AN EQUIVALENCE THEOREM FOR SOME INTEGRAL CONDITIONS WITH GENERAL MEASURES RELATED TO HARDY'S INEQUALITY

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Abstract: It is proved that, besides the usual Muckenhoupt condition, there exist four different scales of conditions for characterizing the Hardy type inequality with general measures for the case $1 < p \le q < \infty$. In fact, an even more general equivalence theorem of independent interest is proved and discussed.

1. Introduction

Let us start by considering the following recent result concerning equivalences between some integral conditions related to Hardy's inequality by A. Gogatishvili, A. Kufner, L.-E. Persson and A. Wedestig in [[3], Theorem 1]:

Theorem 1. *For $-\infty \le a < b \le \infty$, α, β and s positive numbers and f, g measurable functions positive a.e in (a,b), let*

$$F(x) := \int_x^b f(t)dt, \qquad G(x) := \int_a^x g(t)dt$$

and

$$B_1(x;\alpha,\beta) := F(x)^\alpha G(x)^\beta,$$

$$B_2(x;\alpha,\beta,s) := \left(\int_x^b f(t)G(t)^{\frac{\beta-s}{\alpha}}dt\right)^\alpha G(x)^s,$$

$$B_3(x;\alpha,\beta,s) := \left(\int_a^x g(t)F(t)^{\frac{\alpha-s}{\beta}}dt\right)^\beta F(x)^s,$$

$$B_4(x;\alpha,\beta,s) := \left(\int_a^x f(t)G(t)^{\frac{\beta+s}{\alpha}}dt\right)^\alpha G(x)^{-s},$$

$$B_5(x;\alpha,\beta,s) := \left(\int_x^b g(t)F(t)^{\frac{\alpha+s}{\beta}}dt\right)^\beta F(x)^{-s}.$$

The numbers

$$B_1 := \sup_{a<x<b} B_1(x;\alpha,\beta) \quad \textit{and} \quad B_i := \sup_{a<x<b} B_i(x;\alpha,\beta,s), \quad i = 2,3,4,5,$$

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are mutually equivalent. The constants in the equivalence relations can depend on α, β and s.

Remark 1. *The equivalence constants can be explicitly given in each case. For example the following estimates hold:*

$$(\min(1, s/\beta))^{\alpha} \sup_{a<x<b} B_2(x; \alpha, \beta, s) \leq \sup_{a<x<b} B_1(x; \alpha, \beta) \leq (\max(1, s/\beta))^{\alpha} \sup_{a<x<b} B_2(x; \alpha, \beta, s), \tag{1.1}$$

and

$$(\min(1, s/\alpha))^{\beta} \sup_{a<x<b} B_3(x; \alpha, \beta, s) \leq \sup_{a<x<b} B_1(x; \alpha, \beta) \leq (\max(1, s/\alpha))^{\beta} \sup_{a<x<b} B_3(x; \alpha, \beta, s). \tag{1.2}$$

Remark 2. *With $a = 0$, $\alpha = \frac{1}{q}$ and $\beta = \frac{1}{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$ and $f(t)$ and $g(t)$ replaced by $u(t)$ and $v(t)^{1-p'}$, respectively, we have*

$$B_1 := \sup_{a<x<b} \left(\int_x^b u(t)dt \right)^{\frac{1}{q}} \left(\int_0^x v(t)^{1-p'} dt \right)^{\frac{1}{p'}},$$

and the condition $B_1 < \infty$ is just the usual Muckenhoupt condition that is usually used to characterize all weights u and v so that the Hardy inequality

$$\left(\int_0^b \left(\int_0^x f(t)dt \right)^q u(x)dx \right)^{\frac{1}{q}} \leq C \left(\int_0^b f^p(x)v(x)dx \right)^{\frac{1}{p}} \tag{1.3}$$

holds for all measurable functions $f \geq 0$ and for the parameters p, q satisfying $1 < p \leq q < \infty$. Here, by using Theorem 1 we see that we can replace the Muckenhoupt condition by infinite many conditions namely by the corresponding four scales of conditions. For some special cases see also [10], [14], [15] *and c.f. also* [9].

In this paper we will generalize Theorem 1 to the case with general measures (see Theorem 2). According to this Theorem and a well-known result of Muckenhoupt [6] we obtain some new scales for characterizing Hardy type inequalities with general measures for the case $1 < p \leq q < \infty$ (see Theorem 3). As corollaries we obtain a discrete version of Theorem 1 (see Corollary 2) and some new scales of conditions characterizing the weighted discrete Hardy type inequality

$$\left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_k \right)^q u_n \right)^{\frac{1}{q}} \leq C \left(\sum_{n=1}^{\infty} a_n^p v_n \right)^{\frac{1}{p}}, \tag{1.4}$$

to hold for fixed weight sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ and all positive sequences $\{a_k\}_{k=1}^{\infty}$ (see Corollary 3).

The paper is organized as follows: In order not to disturb our discussions later on we present some preliminaries in Section 2. The main results are presented and discussed in Section 3. The proof of Theorem 2 can be found in Section 4. Finally, Section 5 is reserved for some concluding remarks and results.

2. PRELIMINARIES

Throughout this paper we assume that μ and λ are measures on $\mathbb{R}$ satisfying

$$M(x) := \int_{[x,\infty)} d\mu < \infty \quad \text{and} \quad \Lambda(x) := \int_{(-\infty,x]} d\lambda < \infty \tag{2.1}$$

for each $x \in \mathbb{R}$. In particular, observe that both μ and λ are σ-finite.

