is bounded.

Our next step will be to study the Galb for the class of weighted Lorentz spaces.

3. About the Galb of weighted Lorentz spaces

The purpose of this section is to obtain information about the Galb of the spaces $\Lambda^q(w)$ for $0 < q < \infty$ and of the weak type version spaces $\Lambda^{q,\infty}(w)$. Hence, throughout this section

$$f = \sum_{n=1}^{\infty} c_n f_n,$$

where $\|f_n\|_X \leq 1$ and $X = \Lambda^q(w)$ or $X = \Lambda^{q,\infty}(w)$. We shall use the following formula for the decreasing rearrangement of a sum of functions (see [8]):

$$f^*(3t) \leq \sum_n c_n \left(f_n^*(t) + \frac{1}{t}\int_{a_n t}^t f_n^*(s)ds \right), \tag{3}$$

where $\{a_n\}_n$ are positive numbers such that $\sum_n a_n = 1$, and we recall that if $\Lambda^q(w)$ is quasi-Banach, then the primitive of the weight $W(t) = \int_0^t w$ satisfies the Δ_2 condition and, hence, the number 3 in the left hand side of the previous formula gives no problem at all.

We shall also need to use some estimates for the so called Steklov operator acting on decreasing functions. This operator is defined, for $0 < a < 1$, by

$$S_a f(t) = \frac{1}{t}\int_{at}^t f(s)\, ds.$$

Lemma 3.1.

$$\sup_{f \text{dec}} \frac{\left(\frac{1}{t}\int_{at}^t f(s)\, ds\right) W(t)}{\sup_{t>0} f(t)W(t)} = \sup_{t>0} \left(\frac{1}{t}\int_{at}^t \frac{1}{W(s)}\, ds\right) W(t).$$

Proof: The proof follows trivially since the biggest function f with the property that $\sup_{t>0} f(t)W(t) = 1$ is $1/W$. $\square$

The meaning of the two following lemmas is that in estimating the norm of the Steklov operator on Lorentz spaces it is often sufficient to test it only on characteristic functions.

Lemma 3.2.

$$\sup_{f \mathrm{dec}} \frac{\int_0^\infty \left(\frac{1}{t}\int_{at}^t f(s)\,ds\right) w(t)\,dt}{\int_0^\infty f(t)w(t)\,dt} = \sup_{r>0} \frac{1}{W(r)} \int_0^r \left(\int_s^{s/a} \frac{w(t)}{t} dt \right) ds.$$

Proof: The result follows using Fubini and Theorem 2.12 of [9]. □

Lemma 3.3. *If* $q > 1$*, then*

$$A := \sup_{f \mathrm{dec}} \frac{\left(\int_0^\infty \left(\frac{1}{t}\int_{at}^t f(s)\,ds\right)^q w(t)\,dt \right)^{1/q}}{\left(\int_0^\infty f(t)^q w(t)\,dt \right)^{1/q}} < +\infty$$

if and only if

$$B := \sup_r \left(\frac{1}{W(r)} \int_r^{r/a} (r - at)^q \frac{w(t)}{t^q}\, dt \right)^{1/q} < +\infty. \tag{4}$$

Moreover,
a) it holds that $B \leq A \preceq 1 + B^q$ *and*
b) if for some $D > 1$, $W(s/a) \leq DW(s)$ *for all* $s > 0$*, then* $(1-a) + B \preceq A \preceq (1-a) + B(\log D)^{1/q'}$*. From this, we can also conclude that*
c)

$$(1-a) + B \preceq A \preceq (1-a) + B(\log(B/(\sqrt{a} - a)))^{1/q'}.$$

The proof of this lemma will be postponed to the last section, since it is somewhat technical.

Galb $(\Lambda^{q,\infty}(w))$

Let us start with the case $\Lambda^{q,\infty}(w)$ defined by

$$\|f\|_{\Lambda^{q,\infty}(w)} = \sup_{t>0} f^*(t)W^{1/q}(t),$$

and observe that $\Lambda^{q,\infty}(w) = \Lambda^{1,\infty}(w_q)$, where $w_q(t) = W^{1/q-1}(t)w(t)$, and hence, the parameter q is, somehow superfluous. However, it will be important for us the fact that, for every $q > 1$,

$$\Lambda^{q,1}(w) \subseteq \Lambda^q(w) \subseteq \Lambda^{q,\infty}(w),$$

where $\Lambda^{q,1}(w) = \Lambda^1(w_q)$ and w_q as before. Moreover, by real interpolation theory,

$$\Lambda^q(w) = (\Lambda^{q,1}(w), \Lambda^{q,\infty}(w))_{1/q',q}.$$

As a first consequence of (2), we obtain the following result:

Corollary 3.1. *For every* $0 < q < \infty$,

$$\mathrm{Galb}(\Lambda^{q,\infty}(\mathrm{w})) \subseteq \mathrm{L}_{(\mathrm{W}^{1/q})^{-1}} \cap \ell^1.$$

Theorem 3.1. *Let $0 < q < \infty$ and given $0 < a < 1$, let*

$$H(a) = \sup_{t>0} \left(\frac{1}{t} \int_{at}^{t} W^{-1/q}(s)\, ds \right) W^{1/q}(t).$$

Then, if $(c_n)_n \in \ell^1$ and

$$\inf_{\sum_n a_n \le 1} \sum_n c_n H(a_n) < \infty,$$

we have that $(c_n)_n \in \mathrm{Galb}(\Lambda^{q,\infty}(\mathrm{w}))$.

Proof: Using (3), we obtain that

$$\|f\|_{\Lambda^{q,\infty}(w)} \preceq \sum_n c_n \|f_n\|_{\Lambda^{q,\infty}(w)} + \sum_n c_n \sup_{t>0} \left(\frac{1}{t} \int_{a_n t}^{t} f_n^*(s)\, ds \right) W^{1/q}(t),$$

for every sequence $\sum a_n \le 1$, and, by Lemma 3.1, we immediately obtain that

$$\|f\|_{\Lambda^{q,\infty}(w)} \preceq \sum_n c_n + \sum_n c_n H(a_n),$$

from which the result follows. □

Example: If $w(t) = 1$, then $W(t) = t$ and $H(a) = q(1 - a^{(q-1)/q})/(q-1)$. In particular, $H(a) \approx a^{(q-1)/q}$ if $q < 1$, $H(a) = \log 1/a$ if $q = 1$ and $H(a) \approx 1$ if $q > 1$.

