PRODUCT OPERATORS ON MIXED NORM SPACES

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Abstract. Inequalities for product operators on mixed norm Lebesgue spaces and permuted mixed norm Lebesgue spaces are established. They depend only on inequalities for the factors and on the Lebesgue indices involved. Inequalities for the bivariate Laplace transform are given to illustrate the method. Also, an elementary proof is presented for an n-variable Young's inequality in mixed norm spaces.

1. Introduction

The techniques used to study embeddings of mixed norm spaces in [4] and [6] can be extended to work with operators other than the identity. Here we begin this work by considering a select class of operators. Mixed norm spaces are spaces of multivariable functions in which the norm takes advantage of the product structure in the domain. They have a long informal history but were first named and formally studied by Benedek and Panzone in [2]. Permuted mixed norms only arise when studying more than a single space, since they occur when the order in which the factor norms are taken is different in different spaces. The importance of permuted mixed norms was noted in [3] but no systematic study was undertaken until the first author's thesis, [4].

It will be convenient to introduce the two-variable mixed norm spaces needed in Section 2 first, postponing the introduction of n-variable spaces until Section 3. Let λ_1 and λ_2 be σ-finite measures and let $L^0_{\lambda_1\times\lambda_2}$ denote the collection of $(\lambda_1\times\lambda_2)$-measurable functions. Fix indices $p_1, p_2 \in (0,\infty)$.

For any $f \in L^0_{\lambda_1\times\lambda_2}$,

$$\|f\|_{L^{(p_1,p_2)}_{\lambda_1\times\lambda_2}} = \left(\int\left(\int |f(t_1,t_2)|^{p_1}\,d\lambda_1(t_1)\right)^{p_2/p_1} d\lambda_2(t_2)\right)^{1/p_2}.$$

The first variable of the function f is always in the λ_1 measure space, and the order of the indices and measures indicates which variable is the "inner" one. So for any $f \in L^0_{\lambda_1\times\lambda_2}$,

$$\|f\|_{L^{(p_2,p_1)}_{\lambda_2\times\lambda_1}} = \left(\int\left(\int |f(t_1,t_2)|^{p_2}\,d\lambda_2(t_2)\right)^{p_1/p_2} d\lambda_1(t_1)\right)^{1/p_1}.$$

Although these are genuine norms only when $p_1 \geq 1$ and $p_2 \geq 1$ we will refer to them as mixed norms even when some indices are less than 1. Also, the results we present will often extend to the case when one or both of the indices is infinite,

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indicating that the supremum norm is to be taken in that factor. This extension is left to the reader.

The product operators that we consider will be introduced in Section 2. As mentioned above, we restrict our attention to the two-variable case. Although the extension from bivariate operators to multivariate operators may seem straightforward, the delicate arguments needed in the embedding case in [4] and [6] and the advanced techniques introduced in [5] put this extension beyond the scope of the present article.

Convolution operators have some product structure but not enough to make them product operators in general. Convolution is considered in Section 3, where it leads to an elementary proof of a multivariate mixed norm Young's inequality.

We will make frequent use of Minkowski's (integral) inequality in the following mixed norm form: If $0 < p_1 \le p_2 < \infty$, then,

$$\|f\|_{L^{(p_1,p_2)}_{\lambda_1\times\lambda_2}} \le \|f\|_{L^{(p_2,p_1)}_{\lambda_2\times\lambda_1}}, \quad f \in L^+_{\lambda_1\times\lambda_2}.$$

For easy recognition, we will enclose the function in square brackets when applying Minkowski's inequality in integral estimates. It becomes,

$$\left(\int\left(\int [f(t_1,t_2)]^{p_1}\,d\lambda_1(t_1)\right)^{p_2/p_1} d\lambda_2(t_2)\right)^{1/p_2} \le \left(\int\left(\int [f(t_1,t_2)]^{p_2}\,d\lambda_2(t_2)\right)^{p_1/p_2} d\lambda_1(t_1)\right)^{1/p_1}.$$

2. Product operators

Let λ_1, λ_2, μ_1, and μ_2 be σ-finite measures. (We will not need to specify their various underlying spaces.) Let $L^+_{\lambda_1\times\lambda_2}$ denote the collection of non-negative $(\lambda_1\times\lambda_2)$-measurable functions.