First we state and prove the following two technical lemmas involving the measures μ and λ:

Lemma 1. *If $t \geq 0$ and $x \in \mathbb{R}$, then*

$$\lambda\left(\{y : \Lambda(y) \leq t\}\right) \leq \min\left(\Lambda(\infty), t\right),$$

$$\lambda\left(\{y \leq x : \Lambda(y) > t\}\right) \geq \Lambda(x) - t,$$

$$\mu\left(\{y : M(y) \leq t\}\right) \leq \min\left(M(\infty), t\right), \quad \textit{and}$$

$$\mu\left(\{y \geq x : M(y) > t\}\right) \geq M(\infty) - t.$$

Proof. Fix $t \geq 0$ and set $E = \{y : \Lambda(y) \leq t\}$. It is clear that $\lambda(E) \leq \lambda(\mathbb{R}) = \Lambda(\infty)$. Since $\lambda(\emptyset) = 0 \leq t$ we may suppose that $E \neq \emptyset$. Set $z = \sup E$. If $z \in E$ then $E \subset (-\infty, z]$ and hence $\lambda(E) \leq \Lambda(z) \leq t$. If $z \notin E$ then choose $z_n \in E$ such that $z_n \uparrow z$ and observe that $E \subset \cup_n(-\infty, z_n]$. Now $\lambda(E) \leq \lim_{n\to\infty} \Lambda(z_n) \leq t$. This shows that $\lambda(E) \leq t$ and this fact completes the proof of the first inequality.

The second inequality follows from the first one since

$$\begin{aligned} \lambda\left(\{y \leq x : \Lambda(y) > t\}\right) &= \Lambda(x) - \lambda\left(\{y \leq x : \Lambda(y) \leq t\}\right) \\ &\geq \Lambda(x) - \lambda\left(\{y : \Lambda(y) \leq t\}\right) \\ &\geq \Lambda(x) - t. \end{aligned}$$

The inequalities for μ may be proved similarly or else be deduced from those for λ by just making the transformation $y \mapsto -y$. □

Lemma 2. *Let $x \in \mathbb{R}$ and $p > 0$. Then*

$$\min(1, 1/p)\, \Lambda(x)^p \leq \int_{(-\infty,x]} \Lambda^{p-1} d\lambda \leq \max(1, 1/p)\, \Lambda(x)^p. \tag{2.2}$$

Proof. If $p > 1$, then (2.2) reduces to

$$(1/p)\, \Lambda(x)^p \leq \int_{(-\infty,x]} \Lambda^{p-1} d\lambda \leq \Lambda(x)^p.$$

Since Λ is non-decreasing it yields that

$$\int_{(-\infty,x]} \Lambda^{p-1} d\lambda \leq \Lambda(x)^{p-1} \int_{(-\infty,x]} d\lambda = \Lambda(x)^p,$$

giving the right-hand side inequality.

For the left-hand side inequality we use the Fubini theorem and Lemma 1 as follows:

$$\begin{aligned}
\int_{(-\infty,x]} \Lambda^{p-1} d\lambda &= \int_{(-\infty,x]} \int_0^{\Lambda(y)} dt^{p-1} d\lambda(y) \\
&= \int_0^{\Lambda(x)} \int_{\{y\le x:\Lambda(y)>t\}} d\lambda(y) dt^{p-1} \\
&\ge \int_0^{\Lambda(x)} \left(\Lambda\left(x\right)-t\right) dt^{p-1} \\
&= \Lambda\left(x\right) \int_0^{\Lambda(x)} dt^{p-1} - \int_0^{\Lambda(x)} t dt^{p-1} \\
&= \Lambda\left(x\right)^p - (p-1) \int_0^{\Lambda(x)} t^{p-1} dt \\
&= \Lambda\left(x\right)^p - \left(1-\frac{1}{p}\right) \Lambda\left(x\right)^p \\
&= (1/p)\, \Lambda\left(x\right)^p .
\end{aligned}$$

Now we consider (2.2) when $0<p<1$. It reduces to

$$\Lambda(x)^p \le \int_{(-\infty,x]} \Lambda^{p-1} d\lambda \le (1/p)\, \Lambda(x)^p.$$

Since Λ is non-decreasing,

$$\int_{(-\infty,x]} \Lambda^{p-1} d\lambda \ge \Lambda(x)^{p-1} \int_{(-\infty,x]} d\lambda = \Lambda(x)^p,$$

and the left-hand side inequality is proved.

For the right-hand side inequality we use the Fubini theorem and Lemma 2 again to get

$$\begin{aligned}
\int_{(-\infty,x]} \Lambda^{p-1} d\lambda &= \int_{(-\infty,x]} \int_{\Lambda(y)}^{\infty} d\left(-t^{p-1}\right) d\lambda(y) \\
&= \int_0^{\infty} \int_{\{y\le x:\Lambda(y)\le t\}} d\lambda(y) d\left(-t^{p-1}\right) \\
&= \int_0^{\Lambda(x)} \int_{\{y:\Lambda(y)\le t\}} d\lambda(y) d\left(-t^{p-1}\right) + \int_{\Lambda(x)}^{\infty} \int_{\{y\le x\}} d\lambda(y) d\left(-t^{p-1}\right) \\
&\le \int_0^{\Lambda(x)} t d\left(-t^{p-1}\right) + \Lambda(x) \int_{\Lambda(x)}^{\infty} d\left(-t^{p-1}\right) \\
&= -\left(p-1\right) \int_0^{\Lambda(x)} t^{p-1} dt + \Lambda(x) \int_{\Lambda(x)}^{\infty} d\left(-t^{p-1}\right) \\
&= -\left(1-\frac{1}{p}\right) \Lambda(x)^p + \Lambda(x)^p \\
&= (1/p)\, \Lambda(x)^p.
\end{aligned}$$

