AN EQUIVALENCE THEOREM FOR SOME INTEGRAL CONDITIONS WITH GENERAL MEASURES RELATED TO HARDY'S INEQUALITY II

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Abstract: Scales of equivalent weight characterizations for the Hardy type inequality with general measures are proved. The conditions are valid in the case of indices $0 < q < p < \infty$, $p > 1$. We also include a reduction theorem for transferring a three-measure Hardy inequality to the case with two measures.

Key words: Hardy's inequality, Hardy's inequality with general measures, Scales of weight characterizations, Equivalent integral conditions with general measures.
Classifications: 130.000, 220.000 and 360.000.

1. Introduction

Simple necessary and sufficient conditions on σ-finite measures λ and μ for which the Hardy inequality

$$\left(\int_{\mathbf{R}}\left(\int_{(-\infty,x]} f d\lambda\right)^q d\mu(x)\right)^{1/q} \leq C\left(\int_{\mathbf{R}} f^p d\lambda\right)^{1/p} \tag{1.1}$$

holds for all $f \geq 0$ have been known for some time. See [4, 7, 10, 14, 15].

For many applications it is useful to have such conditions available in several equivalent forms. In [3, 5, 8, 16, 17], equivalent forms of these conditions have been given in the case of the weighted Hardy inequality (λ and μ absolutely continuous) and the Hardy inequality for sequences (λ and μ purely atomic). See also [1, 2] for related work on sequences. For general measures we provided, in [9], scales of equivalent conditions in the case $1 < p \leq q < \infty$. Here, in this paper, we continue this work in the case $1 < q < p < \infty$, $p > 1$.

Muckenhoupt [7] in 1972 proved that, in the case $1 \le p = q < \infty$, the inequality

$$\left(\int_0^\infty \left| \int_0^x f(t)dt \right|^q d\mu(x) \right)^{\frac{1}{q}} \le C \left(\int_0^\infty |f(x)|^p \, d\nu(x) \right)^{\frac{1}{p}}, \tag{1.2}$$

where μ and ν are Borel measures, holds if and only if

$$M = \sup_{r>0} \left(\mu[r,\infty)\right)^{\frac{1}{p}} \left(\int_0^r \left(\frac{d\widetilde{\nu}}{dx} \right)^{1-p'} dx \right)^{\frac{1}{p'}} < \infty, \tag{1.3}$$

where $\widetilde{\nu}$ denotes the absolutely continuous part of ν. Moreover, if C is the least constant for which (1.2) holds, then $M \le C \le p^{1/p}(p')^{1/p'} M$ for $1 < p < \infty$ and $C = M$ for $p = 1$. Here $p' = p/(p-1)$ is the conjugate exponent of p. Moreover, Kokilashvili [4] (see also [6]) in 1979 announced the general result (without a proof there) that for $1 \le p \le q < \infty$ the inequality (1.2) holds if and only if

$$MK = MK(p,q) := \sup_{r>0} \left(\mu[r,\infty)\right)^{\frac{1}{q}} \left(\int_0^r \left(\frac{d\widetilde{\nu}}{dx} \right)^{1-p'} dx \right)^{\frac{1}{p'}} < \infty. \tag{1.4}$$

In the sequel we will assume that $f \ge 0$ so that in particular, the absolute value signs in (1.2) can be removed.

From the Muckenhoupt-Kokilashvili condition (1.4) the following more general result was obtained in [9]:

Theorem 1.1. *Let $1 < p \le q < \infty$. Then the inequality*

$$\left(\int_0^\infty \left(\int_0^x f(t)dt \right)^q d\mu(x) \right)^{\frac{1}{q}} \le C \left(\int_0^\infty f(x)^p d\nu(x) \right)^{\frac{1}{p}} \tag{1.5}$$

holds for all ν-measurable functions $f \ge 0$ if and only if, for some $s > 0$,

$$MK_2(s) = \sup_{x>0} \left(\int_{(0,x]} d\lambda \right)^s \left(\int_{[x,\infty)} \left(\int_{(0,x]} d\lambda \right)^{q\left(\frac{1}{p'}-s\right)} d\mu \right)^{\frac{1}{q}} < \infty \tag{1.6}$$