Remark 3.1. *If $H \in L^\infty$, we obtain that $\mathrm{Galb}(\Lambda^{q,\infty}(\mathrm{w})) = \ell^1$ and, of course, this also follows from the fact that $H \in L^\infty$ if and only if $w \in B_q$ which is the case where $\Lambda^{q,\infty}(w)$ is a Banach space (see [20]).*

Corollary 3.2. *If for every $t > 0$ and every $0 < a < 1$,*

$$\frac{1}{t} \int_{at}^{t} W^{-1/q}(s)\, ds \preceq \frac{a}{W^{1/q}(a) W^{1/q}(t)}, \tag{5}$$

then

$$\mathrm{Galb}(\Lambda^{q,\infty}(\mathrm{w})) = \mathrm{WGalb}(\Lambda^{q,\infty}(\mathrm{w})) = \mathrm{FGalb}(\Lambda^{q,\infty}(\mathrm{w})) = \mathrm{L}_{(\mathrm{W}^{1/q})^{-1}} \cap \ell^1.$$

Proof: The embedding $\mathrm{FGalb}(\Lambda^{q,\infty}(\mathrm{w})) \subseteq \mathrm{L}_{(\mathrm{W}^{1/q})^{-1}} \cap \ell^1$ follows from Theorem 2.3 and the opposite embedding follows from Theorem 3.1. Indeed, condition (5) reads that $H(a) \preceq a/W(a)$ and, if $(c_n)_n \in L_{(W^{1/q})^{-1}} \cap \ell^1$, we have that (c_n) tends to zero and hence, we can assume that $(W^{1/q})^{-1}(c_n) \le 1$ for every n. Therefore,

$$\sum_n c_n H((W^{1/q})^{-1}(c_n)) \preceq \sum_n c_n \frac{(W^{1/q})^{-1}(c_n)}{c_n} = \sum_n (W^{1/q})^{-1}(c_n) < \infty,$$

and therefore, $(c_n)_n \in \mathrm{Galb}(\Lambda^{q,\infty}(\mathrm{w}))$ by Theorem 3.1. □

Corollary 3.3. *If $W^{1/q}(s)/s$ is equivalent to a decreasing functions, then*

$$\ell \log \ell \subseteq \mathrm{Galb}(\Lambda^{q,\infty}(w)).$$

Proof: It follows from the trivial fact that $H(a) \leq \log 1/a$ and taking $a_n = c_n$, we obtain the result. □

Observe that if $q = 1$ and $w = 1$, we obtain the well-known fact that $\ell \log \ell \subseteq \mathrm{Galb}(L^{1,\infty}))$.

Galb $(\Lambda^q(w))$
The case $0 < q \leq 1$

Theorem 3.2. *For every $0 < q \leq 1$, it holds that*

$$\mathrm{Galb}(\Lambda^q(w)) \subseteq \ell^q.$$

Proof: Let $\alpha_1 > 0$ be small enough (if necessary) and let us choose, α_k such that

$$W\left(\sum_{j=1}^{k-1} \alpha_j\right) \leq \frac{1}{2} W(\alpha_k).$$

Let $\{A_k\}_{k=1,N}$ be a collection of disjoints sets such that $\alpha_k = |A_k|$ and let us define $\beta_k = W(\alpha_k)$. Obviously β_k is an increasing sequence.

Let us define $f_k = \beta_k^{-1/q} \chi_{A_k}(x)$, so that $\|f_k\|_{\Lambda^q(w)} = 1$ and set

$$f(x) = \sum_{k=1}^{N} c_k f_k.$$

Let us assume, without lost of generality, that $(c_k)_k$ are decreasing and hence, also $\beta_k^{-1/q} c_k$ is decreasing. Let $\gamma_0 = 0$ and $\gamma_k = \sum_{j=1}^{k} \alpha_j$. Then,

$$f^*(t) = \beta_k^{-1/q} c_k,$$

if $\gamma_{k-1} < t < \gamma_k$ and, therefore,

$$\begin{aligned} \int_0^\infty f^*(t)^q w(t)\, dt &= \sum_{k=1}^{N} c_k^q \beta_k^{-1} \int_{\gamma_{k-1}}^{\gamma_k} w(t)\, dt \\ &= \sum_{k=1}^{N} \frac{c_k^q}{W(\alpha_k)} \int_{\gamma_{k-1}}^{\gamma_k} w(t)\, dt > \frac{1}{2} \sum_{k=1}^{N} c_k^q, \end{aligned}$$

from which the result follows. □

Theorem 3.3. *Let $0 < q \leq 1$. Then* $\mathrm{Galb}(\Lambda^q(w)) = \ell^q$ *if and only if $W(t)/t$ is equivalent to a decreasing function.*

Proof: If $W(t)/t$ is equivalent to a decreasing function, then it is known (see [6]) that the space $\Lambda^1(w)$ is a Banach space and since

$$|f|^q \le \sum_n c_n^q |f_n|^q,$$

and $\|f\|^q_{\Lambda^q(w)} = \|f^q\|_{\Lambda^1(w)}$, we obtain that

$$\|f\|^q_{\Lambda^q(w)} \le \sum_n c_n^q \|f_n^q\|_{\Lambda^1(w)} \le \sum_n c_n^q,$$

and therefore $\ell^q \subseteq \mathrm{Galb}(\Lambda^q(\mathrm{w}))$ and hence they coincide. To prove the converse, we observe first, that, if $\mathrm{Galb}(\Lambda^q(\mathrm{w})) = \ell^q$, then

$$\|f\|^q_{\Lambda^q(w)} \le \inf \sum_n \|f_n\|^q_{\Lambda^q(w)},$$

where the infimum extends over all possible decomposition $f = \sum_n f_n$.

Now, we use the same argument that in ([6]): let $k \in \mathbb{N}$ and $s > 0$ and set $f = \chi_{(0,2^k s)}$ and $f_j = \chi_{(js,(j+1)s)}$ with $j = 0, \cdots, 2^k - 1$. Then, since $f = \sum_{j=0}^{2^k-1} f_j$, we obtain that

$$W(2^k s) = \|f\|^q_{\Lambda^q(w)} \preceq \sum_{j=0}^{2^k-1} \|f_j\|^q_{\Lambda^q(w)} = 2^k W(s);$$

that is, $W(2^k s) \le 2^k W(s)$ and hence, if $s < r$ and k is such that $2^{k-1}s < r < 2^k s$,

$$\frac{W(r)}{r} \le \frac{W(2^k s)}{2^{k-1}s} \preceq \frac{2^k W(s)}{2^{k-1}s} \preceq \frac{W(s)}{s},$$

as we wanted to prove. $\square$

Remark 3.2. *In particular, if $X = L^{p,q}$ with $0 < q < \min(p, 1)$, we recover the result proved in* [13].