An operator $K : L^+_{\lambda_1\times\lambda_2} \to L^+_{\mu_1\times\mu_2}$ will be called a *product operator* provided it can be expressed in the form,

$$Kf(x_1,x_2) = \iint k_1(x_1,t_1)k_2(x_2,t_2)f(t_1,t_2)\,d(\lambda_1\times\lambda_2)(t_1,t_2),$$

where k_j is a non-negative $(\mu_j\times\lambda_j)$-measurable function for $j=1,2$. In this case we define $K_j : L^+_{\lambda_j} \to L^+_{\mu_j}$ by

$$K_jf(x) = \int k_j(x,t)f(t)\,d\lambda_j(t), \quad j=1,2.$$

Note that, by Tonelli's theorem,

$$\begin{aligned} Kf(x_1,x_2) &= \int k_1(x_1,t_1)\int k_2(x_2,t_2)f(t_1,t_2)\,d\lambda_2(t_2)\,d\lambda_1(t_1) \\ &= \int k_2(x_2,t_2)\int k_1(x_1,t_1)f(t_1,t_2)\,d\lambda_1(t_1)\,d\lambda_2(t_2). \end{aligned}$$

Suppose p_1, p_2, r_1, and r_2 are positive and let C_j be the least constant, finite or infinite, such that

$$\|K_jf\|_{L^{r_j}_{\mu_j}} \le C_j\|f\|_{L^{p_j}_{\lambda_j}}, \quad f \in L^+_{\lambda_j}.$$

Theorem 2.1. *Fix positive indices p_1, p_2, r_1, and r_2. Suppose K is a product operator, with k_j, K_j, C_j as above for $j = 1, 2$.*

(a) *If $r_1 \geq 1$ then K satisfies the mixed norm inequality,*
$$\|Kf\|_{L^{(r_1,r_2)}_{\mu_1\times\mu_2}} \leq C_1C_2\|f\|_{L^{(p_1,p_2)}_{\lambda_1\times\lambda_2}}, \quad f \in L^+_{\lambda_1\times\lambda_2}.$$

(b) *If $r_1 \geq 1$, $r_2 \geq 1$ and $\min(p_1, r_1) \leq \max(p_2, r_2)$ then K satisfies the permuted mixed norm inequality,*
$$\|Kf\|_{L^{(r_1,r_2)}_{\mu_1\times\mu_2}} \leq C_1C_2\|f\|_{L^{(p_2,p_1)}_{\lambda_2\times\lambda_1}}, \quad f \in L^+_{\lambda_1\times\lambda_2}.$$

Proof. In a slight abuse of notation we write,
$$K_1f(x_1, t_2) = \int k_1(x_1, t_1)f(t_1, t_2)\, d\lambda_1(t_1) \quad \text{and}$$
$$K_2f(t_1, x_2) = \int k_2(x_2, t_2)f(t_1, t_2)\, d\lambda_2(t_2).$$