or

$$MK_3(s) = \sup_{x>0} \left(\int_{[x,\infty)} d\mu \right)^s \left(\int_{(0,x]} \left(\int_{[x,\infty)} d\mu \right)^{p'\left(\frac{1}{q}-s\right)} d\lambda \right)^{\frac{1}{p'}} < \infty \tag{1.7}$$

or

$$MK_4(s) = \sup_{x>0} \left(\int_{(0,x]} d\lambda \right)^{-s} \left(\int_{(0,x]} \left(\int_{(0,x]} d\lambda \right)^{q\left(\frac{1}{p'}+s\right)} d\mu \right)^{\frac{1}{q}} < \infty \tag{1.8}$$

or

$$MK_5(s) = \sup_{x>0} \left(\int_{[x,\infty)} d\mu \right)^{-s} \left(\int_{[x,\infty)} \left(\int_{[x,\infty)} d\mu \right)^{p'\left(\frac{1}{q}+s\right)} d\lambda \right)^{\frac{1}{p'}} < \infty. \tag{1.9}$$

Here $d\lambda = \left(\frac{d\tilde{\nu}}{dx}\right)^{1-p'} dx.$

Moreover, for the best constant C *in (1.5), we have* $C \approx MK_i(s)$, $i = 2, 3, 4, 5,$ *and each* $s > 0$.

By applying Theorem 1.1 with measures μ and λ taken to be purely atomic measures supported on the positive integers, the result for sequences was also stated in [9] (see also [8]).

For a special case, if we let the measures μ and λ be defined by

$$d\mu(t) = \chi_{(a,b)}(t)\, f(t)dt \quad \text{and} \quad d\lambda(t) = \chi_{(a,b)}(t)\, g(t)dt,$$

respectively, where $-\infty \leq a < b \leq \infty$, and f, g are measurable functions positive a.e. in (a, b), then for α, β and s positive numbers Theorem 1.1 reduces to the recent result concerning equivalences between some integral conditions related to Hardy's inequality by A. Gogatishvili, A. Kufner, L.-E. Persson and A. Wedestig in [3, Theorem 1].

Recently some scales of equivalent weight characterizations of the Hardy inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)\, dx\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f(x)^p v(x)\, dx\right)^{\frac{1}{p}} \tag{1.10}$$

for the case $0 < q < p < \infty$, $p > 1$ and $q \neq 1$ were proved by L.-E. Persson, V. D. Stepanov and P. Wall in [11]. They proved that the non-negative weights $u(x)$ and $v(x)$ for which (1.10) holds for all $f(x) \geq 0$ can be characterized by the Mazya-Rozin type conditions $\left(\mathbb{B}^{(1)}_{MR}(s) < \infty\right)$ or by the Persson-Stepanov type conditions $\left(\mathbb{B}^{(1)}_{PS}(s) < \infty\right)$, where, for some $s > 0$,

$$\mathbb{B}^{(1)}_{MR}(s) := \left(\int_0^\infty \left[\int_t^\infty uV^{q\left(1/p'-s\right)}\right]^{r/p} V(t)^{q\left(1/p'-s\right)+rs} u(t)\, dt\right)^{1/r} \tag{1.11}$$

and

$$\mathbb{B}^{(1)}_{PS}(s) := \left(\int_0^\infty \left[\int_0^t uV^{q\left(1/p'+s\right)}\right]^{r/p} V(t)^{q\left(1/p'+s\right)-rs} u(t)\, dt\right)^{1/r} \tag{1.12}$$

respectively. Here $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and $V(x) = \int_0^x v(t)^{1-p'} dt$. To be precise their result reads:

Theorem 1.2. *Let* $0 < q < p < \infty$, $1 < p < \infty$, $q \neq 1$, *and suppose that*

$$0 < \int_x^\infty u(t)\, dt < \infty \text{ and } 0 < V(x) < \infty \text{ for all } x > 0.$$