In general, if w does not satisfy the previous condition we have the following result:

Theorem 3.4. *Given $0 < a < 1$, let*

$$H(a) = \sup_{r>0} \frac{1}{W(r)} \int_0^r \left(\int_s^{s/a} \frac{w(t)}{t} dt \right) ds.$$

Then, if $(c_n)_n \in \ell^q$ and

$$\inf_{\sum_n a_n \le 1} \sum_n c_n^q H(a_n) < \infty,$$

we have that $(c_n)_n \in \mathrm{Galb}(\Lambda^q(\mathrm{w}))$.

Proof: Since $0 < q \leq 1$, we have that $|f|^q \leq \sum_n c_n^q |f_n|^q$, and hence, using (3), we obtain that

$$\|f\|_{\Lambda^q(w)}^q \preceq \sum_n c_n^q \|f_n\|_{\Lambda^q(w)}^q + \sum_n c_n^q \int_0^\infty \left(\frac{1}{t} \int_{a_n t}^t f_n^*(s)^q ds \right) w(t)\, dt,$$

for every sequence $\sum a_n \leq 1$, and, by Lemma 3.2, we immediately obtain

$$\|f\|_{\Lambda^q(w)} \preceq \sum_n c_n^q + \sum_n c_n^q H(a_n),$$

from which the result follows. □

As a corollary of (2) we obtain:

Corollary 3.4.

$$\mathrm{Galb}(\Lambda^q(w)) = \mathrm{WGalb}(\Lambda^q(w)) = \mathrm{FGalb}(\Lambda^q(w)) \subseteq \mathrm{L}_{(W^{1/q})^{-1}} \cap \ell^q.$$

Corollary 3.5. *If for every $r > 0$ and every $0 < a < 1$,*

$$\int_0^r \left(\int_s^{s/a} \frac{w(t)}{t} dt \right) ds \preceq \frac{aW(r)}{W(a)}, \tag{6}$$

then,

$$\mathrm{Galb}(\Lambda^q(w)) = \mathrm{WGalb}(\Lambda^q(w)) = \mathrm{FGalb}(\Lambda^q(w)) = \mathrm{L}_{(W^{1/q})^{-1}} \cap \ell^q.$$

Proof: Condition (6) reads that $H(a) \preceq a/W(a)$ and hence, the proof follows as in Corollary 3.2. □

Let us assume now that W is equivalent to a convex function.

Lemma 3.4. *Let $f_n \geq 0$ and let $g_n \geq 0$ have disjoint supports and satisfy that $f_n^* = g_n^*$ for every n. Then, if W is equivalent to a convex function, we have that*

$$\|\sum_n f_n\|_{\Lambda^1(w)} \leq \|\sum_n g_n\|_{\Lambda^1(w)}.$$

Proof: Let us start by proving that, under the above hypotheses, we have that

$$\int_x^\infty \Big(\sum_n f_n\Big)^* \leq \int_x^\infty \Big(\sum_n g_n\Big)^*,$$

for every $x > 0$. Now, since $\sum_n f_n$ and $\sum_n g_n$ have the same integral, it is enough to prove that

$$\int_0^x \Big(\sum_n g_n\Big)^* \leq \int_0^x \Big(\sum_n f_n\Big)^*.$$

We have that

$$\begin{aligned}
\int_0^x \Big(\sum_n g_n\Big)^* &= \sup\Big\{\int_E \sum_n g_n;\ |E|\le x\Big\}\\
&= \sup\Big\{\sum_n \int_{E_n} g_n;\ \sum_n |E_n|\le x\Big\}\\
&= \sup\Big\{\sum_n \int_0^{x_n} g_n^*;\ \sum_n x_n\le x\Big\}\\
&= \sup\Big\{\sum_n \int_0^{x_n} f_n^*;\ \sum_n x_n\le x\Big\}\\
&= \sup\Big\{\sum_n \int_{E_n} f_n;\ \sum_n |E_n|\le x\Big\}\\
&\le \sup\Big\{\int_E \sum_n f_n;\ |E|\le x\Big\} = \int_0^x \Big(\sum_n f_n\Big)^*
\end{aligned}$$

Finally, since W is equivalent to a convex function, we can assume without lost of generality that w is an increasing function and, hence, using the distribution formula for increasing weights, we know that, there exists a function $c_w(y)$ such that

$$\begin{aligned}
\|\sum_n f_n\|_{\Lambda^1(w)} &= \int_0^\infty \int_{c_w(y)}^\infty \Big(\sum_n f_n\Big)^*(t)dt\,dy\\
&\le \int_0^\infty \int_{c_w(y)}^\infty \Big(\sum_n g_n\Big)^*(t)dt\,dy = \|\sum_n g_n\|_{\Lambda^1(w)}.
\end{aligned}$$

□

Consequently, when computing the $\mathrm{Galb}(\Lambda^1(\mathrm{w}))$ for an increasing weight, we can assume that the functions f_n are disjointly supported. Also:

Theorem 3.5. *If W is a convex function, then, for every $0 < q \le 1$,*

$$\mathrm{Galb}(\Lambda^q(\mathrm{w})) = \Big\{(\mathrm{c_n})_\mathrm{n};\ (\mathrm{c_n^q})_\mathrm{n} \in \mathrm{Galb}(\Lambda^1(\mathrm{w}))\Big\}.$$

Proof: Since

$$\Big(\sum_n c_n f_n\Big)^q \le \sum_n c_n^q f_n^q,$$

it is clear that

$$\Big\{(c_n)_n;\ (c_n^q)_n \in \mathrm{Galb}(\Lambda^1(\mathrm{w}))\Big\} \subseteq \mathrm{Galb}(\Lambda^q(\mathrm{w})).$$

For the converse we observe that if $(c_n)_n \in \mathrm{Galb}(\Lambda^q(\mathrm{w}))$, then $\sum_n c_n f_n \in \Lambda^q(w)$ for every $(f_n)_n$ disjointly supported with $\|f_n\|_{\Lambda^q(w)} \le 1$. Then, since, in this case,

$$\Big(\sum_n c_n f_n\Big)^q = \sum_n c_n^q f_n^q,$$

we obtain that $\sum_n c_n^q g_n \in \Lambda^1(w)$, for every $(g_n)_n$ disjointly supported and with $\|g_n\|_{\Lambda^1(w)} \leq 1$. Since w is increasing, we obtain that $(c_n^q)_n \in \mathrm{Galb}(\Lambda^1(\mathrm{w}))$. □

Theorem 3.6. *If W is a convex function, and, for $0 < a < 1$,*

$$H(a) = \sup_{at \leq r \leq t} \frac{W(t)r}{tW(r)},$$

then, if $(c_n)_n \in \ell^q$ and

$$\inf_{\sum_n a_n \leq 1} \sum_n c_n^q H(a_n) < \infty,$$

we have that $(c_n)_n \in \mathrm{Galb}(\Lambda^q(\mathrm{w}))$.