To prove part (a), use the hypothesis $r_1 \geq 1$ to apply Minkowski's inequality and then invoke the definitions of C_1 and C_2. This yields,
$$\left(\int\left(\int Kf(x_1,x_2)^{r_1}\,d\mu_1(x_1)\right)^{r_2/r_1} d\mu_2(x_2)\right)^{1/r_2}$$
$$= \left(\int\left(\int\left(\int \Big[k_2(x_2,t_2)K_1f(x_1,t_2)\Big]\,d\lambda_2(t_2)\right)^{r_1} d\mu_1(x_1)\right)^{r_2/r_1} d\mu_2(x_2)\right)^{1/r_2}$$
$$\leq \left(\int\left(\int\left(\int \Big[k_2(x_2,t_2)K_1f(x_1,t_2)\Big]^{r_1} d\mu_1(x_1)\right)^{1/r_1} d\lambda_2(t_2)\right)^{r_2} d\mu_2(x_2)\right)^{1/r_2}$$
$$= \left(\int\left(\int k_2(x_2,t_2)\left(\int K_1f(x_1,t_2)^{r_1}\,d\mu_1(x_1)\right)^{1/r_1} d\lambda_2(t_2)\right)^{r_2} d\mu_2(x_2)\right)^{1/r_2}$$
$$\leq C_1\left(\int\left(\int k_2(x_2,t_2)\left(\int f(t_1,t_2)^{p_1}\,d\lambda_1(t_1)\right)^{1/p_1} d\lambda_2(t_2)\right)^{r_2} d\mu_2(x_2)\right)^{1/r_2}$$
$$\leq C_1C_2\left(\int\left(\int f(t_1,t_2)^{p_1}\,d\lambda_1(t_1)\right)^{p_2/p_1} d\lambda_2(t_2)\right)^{1/p_2}.$$

Part (b) will be done in four cases: They arise from the observation that the condition $\min(p_1, r_1) \leq \max(p_2, r_2)$ is satisfied if and only if one of, $p_1 \leq p_2$, $r_1 \leq r_2$, $p_1 \leq r_2$, or $r_1 \leq p_2$ holds. If $p_1 \leq p_2$ then part (a), followed by Minkowski's inequality, yields
$$\|Kf\|_{L^{(r_1,r_2)}_{\mu_1\times\mu_2}} \leq C_1C_2\|f\|_{L^{(p_1,p_2)}_{\lambda_1\times\lambda_2}} \leq C_1C_2\|f\|_{L^{(p_2,p_1)}_{\lambda_2\times\lambda_1}}, \quad f \in L^+_{\lambda_1\times\lambda_2}.$$

If $r_1 \leq r_2$ then we begin by using Minkowski's inequality, and follow with part (a) to get
$$\|Kf\|_{L^{(r_1,r_2)}_{\mu_1\times\mu_2}} \leq \|Kf\|_{L^{(r_2,r_1)}_{\mu_2\times\mu_1}} \leq C_2C_1\|f\|_{L^{(p_2,p_1)}_{\lambda_2\times\lambda_1}}, \quad f \in L^+_{\lambda_1\times\lambda_2}.$$

If $p_1 \le r_2$ then we apply the definition of C_1 followed by Minkowski's inequality and then the definition of C_2 to get

$$
\begin{aligned}
&\left(\int\left(\int Kf(x_1,x_2)^{r_1}\,d\mu_1(x_1)\right)^{r_2/r_1} d\mu_2(x_2)\right)^{1/r_2}\\
&=\left(\int\left(\int\left(\int k_1(x_1,t_1)K_2f(t_1,x_2)\,d\lambda_1(t_1)\right)^{r_1} d\mu_1(x_1)\right)^{r_2/r_1} d\mu_2(x_2)\right)^{1/r_2}\\
&\le C_1\left(\int\left(\int\Big[K_2f(t_1,x_2)\Big]^{p_1} d\lambda_1(t_1)\right)^{r_2/p_1} d\mu_2(x_2)\right)^{1/r_2}\\
&\le C_1\left(\int\left(\int\Big[K_2f(t_1,x_2)\Big]^{r_2} d\mu_2(x_2)\right)^{p_1/r_2} d\lambda_1(t_1)\right)^{1/p_1}\\
&\le C_1C_2\left(\int\left(\int f(t_1,t_2)^{p_2}\,d\lambda_2(t_2)\right)^{p_1/p_2} d\lambda_1(t_1)\right)^{1/p_1}.
\end{aligned}
$$