Then the Hardy inequality (1.10) holds for some finite constant $C \geq 0$ *if and only if any of the constants* $\mathbb{B}^{(1)}_{MR}(s)$ *or* $\mathbb{B}^{(1)}_{PS}(s)$ *is finite for some* $s > 0$. *Moreover, for the best constant* C *in (1.10) we have*

$$C \approx \mathbb{B}^{(1)}_{MR}(s) \approx \mathbb{B}^{(1)}_{PS}(s).$$

Under the conditions of the theorem it is known (Remark on page 93 in [15], see also [11]), that for $s>0$ the Mazya-Rozin constant has equivalent form

$$\mathcal{B}_{MR}^{(1)}(s) := \left(\int_0^\infty \left[\int_t^\infty uV(t)^{q\left(1/p'-s\right)} \right]^{r/q} V^{rs-1} dV(t) \right)^{1/r} \tag{1.13}$$

and, similarly an equivalent form to the Persson-Stepanov constant is

$$\mathcal{B}_{PS}^{(1)}(s) := \left(\int_0^\infty \left[\int_0^t uV(t)^{q\left(1/p'+s\right)} \right]^{r/q} V^{-rs-1} dV(t) \right)^{1/r}. \tag{1.14}$$

Our main result will generalize Theorem 1.2 to the case with general measures. However, our proofs are substantially different. It is important to point out that the inequality (1.1) includes both the weighted integral inequalities of Hardy type and also the corresponding results for sequences.

The paper is organized as follows: In Section 2 the main results and some lemmas are stated, while their proofs can be found in Section 3.

Arithmetic on $[0,\infty]$: By convention $0(\infty)=0^0=0/0=0$. Consequently, the power rule $x^{a+b}=x^a x^b$ can fail for some values of a and b if $x=0$ or $x=\infty$. Special attention must be paid to ensure that difficulties do not arise.

Throughout this paper $A \lesssim B$, $(B \gtrsim A)$, means that $A \le cB$, where $c>0$ is a constant or depends only on inessential parameters. If $B \lesssim A \lesssim B$, then we write $A \approx B$.

2. MAIN RESULTS AND SOME LEMMAS

First we state the following three technical lemmas:

Lemma 2.1. *Let $p \in (1,\infty)$. If $ap+b=cp+d$ and $b+1>0$, then there exists a finite constant C such that the inequality*

$$\int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^a d\mu \right)^p \Lambda(x)^b d\lambda(x) \le C \int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^c d\mu \right)^p \Lambda(x)^d d\lambda(x) \tag{2.1}$$

holds for all σ-finite Borel measures μ and λ such that $\Lambda(x)=\lambda(-\infty,x]<\infty$ for all $x \in \mathbf{R}$.

Lemma 2.1 may be compared with Theorems 8, 9, 10, and 12 in [1].

Lemma 2.2. *Let $p \in (1,\infty)$, suppose that $a>0$ and $b+1<0$, and set $c=a+b/p$. Then there exists a finite constant C such that the inequality*

$$\int_{\mathbf{R}} \left(\int_{(-\infty,x]} \Lambda^a d\mu \right)^p \Lambda(x)^b d\lambda(x) \le C \int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^c d\mu \right)^p d\lambda(x)$$

holds for all σ-finite Borel measures μ and λ such that $\Lambda(x)=\lambda(-\infty,x]<\infty$ for all $x \in \mathbf{R}$.

Definition 2.1. *Let $1<p<\infty$. We say $\lambda \in \mathcal{I}_p(\infty)$ provided*

$$\Lambda(x)^{1-p} - \Lambda(\infty)^{1-p} \le C \int_{[x,\infty)} \Lambda^{-p} d\lambda$$

for some finite constant C.

Note that if $1 < p < q$, then $\mathcal{I}_q(\infty) \subset \mathcal{I}_p(\infty)$. See the remark following Corollary 4.3 of [13].

Also note that all absolutely continuous measures and a great many others are in $\mathcal{I}_p(\infty)$ for all $p > 1$. Example 4.4 of [13] shows that not all measures are.

Lemma 2.3. *Let $p \in (1, \infty)$, suppose that $a > 0$ and $b+1 < 0$, and set $c = a+b/p$. Then there exists a finite constant C such that the inequality*

$$\int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^c d\mu \right)^p d\lambda(x) \le C \int_{\mathbf{R}} \left(\int_{(-\infty,x]} \Lambda^a d\mu \right)^p \Lambda(x)^b \, d\lambda(x)$$

holds for all σ-finite Borel measures μ and λ such that $\Lambda(x) = \lambda(-\infty, x] < \infty$ for all $x \in \mathbf{R}$, $\lambda \in \mathcal{I}_{1+a-c}(\infty)$, and $\Lambda(\infty) = \infty$.