□

Next we record an inequality for μ that corresponds to the inequality in Lemma 2 and since it in fact follows from this lemma we state this as:

Corollary 1. *Let $x \in \mathbb{R}$ and $p > 0$. Then*

$$\min(1, 1/p)\, M(x)^p \leq \int_{[x,\infty)} M^{p-1} d\mu \leq \max(1, 1/p)\, M(x)^p.$$

Proof. Put $\mu(t) = \lambda(-t)$. Then $M(x) = \Lambda(-x)$ and the proof follows from Lemma 2. □

Finally, we state the following result corresponding to Lemma 2 and Corollary 1 for the case $p < 0$:

Lemma 3. *Let $x \in \mathbb{R}$ and $p < 0$. Then*

$$\int_{(x,\infty)} \Lambda^{p-1} d\lambda \leq |1/p| \left(\Lambda(x)^p - \Lambda(\infty)^p\right), \tag{2.3}$$

$$\int_{[x,\infty)} \Lambda^{p-1} d\lambda \leq \Lambda(x)^p + |1/p| \left(\Lambda(x)^p - \Lambda(\infty)^p\right), \tag{2.4}$$

$$\int_{(-\infty,x)} M^{p-1} d\mu \leq |1/p| \left(M(x)^p - M(-\infty)^p\right), \tag{2.5}$$

and

$$\int_{(-\infty,x]} M^{p-1} d\mu \leq M(x)^p + |1/p| \left(M(x)^p - M(\infty)^p\right). \tag{2.6}$$

Proof. For the inequality (2.3) we apply the Fubini theorem and Lemma 1 as follows:

$$\begin{aligned}
\int_{(x,\infty)} \Lambda^{p-1} d\lambda &= \int_{(x,\infty)} \int_{\Lambda(y)}^{\infty} d(-t^{p-1}) d\lambda(y) \\
&= \int_{\Lambda(x)}^{\infty} \int_{\{y>x:\Lambda(y)\leq t\}} d\lambda(y) d(-t^{p-1}) \\
&= \int_{\Lambda(x)}^{\infty} \left(\int_{\{y:\Lambda(y)\leq t\}} d\lambda(y) - \int_{\{y\leq x\}} d\lambda(y) \right) d(-t^{p-1}) \\
&\leq \int_{\Lambda(x)}^{\infty} \left(\min(t, \Lambda(\infty)) - \Lambda(x)\right) d(-t^{p-1}).
\end{aligned}$$

To evaluate the last expression we break it at $\Lambda(\infty)$. We have

$$\begin{aligned}
&\int_{\Lambda(x)}^{\Lambda(\infty)} (t - \Lambda(x))\, d(-t^{p-1}) \\
&= \int_{\Lambda(x)}^{\Lambda(\infty)} t d(-t^{p-1}) - \Lambda(x) \int_{\Lambda(x)}^{\Lambda(\infty)} d(-t^{p-1}) \\
&= (1 - 1/p)\left(\Lambda(x)^p - \Lambda(\infty)^p\right) - \Lambda(x)\left(\Lambda(x)^{p-1} - \Lambda(\infty)^{p-1}\right), \qquad (2.7)
\end{aligned}$$

and, if $\Lambda(\infty) < \infty$, then

$$\int_{\Lambda(\infty)}^{\infty} \left(\Lambda(\infty) - \Lambda(x)\right) d(-t^{p-1}) = \left(\Lambda(\infty) - \Lambda(x)\right) \Lambda(\infty)^{p-1}. \tag{2.8}$$

Adding (2.7) and (2.8) we have the following estimate as required:

$$\begin{aligned}
&\int_{(-\infty,x)} \Lambda^{p-1} d\lambda \\
&\leq (1-1/p)\left(\Lambda(x)^p - \Lambda(\infty)^p\right) - \Lambda(x)\left(\Lambda(x)^{p-1} - \Lambda(\infty)^{p-1}\right) \\
&+ \left(\Lambda(\infty) - \Lambda(x)\right)\Lambda(\infty)^{p-1} \\
&= |1/p|\left(\Lambda(x)^p - \Lambda(\infty)^p\right).
\end{aligned}$$

We also note that for the case $\lambda(\infty) = \infty$ the integral in (2.8) cancels and the estimate above follows directly from (2.7).

The inequality (2.4) follows from (2.3), because

$$\int_{\{x\}} \Lambda^{p-1} d\lambda = \Lambda(x)^{p-1}\lambda(\{x\}) \leq \Lambda(x)^p.$$

Finally, by using the same argument as in the proof of Corollary 1 we find that (2.3) and (2.4) imply (2.5) and (2.6), respectively. The proof is complete. □

3. THE MAIN RESULTS

Our first main result reads:

Theorem 2. *Let M and Λ be defined in (2.1). For fixed positive numbers α, β, s define*

$$A_1(x) := A_1(x;\alpha,\beta) = M(x)^\alpha \Lambda(x)^\beta, \tag{3.1}$$

$$A_2(x) := A_2(x;\alpha,\beta,s) = \left(\int_{[x,\infty)} \Lambda^{(\beta-s)/\alpha} d\mu\right)^\alpha \Lambda(x)^s, \tag{3.2}$$

$$A_3(x) := A_3(x;\alpha,\beta,s) = \left(\int_{(-\infty,x]} M^{(\alpha-s)/\beta} d\lambda\right)^\beta M(x)^s, \tag{3.3}$$

$$A_4(x) := A_4(x;\alpha,\beta,s) = \left(\int_{(-\infty,x]} \Lambda^{(\beta+s)/\alpha} d\mu\right)^\alpha \Lambda(x)^{-s}, \tag{3.4}$$

$$A_5(x) := A_5(x;\alpha,\beta,s) = \left(\int_{[x,\infty)} M^{(\alpha+s)/\beta} d\lambda\right)^\beta M(x)^{-s}. \tag{3.5}$$

The numbers

$$\sup_{x\in\mathbb{R}} A_1(x;\alpha,\beta) \quad \textit{and} \quad \sup_{x\in\mathbb{R}} A_i(x;\alpha,\beta,s), \quad (i = 2,3,4,5)$$

are mutually equivalent. The constants in the equivalence relations can depend on α, β and s.