Proof: By Theorem 3.5, it will be enough to solve the case $q = 1$, and since W is a convex function, it is enough to consider the case where $(f_n)_n$ are disjointly supported. Hence, for every $(a_n)_n$ such that $\sum_n a_n = 1$,

$$\begin{aligned}
\|\sum_n c_n f_n\|_{\Lambda^1(w)} &= \int_0^\infty W\Big(\sum_n \lambda_{f_n}(y/c_n)\Big)\,dy \\
&= \int_0^\infty W\Big(\sum_n a_n \frac{\lambda_{f_n}(y/c_n)}{a_n}\Big)\,dy \\
&\leq \int_0^\infty \sum_n a_n W\Big(\frac{\lambda_{f_n}(y/c_n)}{a_n}\Big)\,dy \\
&\leq \int_0^\infty \sum_n H(a_n) W\Big(\lambda_{f_n}(y/c_n)\Big)\,dy \\
&\leq \sum_n c_n H(a_n) \int_0^\infty W\Big(\lambda_{f_n}(y)\Big)\,dy \leq \sum_n c_n H(a_n),
\end{aligned}$$

and taking the infimum in all the sequences $(a_n)_n$ we obtain the result. □

The case $q > 1$

Theorem 3.7. *For every $q > 1$, $\mathrm{Galb}(\Lambda^q(\mathrm{w})) = \ell^1$ if and only if $w \in B_q$; that is, for every $r > 0$,*

$$r^q \int_r^\infty \frac{w(t)}{t^q} dt \preceq \int_0^r w(t)dt.$$

Proof: It is consequence of the fact (see [16]) that if $q > 1$, $\Lambda^q(w)$ is Banach if and only if $w \in B_q$. □

Theorem 3.8. *Given $0 < a < 1$, let*

$$H(a) = \sup_r \frac{1}{W(r)} \int_r^{r/a} (r - at)^q \frac{w(t)}{t^q}\, dt.$$

Then, if $(c_n)_n \in \ell^1$ and

$$\inf_{\sum_n a_n \leq 1} \sum_n c_n H(a_n) < \infty,$$

we have that $(c_n)_n \in \mathrm{Galb}(\Lambda^q(w))$.

Proof: Using (3), we obtain that

$$\begin{aligned} \|f\|_{\Lambda^q(w)} &\preceq \sum_n c_n \|f_n\|_{\Lambda^q(w)} \\ &+ \sum_n c_n \sup_{\|f\|_{\Lambda^q(w)}=1} \left(\int_0^\infty \left(\frac{1}{t} \int_{a_n t}^t f^*(s)\, ds \right)^q w(t)\, dt \right)^{1/q}, \end{aligned}$$

for every sequence $\sum a_n \leq 1$, and, by Lemma 3.3, we immediately obtain that

$$\|f\|_{\Lambda^q(w)} \preceq \sum_n c_n + \sum_n c_n H(a_n),$$

from which the result follows. □

Remark 3.3. *In particular, if we take $\Lambda^q(w) = L^{1,q}$, which means that $w(t) = t^{q-1}$, then*

$$H(a) \sim \sup_r \frac{1}{r^q} \int_r^{r/a} (r - at)^q \frac{dt}{t} \sim \log \frac{1}{a},$$

and hence, we obtain that $\ell \log \ell \subseteq \mathrm{Galb}(\mathrm{L}^{1,q})$. This estimate is not satisfactory since it is known (see [19]*) that* $\mathrm{Galb}(\mathrm{L}^{1,q}) = \ell(\log \ell)^{1/q'}$.

However, if we use interpolation theory, we can improve the obtained result as follows.

Theorem 3.9. *If $W(t)^{1/q}/t$ is equivalent to a decreasing function, then*

$$\ell(\log \ell)^{1/q'} \subseteq \mathrm{Galb}(\Lambda^q(w)).$$

Proof: To see this, we observe that if $W(t)^{1/q}/t$ is equivalent to a decreasing function, then the space $\Lambda^1(w_q)$ is a Banach space with $w_q(t) = W(t)^{1/q-1} w(t)$ and consequently, $\mathrm{Galb}(\Lambda^1(w_q)) = \ell^1$. On the other hand, by Corollary 3.3, we have that $\ell \log \ell \subseteq \mathrm{Galb}(\Lambda^{1,\infty}(w_q))$ and hence, using interpolation (see [5] and [11]), we obtain the result. □

As a Corollary of Theorem 3.8, we also obtain the following result.

Corollary 3.6. *If, for every* $0 < a < 1$ *and every* $r > 0$*,*

$$\int_r^{r/a} (r - at)^q \frac{w(t)}{t^q}\, dt \le \frac{W(r)a}{W^{1/q}(a)},$$

then

$$\mathrm{Galb}(\Lambda^q(w)) = \mathrm{WGalb}(\Lambda^q(w)) = \mathrm{FGalb}(\Lambda^q(w)) = \mathrm{L}_{(W^{1/q})^{-1}} \cap \ell^1.$$

3.1. **Weak Galb and Finite Galb.** The purpose of this subsection is to obtain information about the Weak and the Finite Galb of the spaces $\Lambda^q(w)$.

Theorem 3.10. *Given* $0 < a < 1$*, let*
a)

$$H(a) = \sup_t \frac{W^{1/q}(t)}{t} \left(\int_{at}^t \left(\frac{u - at}{W(u)} \right)^{q'-1} du \right)^{1/q'},$$

if $q > 1$*, and*
b)

$$H(a) = \sup_{t,r} \frac{W^{1/q}(t)}{t} \frac{(\min(r,t) - at)_+}{W(r)^{1/q}},$$

if $q \le 1$*.*
Then, if $(c_n)_n \in \ell^1$ *and*

$$\inf_{\sum_n a_n \le 1} \sum_n c_n H(a_n) < \infty,$$

we have that $(c_n)_n \in \mathrm{WGalb}(\Lambda^q(w))$*.*

Proof: Using (3), we obtain that

$$\begin{aligned} \|f\|_{\Lambda^{q,\infty}(w)} &\preceq \sum_n c_n \|f_n\|_{\Lambda^q(w)} \\ &+ \sum_n c_n \sup_t \frac{W^{1/q}(t)}{t} \sup_{\|f\|_{\Lambda^q(w)}=1} \frac{1}{t} \int_{a_n t}^t f^*(s)\, ds, \end{aligned}$$

for every sequence $\sum a_n \le 1$, and the result follows, in the case $q > 1$, from the so-called duality Sawyer's formula (see[16]) and, if $q \le 1$, from Theorem 2.12 in [9]. □

Using a completely similar argument to that of the previous Theorem, we can prove the following result.