Let $1 < p < \infty$, $0 < q < p$, and $1/r = 1/q - 1/p$. Suppose that σ, ν and μ are σ-finite measures on the Borel subsets of $\mathbf{R}$. Consider the *three-measure Hardy inequality*

$$\left(\int_{\mathbf{R}} \left| \int_{(-\infty,x]} f d\sigma \right|^q d\mu(x) \right)^{1/q} \le C \left(\int_{\mathbf{R}} |f|^p \, d\nu \right)^{1/p}, \tag{2.2}$$

for all measurable functions f.

Before we formulate the main results we state and motivate a result (Theorem 2.1) showing that this problem can be reduced to the Hardy inequality for two measures studied in [13, Section 3].

In [9] we considered (2.2) in the case that σ is the Lebesgue measure on the interval $(0, \infty)$, in accordance with Muckenhoupt's 1972 paper. His argument there reduces the study of (2.2) to the case that v is absolutely continuous with respect to the Lebesgue measure. The same basic measure theory argument will serve to reduce (2.2) to the case that v is absolutely continuous with respect to σ. See also [12]. We present a variant of this argument that reduces (2.2) to (1.1), ensures that the resulting λ is σ-finite, and makes it clear which absolute continuity of measures is necessary for the validity of (2.2).

Theorem 2.1. *Let $1 < p < \infty$ and $0 < q < \infty$. Let $E = \{x \in \mathbf{R} : \mu[x, \infty) > 0\}$ and define the measure σ_E by $\sigma_E(F) = \sigma(E \cap F)$. A necessary condition for (2.2) is that the measure σ_E is absolutely continuous with repect to ν (i.e. $\sigma_E \ll \nu$). Under this condition, (2.2) holds if and only if (1.1) holds for all non-negative, measurable functions f. Here the measure λ is defined by*

$$d\lambda = \left(\frac{d\sigma_E}{d\nu} \right)^{p'-1} d\sigma_E. \tag{2.3}$$

Remark 2.1. *As a consequence of this reduction theorem it will be sufficient to restrict our attention to the inequality (1.1) henceforth. We leave it to the reader to adapt our main results, given in Theorems 2.2 and 2.3 below, to give scales of equivalent necessary and sufficient conditions for the inequality (2.2) to hold. We also remark that Theorem 2.1 may be used to adapt the results of* [9]*, giving a large number equivalent conditions for (2.2) in the case $1 < p \le q < \infty$.*

The main results read:

Theorem 2.2. *Let $0 < q < p$, $1 < p < \infty$ and $1/r = 1/q - 1/p$. Suppose $\Lambda(x) = \lambda(-\infty, x] < \infty$ for all $x \in \mathbf{R}$. If $b + 1 > 0$, then the inequality (1.1) holds if and only if*

$$A(b) := \left(\int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^{q-1-bq/r} d\mu \right)^{r/q} \Lambda(x)^b \, d\lambda(x) \right)^{1/r} < \infty. \tag{2.4}$$

Theorem 2.3. *Let $0 < q < p$, $1 < p < \infty$ and $1/r = 1/q - 1/p$. Suppose $\Lambda(x) = \lambda(-\infty, x] < \infty$ for all $x \in \mathbf{R}$, $\lambda \in \mathcal{I}_{1+q/r}(\infty)$ and $\Lambda(\infty) = \infty$. If $b + 1 < 0$, then the inequality (1.1) holds if and only if*

$$A^*(b) := \left(\int_{\mathbf{R}} \left(\int_{(-\infty,x]} \Lambda^{q-1-bq/r} d\mu \right)^{r/q} \Lambda(x)^b \, d\lambda(x) \right)^{1/r} < \infty. \tag{2.5}$$

Remark 2.2. *The expression (1.13) is a special case of expression (2.4) with $b = rs - 1$, $d\mu(x) = u(x)\,dx$ and $d\lambda(x) = v(x)\,dx$. Likewise, the expression (1.14) is a special case of expression (2.5) with $b = -rs - 1$, $d\mu(x) = u(x)\,dx$ and $d\lambda(x) = v(x)\,dx$. Due to the equivalent relationship between (1.11) and (1.13) and similarly, between (1.12) and (1.14), Theorems 2.2 and 2.3 give a generalization of Theorem 1.2.*

Finally we state the following useful proposition, which is of independent interest but also used for our proofs.