Remark 3. *The proof of Theorem 2 is carried out in Section 4 by deriving concrete positive constants c_i and d_i so that*

$$c_i \sup_{x\in\mathbb{R}} A_i(x;\alpha,\beta,s) \leq \sup_{x\in\mathbb{R}} A_1(x;\alpha,\beta) \leq d_i \sup_{x\in\mathbb{R}} A_i(x;\alpha,\beta,s), \quad (i = 2,3,4,5).$$

Next we consider the Hardy inequality with general measures and integrable functions f,

$$\left(\int_0^\infty \left|\int_0^x f(t)dt\right|^q d\mu(x)\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty |f(x)|^p\, d\nu(x)\right)^{\frac{1}{p}}, \tag{3.6}$$

where μ and ν are Borel measures and $1 \leq p \leq q < \infty$.

Already Muckenhoupt [6] in 1972 proved that for $1 \leq p < \infty$ the inequality (3.6) (for $p = q$) holds if and only if

$$M = \sup_{r>0} \left(\mu[r,\infty)\right)^{\frac{1}{p}} \left(\int_0^r \left(\frac{d\widetilde{\nu}}{dx}\right)^{1-p'} dx\right)^{\frac{1}{p'}} < \infty, \tag{3.7}$$

where $\widetilde{\nu}$ denotes the absolutely continuous part of ν. Moreover, if C is the least constant for which (3.6) holds, then $M \leq C \leq p^{1/p}(p')^{1/p'}M$ for $1 < p < \infty$ and $C = M$ for $p = 1$. Here $p' = p/(p-1)$ is the conjugate exponent of p. Moreover, Kokilashvili [4] (see also [5]) in 1979 announced the general result (without a proof there but maybe the proofs of Kokilashvili was published somewhere else) that for $1 \leq p \leq q < \infty$ the inequality (3.6) holds if and only if

$$MK = MK(p,q) := \sup_{r>0} \left(\mu[r,\infty)\right)^{\frac{1}{q}} \left(\int_0^r \left(\frac{d\widetilde{\nu}}{dx}\right)^{1-p'} dx\right)^{\frac{1}{p'}} < \infty. \tag{3.8}$$

In the sequel we will assume that $f \geq 0$ so that in particular, the absolute value signs in (3.6) can be removed.

Hence, by applying Theorem 2 with $x = r$, $d\mu = d\nu = 0$ for $x \leq 0$, $d\lambda = \left(\frac{d\widetilde{\nu}}{dx}\right)^{1-p'} dx$ for $x > 0$, $\alpha = 1/q$, $\beta = 1/p'$ in (3.1) and note that

$$MK(p,q) = \sup_{r>0} A_1(r; 1/q, 1/p') \tag{3.9}$$

we obtain the following more general result:

Theorem 3. *Let $1 < p \leq q < \infty$. Then the inequality*

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q d\mu(x)\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f(x)^p d\nu(x)\right)^{\frac{1}{p}} \tag{3.10}$$

holds for all ν-measurable functions $f \geq 0$ if and only if, for some $s > 0$,

$$MK_2(s) = \sup_{x>0} \left(\int_{(0,x]} d\lambda\right)^s \left(\int_{[x,\infty)} \left(\int_{(0,x]} d\lambda\right)^{q\left(\frac{1}{p'}-s\right)} d\mu\right)^{\frac{1}{q}} < \infty \tag{3.11}$$

or

$$MK_3(s) = \sup_{x>0} \left(\int_{[x,\infty)} d\mu\right)^s \left(\int_{(0,x]} \left(\int_{[x,\infty)} d\mu\right)^{p'\left(\frac{1}{q}-s\right)} d\lambda\right)^{\frac{1}{p'}} < \infty \tag{3.12}$$

or

$$MK_4(s) = \sup_{x>0} \left(\int_{(0,x]} d\lambda \right)^{-s} \left(\int_{(0,x]} \left(\int_{(0,x]} d\lambda \right)^{q\left(\frac{1}{p'}+s\right)} d\mu \right)^{\frac{1}{q}} < \infty \tag{3.13}$$

or

$$MK_5(s) = \sup_{x>0} \left(\int_{[x,\infty)} d\mu \right)^{-s} \left(\int_{[x,\infty)} \left(\int_{[x,\infty)} d\mu \right)^{p'\left(\frac{1}{q}+s\right)} d\lambda \right)^{\frac{1}{p'}} < \infty. \tag{3.14}$$

Here $d\lambda = \left(\frac{d\tilde{\nu}}{dx}\right)^{1-p'} dx$.

Moreover, for the best constant C *in (3.10), we have* $C \approx MK_i(s)$, $i = 2, 3, 4, 5$, *and each* $s > 0$.