Theorem 3.11. *Given* $0 < a < 1$*, let*

$$H(a;t) = \left(\int_{at}^t \left(\frac{u - at}{W(u)} \right)^{q'-1} du \right)^{1/q'},$$

if $q>1$ *and*

$$H(a;t)=\sup_r \frac{(\min(r,t)-at)_+}{W(r)^{1/q}},$$

if $q\le 1$.

Then, if $(c_n)_n\in \ell^1$ *and, for every* $t>0$,

$$\inf_{\sum_n a_n\le 1}\sum_n c_n H(a_n;t)<\infty,$$

we have that $(c_n)_n\in \mathrm{FGalb}(\Lambda^q(w))$.

Sometimes, we can use the embedding properties of the Weak and Finite Galb in order to obtain some information about the Galb, as it is shown in the following Corollary:

Corollary 3.7. *Let* $q\ge 1$. *If* $W(s)/s^q$ *is equivalent to a bounded, decreasing function, then*
a)

$$\ell\log\ell=\mathrm{Galb}(\Lambda^{q,\infty}(w))=\mathrm{WGalb}(\Lambda^{q,\infty}(w))=\mathrm{FGalb}(\Lambda^{q,\infty}(w)).$$

b)

$$\ell(\log\ell)^{1/q'}=\mathrm{Galb}(\Lambda^{q}(w))=\mathrm{WGalb}(\Lambda^{q}(w))=\mathrm{FGalb}(\Lambda^{q}(w)).$$

Proof: a) By Corollary 3.3, we know that

$$\ell\log\ell\subseteq\mathrm{Galb}(\Lambda^{q,\infty}(w))\subseteq\mathrm{WGalb}(\Lambda^{q,\infty}(w))\subseteq\mathrm{FGalb}(\Lambda^{q,\infty}(w)).$$

Now, since $W(s)\preceq s^q$, we have that $L^{1,\infty}\subseteq\Lambda^{q,\infty}(w)$ and therefore,

$$\mathrm{FGalb}(\Lambda^{q,\infty}(w))\subseteq\mathrm{FGalb}(L^{1,\infty}).$$

but, since it is known (see [13]) that $\mathrm{FGalb}(L^{1,\infty})=\ell\log\ell$, we obtain the result.
b) The proof of this part is completely similar since by Theorem 3.9, we have that

$$\ell(\log\ell)^{1/q'}\subseteq\mathrm{Galb}(\Lambda^{q}(w)),$$

and since $W(s)\preceq s^q$, we have that $L^{1,q}\subseteq\Lambda^q(w))$.

Finally, we have to use (see [19]) that $\mathrm{FGalb}(L^{1,q})=\ell(\log\ell)^{1/q'}$. □

4. Some examples and applications

If we apply our result to obtain the Galb in the classical case $L^{p,q}$ we obtain the following result:

Corollary 4.1. *1) If $q > 1$ and $p > 1$,* $\mathrm{Galb}(\mathrm{L^{p,q}}) = \ell^1$.
2) If $p = 1$, $\ell \log \ell = \mathrm{Galb}(\mathrm{L^{1,\infty}})$.
3) If $p < 1$, $\mathrm{Galb}(\mathrm{L^{p,\infty}}) = \ell^{\mathrm{p}}$.
4) If $0 < q \leq 1$ and $q \leq p$, $\mathrm{Galb}(\mathrm{L^{p,q}}) = \ell^{\mathrm{q}}$.
5) If $0 < p \leq 1$ and $p \leq q \leq \infty$, $\mathrm{Galb}(\mathrm{L^{p,q}}) = \ell^{\mathrm{p}}$.
6) If $q > 1$, $\ell(\log \ell)^{1/q'} = \mathrm{Galb}(\mathrm{L^{1,q}})$.

Proof: 1) is clear because those spaces are Banach spaces, and 2) has been already mentioned several times. 3) is consequence of (5), 4) of Theorem 3.3 and 5) is consequence of Theorems 3.5 and 3.6 for the case $q \leq 1$ and for the case $q > 1$, we have to proceed by interpolation since we already know that $\mathrm{Galb}(\mathrm{L^{p,\infty}}) = \mathrm{Galb}(\mathrm{L^{p,1}}) = \ell^{\mathrm{p}}$. Finally the embedding $\ell(\log \ell)^{1/q'} \subset \mathrm{Galb}(\mathrm{L^{1,q}})$ in 6) is consequence of Theorem 3.9 and for the converse we have to refer to [19]. □

Another example:

Corollary 4.2. *If $W(t) = t\Big(1 + \log^+ \frac{1}{t}\Big)^{-\alpha}$ with $\alpha > 0$, then, for every $0 < q \leq 1$,*

$$\ell^q (\log \ell)^\alpha = \mathrm{Galb}(\Lambda^{\mathrm{q}}(\mathrm{w}))$$

Proof: The result follows as a consequence of Theorems 3.6 and (2), since one can easily check that, in this case, $H(a) \preceq (1 + \log^+ 1/a)^\alpha$, and $L_{\varphi_X^{-1}} = \ell^q (\log \ell)^\alpha$. □

If T is of restricted weak type (p, p) with constant $1/(p-1)$ as it happens with the Carleson operator given in the introduction, then

$$(T\chi_E)^*(t) \leq \frac{|E|}{t}\left(1 + \log^+ \frac{t}{|E|}\right) \leq |E|\left(1 + \log^+ \frac{1}{|E|}\right)\frac{1}{t}\left(1 + \log^+ t\right);$$

that is,

$$||T\chi_E||_X \leq D(|E|),$$

where $X = \Lambda^{1,\infty}(w)$ with $W(t) = t/(1 + \log^+ t)$ and $D(s) = s(1 + log^+ \frac{1}{s})$. Also, when dealing with the bilinear Hilbert transform, the space that appears naturally is $X = \Lambda^{p,\infty}(w)$ with $p = 2/3$ and $W(t) = t/(1 + \log^+ t)^{4/3}$, see [10]. These examples motivate the study of the Galb of the above spaces X.