Proposition 2.1. *Let $x \in \mathbf{R}$. Then for $p > 0$*

$$\min(1, 1/p)\,\Lambda(x)^p \le \int_{(-\infty,x]} \Lambda^{p-1} d\lambda \le \max(1, 1/p)\,\Lambda(x)^p, \tag{2.6}$$

$$\min(1, 1/p)\,M(x)^p \le \int_{[x,\infty)} M^{p-1} d\mu \le \max(1, 1/p)\,M(x)^p \tag{2.7}$$

and for $p < 0$

$$\int_{(x,\infty)} \Lambda^{p-1} d\lambda \le |1/p| \left(\Lambda(x)^p - \Lambda(\infty)^p \right), \tag{2.8}$$

$$\int_{[x,\infty)} \Lambda^{p-1} d\lambda \le \Lambda(x)^p + |1/p| \left(\Lambda(x)^p - \Lambda(\infty)^p \right), \tag{2.9}$$

$$\int_{(-\infty,x)} M^{p-1} d\mu \le |1/p| \left(M(x)^p - M(-\infty)^p \right), \tag{2.10}$$

and

$$\int_{(-\infty,x]} M^{p-1} d\mu \le M(x)^p + |1/p| \left(M(x)^p - M(\infty)^p \right). \tag{2.11}$$

Proof: The detailed proofs of (2.6), (2.7), and (2.8)-(2.11) can be found in Lemma 1, Corollary 1 and Lemma 3 of [9], respectively.

3. Proofs

Proof of Lemma 2.1:

The λ-measure of $\{x \in \mathbf{R} : \mathbf{\Lambda}(x) = 0\}$ is zero, $0 < \mathbf{\Lambda}(x) < \infty$ for λ-almost every x. Therefore, for λ-almost every x and all $t \in [x, \infty)$, $\Lambda(t)^a = \Lambda(t)^c \Lambda(t)^{a-c}$. Since Λ is non-decreasing, in the case $a \le c$ we have

$$\begin{aligned}
\int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^a d\mu \right)^p \Lambda(x)^b d\lambda(x) &= \int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^c \Lambda^{a-c} d\mu \right)^p \Lambda(x)^b d\lambda(x) \\
&\le \int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^c d\mu \right)^p \Lambda(x)^{p(a-c)} \Lambda(x)^b d\lambda(x) \\
&= \int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^c d\mu \right)^p \Lambda(x)^d d\lambda(x).
\end{aligned}$$

Now suppose that $a > c$. Set

$$G(x) = \int_{[x,\infty)} \Lambda^c d\mu, \tag{3.1}$$

apply (2.6) of Proposition 2.1 and interchange the order of integration to get

$$\begin{aligned}
\int_{[x,\infty)} \Lambda^a d\mu &\approx \int_{[x,\infty)} \int_{(-\infty,t]} \Lambda^{a-c-1} d\lambda \Lambda(t)^c d\mu(t) \\
&\le \int_{[x,\infty)} \left(\int_{(-\infty,x]} \Lambda^{a-c-1} d\lambda + \int_{[x,t]} \Lambda^{a-c-1} d\lambda \right) \Lambda(t)^c d\mu(t) \\
&\approx G(x)\Lambda(x)^{a-c} + \int_{[x,\infty)} \int_{[x,t]} \Lambda^{a-c-1} d\lambda \Lambda(t)^c d\mu(t) \\
&= G(x)\Lambda(x)^{a-c} + \int_{[x,\infty)} \int_{[y,\infty)} \Lambda^c d\mu \Lambda(y)^{a-c-1} d\lambda(y) \\
&= G(x)\Lambda(x)^{a-c} + \int_{[x,\infty)} G\Lambda^{a-c-1} d\lambda.
\end{aligned}$$

Thus, by Minkowski's inequality, we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^a d\mu \right)^p \Lambda(x)^b d\lambda(x) \right)^{1/p} \\
\lesssim\ &\left(\int_{\mathbf{R}} \left(G(x)\Lambda(x)^{a-c} + \int_{[x,\infty)} G\Lambda^{a-c-1} d\lambda \right)^p \Lambda(x)^b d\lambda(x) \right)^{1/p} \\
\le\ &\left(\int_{\mathbf{R}} \left(G(x)\Lambda(x)^{a-c} \right)^p \Lambda(x)^b d\lambda(x) \right)^{1/p} + \\
&\left(\int_{\mathbf{R}} \left(\int_{[x,\infty)} G\Lambda^{a-c-1} d\lambda \right)^p \Lambda(x)^b d\lambda(x) \right)^{1/p}.
\end{aligned}$$