Remark 4. *We see that* $MK_2(\frac{1}{p'}) = MK$ *(when* $d\lambda = \left(\frac{d\tilde{\nu}}{dx}\right)^{1-p'} dx$*) so that (3.11) may be regarded as a generalization of the usual Muckenhoupt-Kokilashvili condition (3.8). Similarly, we have that* $MK_4(\frac{1}{p})$ *coincides with an alternative condition, which for the continuous case recently was pointed out by L.-E. Persson and V. D. Stepanov (see* [10]*).*

Remark 5. *The equivalence constants in the relations* $C \approx MK_i(s)$, $i = 2, ..., 5$ *are known in each case so this can help us to give a better estimate of the best constant* C *in (3.10).*

Next we note that by applying Theorem 2 with measures μ and λ taken to be purely atomic measures supported on the positive integers we obtain the following equivalence result for sequences:

Corollary 2. *Let* α, β *and* s *be positive numbers and* $\{a_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty}$ *denote positive sequences. Moreover, let us denote*

$$A_n = \sum_{k=n}^{\infty} a_k \quad \text{and} \quad B_n = \sum_{k=1}^{n} b_k$$

and

$$D_1(n;\alpha,\beta) := A_n^{\alpha} B_n^{\beta},$$

$$D_2(n;\alpha,\beta,s) := \left(\sum_{k=n}^{\infty} a_k B_k^{\frac{\beta-s}{\alpha}} \right)^{\alpha} B_n^{s},$$

$$D_3(n;\alpha,\beta,s) := \left(\sum_{k=1}^{n} b_k A_k^{\frac{\alpha-s}{\beta}} \right)^{\beta} A_n^{s},$$

$$D_4(n;\alpha,\beta,s) := \left(\sum_{k=1}^{n} a_k B_k^{\frac{\beta+s}{\alpha}} \right)^{\alpha} B_n^{-s},$$

$$D_5(n;\alpha,\beta,s) := \left(\sum_{k=n}^{\infty} b_k A_k^{\frac{\alpha+s}{\beta}} \right)^{\beta} A_n^{-s}.$$

The numbers

$$D_1 := \sup_{1<n<\infty} D_1(n;\alpha,\beta) \text{ and } D_i := \sup_{1<n<\infty} D_i(n;\alpha,\beta,s),\ i = 2, 3, 4, 5$$

are mutually equivalent. The constants in the equivalence relations can depend on α, β, *and* s.

Remark 6. *A direct proof of Corollary 2 was recently presented by C. Okpoti* [8] *(see also Remark 9).*

G. Bennett [1] characterized the inequality (1.4) to hold (for all positive sequences $\{a_k\}_{k=1}^{\infty}$) by the condition

$$B_S := \sup_{n\geq 1}\left(\sum_{k=n}^{\infty} u_k\right)^{\frac{1}{q}}\left(\sum_{k=1}^{n} v_k^{1-p'}\right)^{\frac{1}{p'}} < \infty \tag{3.15}$$

and also some equivalent conditions (see also [2]). Independently, the characterization (3.15) was found by G. Sinnamon [13]. We note also that

$$D_1 := \sup_{1<n<\infty} D_1(n; 1/q, 1/p') = B_S,$$

where $D_1(n; 1/q, 1/p')$ is one of the equivalent expressions in Corollary 2. Thus, we obtain the following scales for characterizing the discrete Hardy inequality (1.4):

Corollary 3. *Let* $1 < p \leq q < \infty$. *Then the inequality (1.4) holds for all arbitrary non-negative sequences* $\{a_k\}_{k=1}^{\infty}$ *if and only if, for some* $s > 0$,

$$D_2(s) := \sup_{n\geq 1}\left(\sum_{k=1}^{n} v_k^{1-p'}\right)^{s}\left(\sum_{k=n}^{\infty} u_k\left(\sum_{m=1}^{k} v_m^{1-p'}\right)^{q\left(\frac{1}{p'}-s\right)}\right)^{\frac{1}{q}} < \infty$$

or

$$D_3(s) := \sup_{n\geq 1}\left(\sum_{k=n}^{\infty} u_k\right)^{s}\left(\sum_{k=1}^{n} v_k^{1-p'}\left(\sum_{m=k}^{\infty} u_m\right)^{p'\left(\frac{1}{q}-s\right)}\right)^{\frac{1}{p'}} < \infty$$

or

$$D_4(s) := \sup_{n\geq 1}\left(\sum_{k=1}^{n} v_k^{1-p'}\right)^{-s}\left(\sum_{k=1}^{n} u_k\left(\sum_{m=1}^{k} v_m^{1-p'}\right)^{q\left(\frac{1}{p'}+s\right)}\right)^{\frac{1}{q}} < \infty$$

or

$$D_5(s) := \sup_{n\geq 1}\left(\sum_{k=n}^{\infty} u_k\right)^{-s}\left(\sum_{k=n}^{b} v_k^{1-p'}\left(\sum_{m=k}^{\infty} u_m\right)^{p'\left(\frac{1}{q}+s\right)}\right)^{\frac{1}{p'}} < \infty.$$

Moreover, for the best constant C *in (1.4) it yields that* $C \approx D_i(s)$, $i = 2, ..., 5$ *and each* $s > 0$.

Remark 7. *Another proof of Corollary 3 can be found in the Licentiate thesis of C. Okpoti* [7]. *This result is a generalization of a previous result of C. Okpoti, L.-E. Persson and A. Wedestig (*[9]*, Theorem 1). At the endpoints of these scales we rediscover some previous results of G. Bennett* [2].

4. Proofs

Proof of Theorem 2:

In the calculations below by writing sup we always mean $\sup_{x\in\mathbb{R}}$ (i.e. supremum over all $x \in \mathbb{R}$).

The proof consists of considering a number of cases, which, in particular, give us explicit expressions of the equivalence constants in all cases.