Corollary 4.3. *If $W(t) = t\Big(1 + \log^+ t\Big)^{-\alpha}$ with $\alpha > 0$, then, for every $0 < q < 1$,*

$$\ell^q = \mathrm{Galb}(\Lambda^{\mathrm{q},\infty}(\mathrm{w}))$$

and, for $q = 1$,

$$\ell(\log \ell) = \mathrm{Galb}(\Lambda^{1,\infty}(\mathrm{w})).$$

Proof: The result follows as a consequence of (5). □

Finally, it is important to mention that, in general the Galb(X), the WGalb(X) and the FGalb(X) do not coincide as it is shown in the following proposition:

Proposition 4.1. *If* $0 < q < p < 1$*, then*

1)

$$\mathrm{Galb}(\mathrm{L}^{\mathrm{p,q}}(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})) = \ell^{\mathrm{q}},$$

2)

$$\mathrm{WGalb}(\mathrm{L}^{\mathrm{p,q}}(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R}))) = \ell^{\mathrm{p}},$$

3)

$$\mathrm{FGalb}(\mathrm{L}^{\mathrm{p,q}}(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R}))) = \ell^1.$$

Proof: 1) Recall that the Galb of $L^{p,q}$ is ℓ^q, while the Galb of L^1 is ℓ^1. In particular, if $(c_n)_n$ is in ℓ^q and if $\|f_n\|_{L^{p,q}\cap L^1} \leq 1$, then

$$\left\|\sum_n c_n f_n\right\|_{L^{p,q}} \leq c\left(\sum_n |c_n|^q\right)^{1/q},$$

$$\left\|\sum_n c_n f_n\right\|_{L^1} \leq c\sum_n |c_n| \leq c\left(\sum_n |c_n|^q\right)^{1/q}.$$

Hence ℓ^q is contained in the Galb of $L^{p,q} \cap L^1$. In order to prove the converse, given a positive sequence $(c_n)_n$, choose a strictly increasing sequence of integers $k_n \geq 1$ so that $2^{-k_n/p}|c_n|$ is decreasing. Then choose disjoint sets $\{A_n\}_n$ with $|A_n| = 2^{k_n}$. Finally let $f_n = |A_n|^{-1/p}\chi_{A_n}$, so that

$$\begin{aligned}\|f_n\|_{L^{p,q}\cap L^1} &= \max\left\{|A_n|^{-1/p}\|\chi_{A_n}\|_{L^{p,q}}, |A_n|^{-1/p}\|\chi_{A_n}\|_{L^1}\right\}\\ &= \max\left\{1, |A_n|^{1-1/p}\right\} = 1.\end{aligned}$$

Then, it is routine to calculate the rearrangement of the simple function $\sum_n c_n f_n$ and, using the fact that $|A_n|$ is rapidly increasing, to get

$$\begin{aligned}&\left\|\sum_n c_n f_n\right\|_{L^{p,q}}\\ &= \left(\sum_n (p/q)|c_n|^q |A_n|^{-q/p}\left(\left(\sum_{j=1}^n |A_j|\right)^{q/p} - \left(\sum_{j=1}^{n-1}|A_j|\right)^{q/p}\right)\right)^{1/q}\\ &\succeq \left(\sum |c_n|^q\right)^{1/q},\end{aligned}$$

and we conclude that if $(c_n)_n$ is in the Galb of $L^{p,q} \cap L^1$, then $(c_n)_n$ is in ℓ^q.

2) Let $(c_n)_n \in \ell^p$ and let $\|f_n\|_{L^{p,q}\cap L^1} = 1$. Then, since $\|f_n\|_{L^{p,\infty}} \preceq \|f_n\|_{L^{p,q}}$ and $\mathrm{Galb}(\mathrm{L^{p,\infty}}) = \ell^{\mathrm{p}}$,

$$\left\| \sum_n c_n f_n \right\|_{L^{p,\infty}} \preceq \left(\sum_n |c_n|^p \right)^{1/p}.$$

Similarly, $\|f_n\|_{L^{1,\infty}} \preceq \|f_n\|_{L^1}$ and since $\mathrm{Galb}(\mathrm{L}^{1,\infty}) = \ell \log \ell$,

$$\left\| \sum_n c_n f_n \right\|_{L^{1,\infty}} \preceq ||(c_n)_n||_{\ell \log \ell} \preceq \left(\sum_n |c_n|^p \right)^{1/p}.$$

Hence ℓ^p is contained in $\mathrm{WGalb}(\mathrm{L^{p,q}} \cap \mathrm{L}^1)$. In order to prove the converse, given a finite sequence $(c_n)_n$, choose k and a sequence of disjoint sets $\{A_n\}_n$ with $|A_n| = k\,|c_n|^p \geq 1$. Finally let $f_n = |A_n|^{-1/p} \chi_{A_n}$, so that $\|f_n\|_{L^{p,q}\cap L^1} = 1$. Then $|\sum_n c_n f_n|$ is equal to $k^{-1/p}$ on a set of measure $\sum_n |A_n| = k \sum_n |c_n|^p$. In particular, the norm of $\sum_n c_n f_n$ in $L^{p,\infty}$ is $(\sum_n |c_n|^p)^{1/p}$, and hence $(c_n)_n \in \ell^p$.
3) First observe that if $(c_n)_n \in \mathrm{FGalb}(\mathrm{L^{p,q}} \cap \mathrm{L}^1)$, then at least $\sum_n |c_n| < +\infty$. In order to prove the converse, observe that if $\sum_n |c_n| < +\infty$ and if $\|f_n\|_{L^{p,q}\cap L^1} = 1$, then $\sum_n c_n f_n(x)$ converges in L^1 and therefore it is finite almost everywhere. □

5. Proof of Lemma 3.3

a) The necessary condition is trivial since it is just taking the supremum over all functions of the form $\chi_{(0,r)}$. To prove the converse, let us fix a decreasing function f and an $r > 0$. Let $r_j = r/a^j$ and write

$$\int_0^\infty \left(\frac{1}{t} \int_{at}^t f \right)^q w(t)\, dt \leq 2^{q-1} \sum_{j=-\infty}^{\infty} U_j + V_j$$

where

$$U_j = \int_{r_j}^{r_{j+1}} \left(\int_{r_j}^t f \right)^q \frac{w(t)}{t^q}\, dt \quad \text{and} \quad V_j = \int_{r_j}^{r_{j+1}} \left(\int_{at}^{r_j} f \right)^q \frac{w(t)}{t^q}\, dt.$$

Since f is decreasing we have for each j,

$$\begin{aligned} U_j &= q\int_{r_j}^{r_{j+1}}\int_{r_j}^{t}\left(\int_{r_j}^{s} f\right)^{q-1} f(s)\,ds\frac{w(t)}{t^q}\,dt \\ &\le q\int_{r_j}^{r_{j+1}}\int_{r_j}^{t}\left(\frac{s-r_j}{s-as}\int_{as}^{s} f\right)^{q-1} f(s)\,ds\frac{w(t)}{t^q}\,dt \\ &= q(1-a)^{1-q}\int_{r_j}^{r_{j+1}}\int_{r_j}^{t}(s-r_j)^{q-1}\int_{g(t)}^{g(s)} dy\,ds\frac{w(t)}{t^q}\,dt \\ &+ q(1-a)^{1-q}\int_{r_j}^{r_{j+1}}\int_{r_j}^{t}(s-r_j)^{q-1}\,dsg(t)\frac{w(t)}{t^q}\,dt \\ &\equiv (1-a)^{1-q}U_j^{(1)} + (1-a)^{1-q}U_j^{(2)} \end{aligned}$$

where $g(s) = \left(\frac{1}{s}\int_{as}^{s} f\right)^{q-1} f(s)$. Note that g is also decreasing.