To prove (2.1), it is enough to prove

$$\int_{\mathbf{R}} \left(G(x)\Lambda(x)^{a-c} \right)^p \Lambda(x)^b d\lambda(x) \lesssim \int_{\mathbf{R}} G^p \Lambda^d d\lambda \tag{3.2}$$

and

$$\int_{\mathbf{R}}\left(\int_{[x,\infty)} G\Lambda^{a-c-1}d\lambda\right)^{p}\Lambda\left(x\right)^{b}d\lambda\left(x\right)\lesssim\int_{\mathbf{R}}G^{p}\Lambda^{d}d\lambda. \tag{3.3}$$

The first inequality, (3.2), is trivially valid so we focus on the second one, (3.3). Set $\alpha=\left(b+p\right)/\left(1-p\right)$ and observe that $\alpha+1<0$. Thus

$$\int_{[x,\infty)}\Lambda^{\alpha}d\lambda\lesssim\Lambda(x)^{\alpha+1}.$$

Applying the One Hardy Inequality from [13] we get

$$\begin{aligned}
&\int_{\mathbf{R}}\left(\int_{[x,\infty)} G\Lambda^{a-c-1}d\lambda\right)^{p}\Lambda\left(x\right)^{b}d\lambda\left(x\right)\\
\lesssim\ &\int_{\mathbf{R}}\left(\frac{\int_{[x,\infty)}\left(G\Lambda^{a-c-1-\alpha}\right)\Lambda^{\alpha}d\lambda}{\int_{[x,\infty)}\Lambda^{\alpha}d\lambda}\right)^{p}\Lambda\left(x\right)^{\alpha}d\lambda\left(x\right)\\
\lesssim\ &\int_{\mathbf{R}}\left(G\Lambda^{a-c-1-\alpha}\right)^{p}\Lambda^{\alpha}d\lambda\\
=\ &\int_{\mathbf{R}}G^{p}\Lambda^{d}d\lambda,
\end{aligned}$$

i.e. also (3.3) holds and the proof is complete.

Proof of Lemma 2.2:

First observe that $a-c=-b/p>0$ since $b<-1$. Because $a>0$ and Λ is never infinite, this yields

$$\int_{(-\infty,x]}\Lambda^{a}d\mu=\int_{(-\infty,x]}\Lambda^{a-c}\Lambda^{c}d\mu\approx\int_{(-\infty,x]}\int_{(-\infty,t]}\Lambda^{a-c-1}d\lambda\Lambda\left(t\right)^{c}d\mu\left(t\right).$$

Interchanging the order of integration and, with G defined by (3.1), we get

$$\begin{aligned}
\int_{(-\infty,x]}\int_{(-\infty,t]}\Lambda^{a-c-1}d\lambda\Lambda\left(t\right)^{c}d\mu\left(t\right) &= \int_{(-\infty,x]}\int_{[y,x]}\Lambda^{c}d\mu\Lambda\left(y\right)^{a-c-1}d\lambda\left(y\right)\\
&\leq \int_{(-\infty,x]}\int_{[y,\infty)}\Lambda^{c}d\mu\Lambda\left(y\right)^{a-c-1}d\lambda\left(y\right)\\
&= \int_{(-\infty,x]}G\Lambda^{a-c-1}d\lambda.
\end{aligned}$$

Let $\beta=\left(b+p\right)/\left(1-p\right)$ and observe that $\beta>-1$. It follows that

$$\int_{(-\infty,x]}\Lambda^{\beta}d\lambda\approx\Lambda^{\beta+1}.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbf{R}}\left(\int_{(-\infty,x]}\Lambda^{a}d\mu\right)^{p}\Lambda\left(x\right)^{b}d\lambda\left(x\right)\\
\lesssim\ &\int_{\mathbf{R}}\left(\int_{(-\infty,x]}G\Lambda^{a-c-1}d\lambda\right)^{p}\Lambda\left(x\right)^{b}d\lambda\left(x\right)\\
\lesssim\ &\int_{\mathbf{R}}\left(\frac{\int_{(-\infty,x]}\left(G\Lambda^{a-c-1-\beta}\right)\Lambda^{\beta}d\lambda}{\int_{(-\infty,x]}\Lambda^{\beta}d\lambda}\right)^{p}\Lambda\left(x\right)^{\beta}d\lambda\left(x\right).
\end{aligned}$$