1. (i). If $s \le \beta$, then $A_2(x) \ge A_1(x)$ and hence $\sup A_2 \ge \sup A_1$: Since $\Lambda^{(\beta-s)/\alpha}$ is non-decreasing we get that

$$
\begin{aligned}
A_2(x) &= \left(\int_{[x,\infty)} \Lambda^{(\beta-s)/\alpha} d\mu\right)^{\alpha} \Lambda(x)^s \\
&\ge \left(\int_{[x,\infty)} d\mu\right)^{\alpha} \Lambda(x)^{(\beta-s)} \Lambda(x)^s = A_1(x).
\end{aligned}
\tag{4.1}
$$

(ii). If $s > \beta$, then $\sup A_2 \ge (\beta/s)^{\alpha} \sup A_1$: For any x we may use Corollary 1 with $p = s/\beta$ to get

$$
\begin{aligned}
A_2(x) &= \left(\int_{[x,\infty)} \left(M^{\alpha}\Lambda^{\beta}\right)^{(\beta-s)/\alpha\beta} M^{(s-\beta)/\beta} d\mu\right)^{\alpha} \Lambda(x)^s \\
&\ge \left(\sup A_1\right)^{(\beta-s)/\beta} \left(\int_{[x,\infty)} M^{(s-\beta)/\beta} d\mu\right)^{\alpha} \Lambda(x)^s \\
&\ge \left(\sup A_1\right)^{(\beta-s)/\beta} (\beta/s)^{\alpha} M(x)^{\alpha s/\beta} \Lambda(x)^s \\
&= (\beta/s)^{\alpha} \left(\sup A_1\right)^{(\beta-s)/\beta} A_1(x)^{s/\beta}.
\end{aligned}
$$

Taking the supremum we have

$$
\sup A_2 \ge (\beta/s)^{\alpha} \left(\sup A_1\right)^{(\beta-s)/\beta} \left(\sup A_1\right)^{s/\beta} = (\beta/s)^{\alpha} \sup A_1. \tag{4.2}
$$

According to (4.1) and (4.2), for $s > 0$, $\alpha, \beta > 0$ we have

$$
\sup A_1(x;\alpha,\beta) \le \left(\max(1, s/\beta)\right)^{\alpha} \sup A_2(x;\alpha,\beta,s). \tag{4.3}
$$

(iii). If $s \ge \beta$, then $A_2(x) \le A_1(x)$ and hence $\sup A_2 \le \sup A_1$: Since $\Lambda^{(\beta-s)/\alpha}$ is non-increasing,

$$
\begin{aligned}
A_2(x) &= \left(\int_{[x,\infty)} \Lambda^{(\beta-s)/\alpha} d\mu\right)^{\alpha} \Lambda(x)^s \\
&\le \left(\int_{[x,\infty)} d\mu\right)^{\alpha} \Lambda(x)^{(\beta-s)} \Lambda(x)^s \\
&= M(x)^{\alpha} \Lambda(x)^{\beta} = A_1(x).
\end{aligned}
\tag{4.4}
$$

(iv). If $s < \beta$, then $\sup A_2 \leq (\beta/s)^{\alpha} \sup A_1$: For any x we may use Corollary 1 with $p = s/\beta$ to get

$$\begin{aligned} A_2(x) &= \left(\int_{[x,\infty)} \left(M^{\alpha}\Lambda^{\beta}\right)^{(\beta-s)/\alpha\beta} M^{(s-\beta)/\beta} d\mu \right)^{\alpha} \Lambda(x)^s \\ &\leq (\sup A_1)^{(\beta-s)/\alpha} \left(\int_{[x,\infty)} M^{(s-\beta)/\beta} d\mu \right)^{\alpha} \Lambda(x)^s \\ &\leq (\sup A_1)^{(\beta-s)/\alpha} (\beta/s)^{\alpha} M(x)^{\alpha s/\beta} \Lambda(x)^s \\ &= (\sup A_1)^{(\beta-s)/\alpha} (\beta/s)^{\alpha} A_1(x)^{s/\beta}. \end{aligned}$$

Taking supremum we have

$$\sup A_2 \leq (\beta/s)^{\alpha} \sup A_1. \tag{4.5}$$

In view of (4.4) and (4.5), for $s > 0$, $\alpha, \beta > 0$ we have

$$\sup A_2(x; \alpha, \beta, s) \leq (\max(1, \beta/s))^{\alpha} \sup A_1(x; \alpha, \beta). \tag{4.6}$$

Combining (4.3) and (4.6) gives us

$$\sup A_1(x; \alpha, \beta) \approx \sup A_2(x; \alpha, \beta, s).$$

2. By making completely similar calculations as above we get that

$$\sup A_1(x; \alpha, \beta) \leq (\max(1, s/\alpha))^{\beta} \sup A_3(x; \alpha, \beta, s) \tag{4.7}$$

and

$$\sup A_3(x; \alpha, \beta, s) \leq (\max(1, \alpha/s))^{\beta} \sup A_1(x; \alpha, \beta). \tag{4.8}$$

In view of (4.7) and (4.8) we have

$$\sup A_1(x; \alpha, \beta) \approx \sup A_3(x; \alpha, \beta, s)$$

and the claimed equivalence holds in this case too.