Let λ_g denote the distribution function of g. To estimate $U_j^{(1)}$, we expand the region of integration by observing that

$$\left\{\begin{matrix} r_j < t < r_{j+1} \\ r_j < s < t \\ g(t) < y < g(s) \end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} g(r_{j+1}) \le y \le g(r_j) \\ \lambda_g(y) \le t \le \lambda_g(y)/a \\ at \le s \le \lambda_g(y) \end{matrix}\right\}.$$

Performing the inner, ds, integral and using the hypothesis (4) we see that

$$U_j^{(1)} \le \int_{g(r_{j+1})}^{g(r_j)}\int_{\lambda_g(y)}^{\lambda_g(y)/a}(\lambda_g(y)-at)^q\frac{w(t)}{t^q}\,dt\,dy \le B^q\int_{g(r_{j+1})}^{g(r_j)}\left(\int_0^{\lambda_g(y)} w\right)dy.$$

The estimate for $U_j^{(2)}$ is simpler,

$$U_j^{(2)} = \int_{r_j}^{r_{j+1}}(t-r_j)^q\,dsg(t)\frac{w(t)}{t^q}\,dt \le \int_{r_j}^{r_{j+1}} gw.$$

Now

$$\sum_{j=-\infty}^{\infty} U_j^{(1)} + U_j^{(2)} \le B^q\int_0^\infty\left(\int_0^{\lambda_g(y)} w\right)dy + \int_0^\infty gw = (B^q+1)\int_0^\infty gw.$$

The estimate for V_j begins similarly. With g as above,

$$\begin{aligned} V_j &= q\int_{r_j}^{r_{j+1}}\int_{at}^{r_j}\left(\int_{at}^{s} f\right)^{q-1} f(s)\,ds\frac{w(t)}{t^q}\,dt \\ &\le q\int_{r_j}^{r_{j+1}}\int_{at}^{r_j}\left(\frac{s-at}{s-as}\int_{as}^{s} f\right)^{q-1} f(s)\,ds\frac{w(t)}{t^q}\,dt \\ &= q(1-a)^{1-q}\int_{r_j}^{r_{j+1}}\int_{at}^{r_j}(s-at)^{q-1}\int_{g(r_j)}^{g(s)} dy\,ds\frac{w(t)}{t^q}\,dt \\ &+ q(1-a)^{1-q}g(r_j)\int_{r_j}^{r_{j+1}}\int_{at}^{r_j}(s-at)^{q-1}\,ds\frac{w(t)}{t^q}\,dt \\ &\equiv (1-a)^{1-q}V_j^{(1)} + (1-a)^{1-q}V_j^{(2)}. \end{aligned}$$

Interchange and expand the region of integration for $V_j^{(1)}$ by observing that

$$\left\{\begin{array}{c} r_j < t < r_{j+1} \\ at < s < r_j \\ g(r_j) < y < g(s) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} g(r_j) \le y \le g(r_{j-1}) \\ \lambda_g(y) \le t \le \lambda_g(y)/a \\ at \le s \le \lambda_g(y) \end{array}\right\}.$$

Performing the inner, ds, integral and using the hypothesis (4) yields

$$V_j^{(1)} \le \int_{g(r_j)}^{g(r_{j-1})}\int_{\lambda_g(y)}^{\lambda_g(y)/a}(\lambda_g(y)-at)^q\frac{w(t)}{t^q}\,dt\,dy \le B^q\int_{g(r_j)}^{g(r_{j-1})}\left(\int_0^{\lambda_g(y)} w\right)dy.$$

Thus,

$$\sum_{j=-\infty}^{\infty} V_j^{(1)} \le B^q\int_0^\infty\left(\int_0^{\lambda_g(y)} w\right)dy = B^q\int_0^\infty gw.$$

To estimate $V_j^{(2)}$ we use the fact that $g(r_j)$ is a decreasing sequence. For each $k > 1$,

$$\sum_{j=-\infty}^{k-1}\int_{r_j}^{r_{j+1}}(r_j-at)^q\frac{w(t)}{t^q}\,dt \le (1-a)^q\int_0^{r_k} w$$

and, by (4),

$$\int_{r_k}^{r_{k+1}}(r_k-at)^q\frac{w(t)}{t^q}\,dt \le B^q\int_0^{r_k} w.$$

It follows that

$$\sum_{j=-\infty}^{k}\int_{r_j}^{r_{j+1}}(r_j-at)^q\frac{w(t)}{t^q}\,dt \le \sum_{j=-\infty}^{k}((1-a)^q+B^q)\int_{r_{j-1}}^{r_j} w$$

for all k and, because $g(r_j)$ is a decreasing sequence,

$$\sum_{j=-\infty}^{\infty} V_j^{(2)} \le \sum_{j=-\infty}^{\infty} g(r_j)((1-a)^q + B^q)\int_{r_{j-1}}^{r_j} w \le ((1-a)^q + B^q)\int_0^\infty gw.$$

Combining the inequalities above, we get,

$$\begin{aligned} &\int_0^\infty \left(\frac{1}{t}\int_{at}^t f\right)^q w(t)\,dt \preceq (1+B^q)\int_0^\infty gw \\ \le\ & (1+B^q)\left(\int_0^\infty \left(\frac{1}{t}\int_{at}^t f\right)^q w(t)\,dt\right)^{1/q'}\left(\int_0^\infty f^q w\right)^{1/q}, \end{aligned}$$

and we conclude (by approximating f by integrable functions if necessary) that $A \preceq 1 + B^q$.