Finally, the One Hardy Inequality in [13] bounds the last integral from above by a multiple of

$$\int_{\mathbf{R}}\left(G\Lambda^{a-c-1-\beta}\right)^{p}\Lambda^{\beta}d\lambda=\int_{\mathbf{R}}G^{p}d\lambda$$

and the proof follows.

Proof of Lemma 2.3:

First observe that $c-a=b/p<0$. For λ-almost every x, $\Lambda\left(x\right)>0$ and thus $0<\Lambda\left(t\right)<\infty$ for all $t>x$. Therefore, the hypotheses $\lambda\in\mathcal{I}_{1+a-c}\left(\infty\right)$ and $\Lambda\left(\infty\right)=\infty$ yield

$$\int_{[x,\infty)}\Lambda^{c}d\mu=\int_{[x,\infty)}\Lambda^{c-a}\Lambda^{a}d\mu\lesssim\int_{[x,\infty)}\int_{[t,\infty)}\Lambda^{c-a-1}d\lambda\Lambda\left(t\right)^{a}d\mu\left(t\right).$$

Interchanging the order of integration shows that

$$\begin{aligned}
\int_{[x,\infty)}\int_{[t,\infty)}\Lambda^{c-a-1}d\lambda\Lambda\left(t\right)^{a}d\mu\left(t\right) &= \int_{[x,\infty)}\int_{[x,y]}\Lambda^{a}d\mu\Lambda\left(y\right)^{c-a-1}d\lambda\left(y\right)\\
&\le \int_{[x,\infty)}\int_{(-\infty,y]}\Lambda^{a}d\mu\Lambda\left(y\right)^{c-a-1}d\lambda\left(y\right)\\
&= \int_{[x,\infty)}H\Lambda^{c-a-1}d\lambda,
\end{aligned}$$

where $H(y)=\int_{(-\infty,y]}\Lambda^{a}d\mu$.

Since $p'>1$ and $\Lambda(\infty)=\infty$, (2.9) yields

$$\int_{[x,\infty)}\Lambda^{-p'}d\lambda\lesssim\Lambda(x)^{1-p'}.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbf{R}}\left(\int_{[x,\infty)}\Lambda^{c}d\mu\right)^{p}d\lambda\left(x\right)\\
\lesssim\ &\int_{\mathbf{R}}\left(\int_{[x,\infty)}H\Lambda^{c-a-1}d\lambda\right)^{p}d\lambda\left(x\right)\\
\lesssim\ &\int_{\mathbf{R}}\left(\frac{\int_{[x,\infty)}\left(H\Lambda^{c-a-1+p'}\right)\Lambda^{-p'}d\lambda}{\int_{[x,\infty)}\Lambda^{-p'}d\lambda}\right)^{p}\Lambda\left(x\right)^{-p'}d\lambda\left(x\right).
\end{aligned}$$

The One Hardy inequality from [13] bounds the last integral from above by a multiple of

$$\int_{\mathbf{R}} \left(H\Lambda^{a-c-1+p'} \right)^p \Lambda^{-p'} d\lambda = \int_{\mathbf{R}} H^p \Lambda^b d\lambda$$

and the proof is complete.

Proof of Theorem 2.1:

Suppose that (2.2) holds for some constant C and let F be a measurable subset of $\mathbf{R}$ such that $\nu(F) = 0$. With $f = \chi_F$, the right hand side of (2.2) is zero and therefore so is the left hand side. It follows that

$$\sigma\left((-\infty, x] \cap F\right) = 0$$

for μ-almost every $x \in \mathbf{R}$. Let $y = \sup E \subseteq (-\infty, \infty]$. (We ignore the trivial case $y = -\infty$, which occurs only if μ is the zero measure.) If y is an atom for μ, then $E = (-\infty, y]$ so

$$\sigma\left(E \cap F\right) = \sigma\left((-\infty, y] \cap F\right) = 0.$$

If y is not an atom for μ then $E = (-\infty, y)$. In this case, let y_n be a strictly increasing sequence of real numbers that converge to y. For each integer n, the interval $[y_n, y)$ has positive μ-measure and must contain a point x such that $\sigma\left((-\infty, x] \cap F\right) = 0$. Thus $\sigma\left((-\infty, y_n] \cap F\right) = 0$, and so

$$\sigma\left(E \cap F\right) = \sigma\left(\bigcup_{n=1}^{\infty} (-\infty, y_n] \cap F\right) = 0.$$

This shows that $\sigma_E \ll \nu$.