3. (i). We have $\sup A_4 \leq \max(1, (\beta+s)/\alpha)^{\alpha} \max(1, \alpha/s)^{\alpha} \sup A_1$: Fix x and begin by applying Lemma 2 with $p = (\beta+s)/\alpha$ and, after that, interchange the

order of integration and apply Lemma 2 again, this time with $p = s/\alpha$.

$$\begin{aligned}
A_4(x) &= \left(\int_{(-\infty,x]} \Lambda^{(\beta+s)/\alpha} d\mu\right)^{\alpha} \Lambda(x)^{-s} \\
&\leq \left(\int_{(-\infty,x]} \max\left(1, (\beta+s)/\alpha\right) \int_{(-\infty,y]} \Lambda^{(\beta+s-\alpha)/\alpha} d\lambda d\mu(y)\right)^{\alpha} \Lambda(x)^{-s} \\
&= \max\left(1, (\beta+s)/\alpha\right)^{\alpha} \left(\int_{(-\infty,x]} \int_{[z,x]} d\mu \Lambda(z)^{(\beta+s-\alpha)/\alpha} d\lambda(z)\right)^{\alpha} \Lambda(x)^{-s} \\
&\leq \max\left(1, (\beta+s)/\alpha\right)^{\alpha} \left(\int_{(-\infty,x]} \left(M^{\alpha}\Lambda^{\beta}\right)^{1/\alpha} \Lambda^{(s-\alpha)/\alpha} d\lambda\right)^{\alpha} \Lambda(x)^{-s} \\
&\leq \max\left(1, (\beta+s)/\alpha\right)^{\alpha} \sup A_1 \left(\int_{(-\infty,x]} \Lambda^{(s-\alpha)/\alpha} d\lambda\right)^{\alpha} \Lambda(x)^{-s} \\
&\leq \max\left(1, (\beta+s)/\alpha\right)^{\alpha} \sup A_1 \max\left(1, \alpha/s\right)^{\alpha} \Lambda(x)^{s} \Lambda(x)^{-s} \\
&= \max\left(1, (\beta+s)/\alpha\right)^{\alpha} \max\left(1, \alpha/s\right)^{\alpha} \sup A_1.
\end{aligned}$$

Taking supremum over all x, we have

$$\sup A_4 \leq \max\left(1, (\beta+s)/\alpha\right)^{\alpha} \max\left(1, \alpha/s\right)^{\alpha} \sup A_1. \tag{4.9}$$

(ii). We have $\sup A_1 \leq \left[\max\left(s/\alpha, \alpha/s\right) + (\alpha/\beta)\max\left(1, s/\alpha\right)\right]^{\alpha} \sup A_4$: We use the fact that if a function H is non-increasing on $\mathbb{R}$, then there exists a sequence of non-negative functions h_n such that $\int_{[x,\infty)} h_n d\mu$ increases to $H(x)$ for μ-almost every $x \in \mathbb{R}$. See Lemma 1.2 of [12].

Take $H = \Lambda^{-(\beta+s)/\alpha}$ and choose such a sequence $\{h_n\}_{n=1}^{\infty}$. Then, for any $x \in \mathbb{R}$, by using Lemma 2, Lemma 3, interchanging the order of integration and making some obvious estimates we obtain that

$$\begin{aligned}
&\int_{[x,\infty)} \Lambda(y)^{(\beta+s)/\alpha} \int_{[x,\infty)} h_n d\mu d\mu(y) = \int_{[x,\infty)} \int_{[x,z]} \Lambda^{(\beta+s)/\alpha} d\mu h_n(z) d\mu(z) \\
&\leq \int_{[x,\infty)} \left(\left(\left(\int_{(-\infty,z]} \Lambda^{(\beta+s)/\alpha} d\mu\right)^{\alpha} \Lambda(z)^{-s}\right)^{1/\alpha} \Lambda(z)^{s/\alpha} h_n(z) d\mu(z)\right. \\
&\leq (\sup A_4)^{1/\alpha} \int_{[x,\infty)} \Lambda(z)^{s/\alpha} h_n(z) d\mu(z) \\
&\leq \max(1, s/\alpha)(\sup A_4)^{1/\alpha} \int_{[x,\infty)} \int_{(-\infty,z]} \Lambda^{(s-\alpha)/\alpha} d\lambda h_n(z) d\mu(z) \\
&= \max(1, s/\alpha)(\sup A_4)^{1/\alpha} \int_{[x,\infty)} \int_{(-\infty,x]} \Lambda^{(s-\alpha)/\alpha} d\lambda h_n(z) d\mu(z) \\
&+ \max(1, s/\alpha)(\sup A_4)^{1/\alpha} \int_{[x,\infty)} \int_{(x,z]} \Lambda^{(s-\alpha)/\alpha} d\lambda h_n(z) d\mu(z) \\
&\leq \max(1, s/\alpha) \max(1, \alpha/s)(\sup A_4)^{1/\alpha} \Lambda(x)^{s/\alpha} \int_{[x,\infty)} h_n(z) d\mu(z)
\end{aligned}$$

$$
\begin{aligned}
&+\max(1,s/\alpha)(\sup A_4)^{1/\alpha}\int_{(x,\infty)}\int_{[y,\infty)} h_n d\mu \Lambda(y)^{(s-\alpha)/\alpha} d\lambda(y)\\
&\leq \max(s/\alpha,\alpha/s)(\sup A_4)^{1/\alpha}\Lambda(x)^{s/\alpha}\Lambda(x)^{-(\beta+s)/\alpha}\\
&+\max(1,s/\alpha)(\sup A_4)^{1/\alpha}\int_{(x,\infty)}\Lambda(y)^{-(\beta+s)/\alpha}\Lambda(y)^{(s-\alpha)/\alpha} d\lambda(y)\\
&\leq \max(s/\alpha,\alpha/s)(\sup A_4)^{1/\alpha}\Lambda(x)^{-\beta/\alpha}+(\alpha/\beta)\max(1,s/\alpha)\left(\sup A_4\right)^{1/\alpha}\Lambda(x)^{-\beta/\alpha}\\
&=\left[\max(s/\alpha,\alpha/s)+(\alpha/\beta)\max(1,s/\alpha)\right]\left(\sup A_4\right)^{1/\alpha}\Lambda(x)^{-\beta/\alpha}.
\end{aligned}
$$