To prove b), we shall use some of the ideas of Stepanov and Ushakova [21], Theorem 3. For the converse, we apply Theorem 3.1 of [17], although Theorem 1 of [16] will also do.

$$\begin{aligned} A &= \sup_{\substack{\|f\|_{L^q(w)}\le 1 \\ f\downarrow}} \left(\int_0^\infty \left(\frac{1}{t}\int_{at}^t f(s)\,ds\right)^q w(t)\,dt\right)^{1/q} \\ &= \sup_{\substack{\|f\|_{L^q(w)}\le 1 \\ f\downarrow}} \sup_{\|g\|_{L^{q'}(w)}\le 1} \int_0^\infty \frac{1}{t}\int_{at}^t f(s)\,ds g(t)w(t)\,dt \\ &= \sup_{\|g\|_{L^{q'}(w)}\le 1} \sup_{\substack{\|f\|_{L^q(w)}\le 1 \\ f\downarrow}} \int_0^\infty f(s)\left(\frac{1}{w(s)}\int_s^{s/a} g(t)w(t)\,\frac{dt}{t}\right)w(s)\,ds \\ &\approx \sup_{\|g\|_{L^{q'}(w)}\le 1} \left(\int_0^\infty \left(\frac{\int_0^x G(s)w(s)\,ds}{\int_0^x w(s)\,ds} + \frac{\int_0^\infty G(s)w(s)\,ds}{\int_0^\infty w(s)\,ds}\right)^{q'} w(x)\,dx\right)^{1/q'} \end{aligned}$$

where $G(s) = \frac{1}{w(s)}\int_s^{s/a} g(t)w(t)\,\frac{dt}{t}$. Now

$$\begin{aligned} \int_0^x Gw &= \int_0^x \int_s^{s/a} g(t)w(t)\,\frac{dt}{t}\,ds = \int_0^{x/a} \frac{1}{t}\int_{at}^{\min(x,t)} ds g(t)w(t)\,dt \\ &= (1-a)\int_0^x g(t)w(t)\,dt + \int_x^{x/a}\left(\frac{x}{t} - a\right)g(t)w(t)\,dt \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty Gw &= \int_0^\infty \frac{1}{w(s)}\int_s^{s/a} g(t)w(t)\,\frac{dt}{t} w(s)\,ds \\ &= \int_0^\infty \int_s^{s/a} g(t)w(t)\,\frac{dt}{t}\,ds = (1-a)\int_0^\infty gw. \end{aligned}$$

Therefore $A \approx A_1 + A_2 + A_3$ where

$$\begin{aligned} A_1 &= (1-a) \sup_{\|g\|_{L^{q'}(w)} \le 1} \left(\int_0^\infty \left(\frac{\int_0^x g(t) w(t)\, dt}{\int_0^x w(t)\, dt} \right)^{q'} w(x)\, dx \right)^{1/q'}, \\ A_2 &= (1-a) \sup_{\|g\|_{L^{q'}(w)} \le 1} \left(\int_0^\infty \left(\frac{\int_0^\infty g(t) w(t)\, dt}{\int_0^\infty w(t)\, dt} \right)^{q'} w(x)\, dx \right)^{1/q'}, \\ A_3 &= \sup_{\|g\|_{L^{q'}(w)} \le 1} \left(\int_0^\infty \left(\frac{\int_x^{x/a} \left(\frac{x}{t} - a \right) g(t) w(t)\, dt}{\int_0^x w(t)\, dt} \right)^{q'} w(x)\, dx \right)^{1/q'}. \end{aligned}$$

The first two are easy. Hardy's inequality says that $A_1 = (1-a)q$ and Hölder's inequality yields $A_2 = 1-a$. For A_3 we use Theorem 4.4 of [12]. By replacing x by s and $g(t)w(t)$ by $f(t)t$ we recognize A_3 as the best constant in the inequality

$$\begin{aligned} &\left(\int_0^\infty \left(\int_s^{s/a} (s - ay) f(y)\, dy \right)^{q'} \left(\int_0^s w \right)^{-q'} w(s)\, ds \right)^{1/q'} \\ \le\ & A_3 \left(\int_0^\infty f(t)^{q'} t^{q'} w(t)^{1-q'}\, dt \right)^{1/q'}. \end{aligned}$$

It is trivial to check that the so called GHO condition in [12] holds for the kernel $k(s,y) = s - ay$ and so $A_3 \approx \max(A_{3,1}, A_{3,2})$ where

$$A_{3,1} = \sup_{s \le x \le s/a} \left(\int_s^x (t-s)^{q'} \left(\int_0^t w \right)^{-q'} w(t)\, dt \right)^{1/q'} \left(\int_x^{s/a} t^{-q} w(t)\, dt \right)^{1/q}.$$

and

$$A_{3,2} = \sup_{s \le x \le s/a} \left(\int_s^x \left(\int_0^t w \right)^{-q'} w(t)\, dt \right)^{1/q'} \left(\int_x^{s/a} (s - at)^q t^{-q} w(t)\, dt \right)^{1/q}.$$

Since

$$B = \sup_r \frac{\left(\int_r^{r/a} (r - at)^q \frac{w(t)}{t^q}\, dt \right)^{1/q}}{\left(\int_0^r w(t)\, dt \right)^{1/q}}$$

we have

$$\begin{aligned} A_{3,2} &\le \sup_s \left(\int_s^\infty \left(\int_0^t w \right)^{-q'} w(t)\, dt \right)^{1/q'} \left(\int_s^{s/a} \left(s - at \right)^q w(t)\, dt \right)^{1/q} \\ &\preceq B. \end{aligned}$$

Also

$$\begin{aligned} A_{3,1} &= \sup_{s\le x\le s/a}\left(\int_s^x\left(\int_x^{s/a}(t-s)^q y^{-q}w(y)\,dy\right)^{q'-1}\left(\int_0^t w\right)^{-q'} w(t)\,dt\right)^{1/q'} \\ &\le \sup_{s\le x\le s/a}\left(\int_s^x\left(\int_t^{t/a}(t-ay)^q y^{-q}w(y)\,dy\right)^{q'-1}\left(\int_0^t w\right)^{-q'} w(t)\,dt\right)^{1/q'} \\ &\le B\sup_{s\le x\le s/a}\left(\int_s^x\left(\int_0^t w\right)^{-1} w(t)\,dt\right)^{1/q'} \\ &\le B\sup_{s\le x\le s/a}\left(\log\left(\int_0^x w\right)-\log\left(\int_0^s w\right)\right)^{1/q'} \\ &\le B(\log(D))^{1/q'} \end{aligned}$$

This completes the proof of b). To prove c), it is enough to show that the condition

$$\int_0^{s/a} w \le D\int_0^s w, \quad s>0,$$

always holds with $D=(B/(\sqrt{a}-a))^{2q}$. Now, for any $r>0$, $r/a\ge r/\sqrt{a}$ so

$$B^q\int_0^r w \ge \int_r^{r/\sqrt{a}}(r-at)^q t^{-q}w(t)\,dt \ge (\sqrt{a}-a)^q\int_0^{r/\sqrt{a}} w.$$

Now

$$\int_0^{s/a} w \le (B/(\sqrt{a}-a))^q\int_0^{s/\sqrt{a}} w \le (B/(\sqrt{a}-a))^{2q}\int_0^s w.$$

This completes the proof. □

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