Now suppose that $\sigma_E \ll \nu$ and let $h = d\sigma_E/d\nu$ be the Radon-Nikodym derivative of σ_E with respect to ν. If f is a non-negative measurable function, then, by (2.3),

$$\int_{(-\infty,x]} f d\lambda = \int_{(-\infty,x]} f h^{p'-1} d\sigma_E \le \int_{(-\infty,x]} f h^{p'-1} d\sigma$$

and

$$\int_{\mathbf{R}} \left(f h^{p'-1}\right)^p d\nu = \int_{\mathbf{R}} f^p h^{p'-1} h d\nu = \int_{\mathbf{R}} f^p h^{p'-1} d\sigma_E = \int_{\mathbf{R}} f^p d\lambda.$$

Therefore, if (2.2) holds, then we may apply it with f replaced by $fh^{p'-1}$ to deduce (1.1).

On the other hand, suppose that (1.1) holds and fix a measurable function f. By the Lebesgue decomposition theorem we can write $\nu = \nu_0 + \widetilde{\nu}$ so that ν_0 is singular with respect to σ_E and $\widetilde{\nu}$ is absolutely continuous with respect to σ_E. Setting $g = d\widetilde{\nu}/d\sigma_E$, the Radon-Nykodym derivative of $\widetilde{\nu}$ with respect to σ_E, we have

$$d\sigma_E = hd\nu = hd\nu_0 + hd\widetilde{\nu} = hd\nu_0 + hgd\sigma_E.$$

Therefore, $h = 0$ ν_0-almost everywhere and $hg = 1$ σ_E-almost everywhere. In particular, $0 < h < \infty$ σ_E-almost everywhere.

If $x \in E$, then $(-\infty, x] \subset E$ and, thus,

$$\left| \int_{(-\infty,x]} f d\sigma \right| \le \int_{(-\infty,x]} |f|\, d\sigma_E = \int_{(-\infty,x]} |f|\, h^{1-p'} d\lambda.$$

Moreover,

$$\int_{\mathbf{R}} \left(|f|\, h^{1-p'}\right)^p d\lambda = \int_{\mathbf{R}} |f|^p\, g d\sigma_E = \int_{\mathbf{R}} |f|^p\, d\widetilde{\nu} \le \int_{\mathbf{R}} |f|^p\, d\nu.$$

Since the complement of E has zero μ-measure, these estimates combined with (1.1) show that (2.2) holds. This completes the proof.

Proof of Theorem 2.2:
By Corollary 3.5 of [13], inequality (2.2) holds if and only if

$$A(0) := \left(\int_{\mathbf{R}} \left(\int_{[x,\infty)} \Lambda^{q-1} d\mu \right)^{r/q} d\lambda(x) \right)^{1/r} < \infty.$$

Therefore it is enough to show that $A(b_1) \lesssim A(b_2)$ for any b_1 and b_2 greater than -1. This follows from Lemma 2.1 with p replaced by r/q and the proof is complete.

Proof of Theorem 2.3:
It is enough to show that for any $b < -1$, $A^*(b) \lesssim A(0)$ and $A(0) \lesssim A^*(b)$. Since $q - 1 - bq/r > q - 1 + q/r = q/p' > 0$ the first estimate follows from Lemma 2.2 with p replaced by r/q. To see that the second follows from Lemma 2.3 we observe that $\lambda \in \mathcal{I}_{1+q/r}(\infty)$ and $1 - bq/r > 1 + q/r$ implies $\lambda \in \mathcal{I}_{1-bq/r}(\infty)$. The proof is complete.

Remark 3.1. *Comparing the proof of* [11, Theorem 1.2] *and that of Theorems 2.2 and 2.3, we observe that the more general situation in fact leads to simpler proofs.*

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