Now, letting $n\to\infty$ and using the Monotone Convergence Theorem, we have

$$
\begin{aligned}
A_1(x) &= \left(\lim_{n\to\infty}\int_{[x,\infty)}\Lambda(y)^{(\beta+s)/\alpha}\int_{[y,\infty)} h_n d\mu d\mu(y)\right)^{\alpha}\Lambda(x)^{\beta}\\
&\leq \left[\max(s/\alpha,\alpha/s)+(\alpha/\beta)\max(1,s/\alpha)\right]^{\alpha}(\sup A_4)\Lambda(x)^{-\beta}\Lambda(x)^{\beta}\\
&= \left[\max(s/\alpha,\alpha/s)+(\alpha/\beta)\max(1,s/\alpha)\right]^{\alpha}\sup A_4.
\end{aligned}
$$

Taking supremum over all x, we have

$$\sup A_1 \leq \left[\max(s/\alpha,\alpha/s)+(\alpha/\beta)\max(1,s/\alpha)\right]^{\alpha}\sup A_4. \tag{4.10}$$

Combining (4.9) and (4.10) we have

$$\sup A_1(x;\alpha,\beta)\approx \sup A_4(x;\alpha,\beta,s).$$

4. By making completely similar calculations as above we get

$$\sup A_1 \leq \left(\max(s/\beta,\beta/s)+(\beta/\alpha)\max\left(1,s/\beta\right)\right)^{\beta}\sup A_5 \tag{4.11}$$

and

$$\sup A_5 \leq \max\left(1,\left(\alpha+s\right)/\beta\right)^{\beta}\max\left(1,\beta/s\right)^{\beta}\sup A_1. \tag{4.12}$$

In view of (4.11) and (4.12) we have

$$\sup A_1(x;\alpha,\beta)\approx \sup A_5(x;\alpha,\beta,s)$$

and the claimed equivalence holds in this case too.

The proof is complete.■

Proof of Theorem 3:

It is a known fact that for the case $1<p\leq q<\infty$ the condition (3.8) characterizes the inequality (3.10) to hold for measurable functions $f\geq 0$. As noted already in Section 3 (see (3.9)) that the condition (3.8) in fact is equivalent to the condition

$$\sup_{r>0} A_1(r;1/q,1/p')<\infty,$$

where A_1 is defined by (3.1). Hence, by just using Theorem 2 for this case we find that (3.10) is also equivalent to

$$MK_i=\sup_{x>0} A_i\left(x;1/q,1/p',s\right)<\infty,\quad i=2,...,5.$$

The claim concerning the best constant C in (3.10) follows accordingly. The proof is complete.■

5. Concluding remarks and results

Remark 8. *According to our proof of Theorem 2 the equivalence constants in this Theorem can be given explicitly in each case. For example we have:*

$$(5.1)\qquad (\min(1,s/\beta))^{\alpha}\sup A_2(x;\alpha,\beta,s)\le \sup A_1(x;\alpha,\beta)\le (\max(1,s/\beta))^{\alpha}\sup A_2(x;\alpha,\beta,s),$$

and

$$(5.2)\qquad (\min(1,s/\alpha))^{\beta}\sup A_3(x;\alpha,\beta,s)\le \sup A_1(x;\alpha,\beta)\le (\max(1,s/\alpha))^{\beta}\sup A_3(x;\alpha,\beta,s).$$

Hence, we get the same equivalence constants in our case with general measures as for the previously proved continuous case (see [3]*).*

Remark 9. *The discrete result corresponding to Theorem 1 was recently proved by C. Okpoti* [8] *(c.f. our Corollary 2). Also in this case he obtained equivalence constants in the relations corresponding to (1.1) and (1.2) (or (5.1) and (5.2)) and are the same. However, in the other cases there are differences. More precise information about this and also some related results can be found in* [8]*.*

Definition 1. *In view of the above equivalences we say that a pair of measures* (μ,λ) *is in class* $WP(\alpha,\beta)$ *provided*

$$\sup_{x\in\mathbb{R}} A_1(x;\alpha,\beta)<\infty$$

or, equivalently,

$$\sup_{x\in\mathbb{R}} A_j(x;\alpha,\beta,s)<\infty$$

for some $s>0$ *and some* j *among* $2,3,4,5$.

Proposition 1. *Suppose* $1<p\le q<\infty$*. Then the Hardy inequality*

$$(5.3)\qquad \left(\int_{\mathbb{R}}\left|\int_{(-\infty,x]} f\,d\lambda\right|^{q} d\mu(x)\right)^{1/q}\le C\left(\int_{\mathbb{R}}|f|^{p}\,d\lambda\right)^{1/p}$$

holds for all λ*-measurable* f *if and only if* $(\mu,\lambda)\in WP(1/q,1/p')$.

Proof. By Corollary 3.4 of [11], the inequality (5.3) holds if and only if

$$\sup_{x\in\mathbb{R}} A_4(x;1/q,1/p',1/p)<\infty,$$

one of the equivalent conditions for $WP(1/q,1/p')$. □

Proposition 2. *Suppose* $1<p\le q<\infty$*. Then* $(\mu,\lambda)\in WP(1/q,1/p')$ *if and only if the cone of non-increasing functions in* $L^p(d\lambda)$ *is contained in* $L^q(\Lambda^q d\mu)$.

Proof. The proof follows from Theorem 3.2 of [11] and Proposition 1. □

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