If $r_1 \le p_2$ the process is somewhat lengthy, using the definitions of C_1 and C_2 as well as three applications of Minkowski's inequality. First, apply Minkowski's inequality with $r_1 \ge 1$, followed by the definition of C_2 to get

$$
\begin{aligned}
&\left(\int\left(\int Kf(x_1,x_2)^{r_1}\,d\mu_1(x_1)\right)^{r_2/r_1} d\mu_2(x_2)\right)^{1/r_2}\\
&=\left(\int\left(\int\left(\int\Big[k_2(x_2,t_2)K_1f(x_1,t_2)\Big]\,d\lambda_2(t_2)\right)^{r_1} d\mu_1(x_1)\right)^{r_2/r_1} d\mu_2(x_2)\right)^{1/r_2}\\
&\le\left(\int\left(\int\left(\int\Big[k_2(x_2,t_2)K_1f(x_1,t_2)\Big]^{r_1} d\mu_1(x_1)\right)^{1/r_1} d\lambda_2(t_2)\right)^{r_2} d\mu_2(x_2)\right)^{1/r_2}\\
&=\left(\int\left(\int k_2(x_2,t_2)\left(\int K_1f(x_1,t_2)^{r_1}\,d\mu_1(x_1)\right)^{1/r_1} d\lambda_2(t_2)\right)^{r_2} d\mu_2(x_2)\right)^{1/r_2}\\
&\le C_2\left(\int\left(\int\Big[K_1f(x_1,t_2)\Big]^{r_1} d\mu_1(x_1)\right)^{p_2/r_1} d\lambda_2(t_2)\right)^{1/p_2}.
\end{aligned}
$$

Now Minkowski's inequality with $p_2 \ge r_1$ shows that the last expression is no larger than

$$
C_2\left(\int\left(\int\Big[K_1f(x_1,t_2)\Big]^{p_2} d\lambda_2(t_2)\right)^{r_1/p_2} d\mu_1(x_1)\right)^{1/r_1}.
$$

After expanding $K_1 f(x_1, t_2)$ in this expression we may apply Minkowski's inequality with $p_2 \geq 1$ to get

$$C_2 \left(\int \left(\int \left(\int \left[k_1(x_1, t_1) f(t_1, t_2) \right] d\lambda_1(t_1) \right)^{p_2} d\lambda_2(t_2) \right)^{r_1/p_2} d\mu_1(x_1) \right)^{1/r_1}$$

$$\leq C_2 \left(\int \left(\int \left(\int \left[k_1(x_1, t_1) f(t_1, t_2) \right]^{p_2} d\lambda_2(t_2) \right)^{1/p_2} d\lambda_1(t_1) \right)^{r_1} d\mu_1(x_1) \right)^{1/r_1}$$

$$\leq C_2 \left(\int \left(\int k_1(x_1, t_1) \left(\int f(t_1, t_2)^{p_2} \, d\lambda_2(t_2) \right)^{1/p_2} d\lambda_1(t_1) \right)^{r_1} d\mu_1(x_1) \right)^{1/r_1}$$

$$\leq C_1 C_2 \left(\int \left(\int f(t_1, t_2)^{p_2} \, d\lambda_2(t_2) \right)^{p_1/p_2} d\lambda_1(t_1) \right)^{1/p_1},$$

where the last inequality uses the definition of C_1. These estimates complete the proof of the fourth and last case. □

The index conditions in Theorem 2.1(b) are somewhat stronger than needed. It is enough to assume that one (or more) of the following four conditions holds: $p_1 \leq r_2$; $1 \leq r_1 \leq p_2$; $1 \leq r_1$ and $p_1 \leq p_2$; or $1 \leq r_2$ and $r_1 \leq r_2$. The statement of the theorem remains valid when some indices are infinite, provided the index conditions are met, but the proofs would have to be modified to accommodate occurrences of the supremum norm.

The inequalities of Theorem 2.1 are stated for non-negative functions only but it is routine to extend the operator K to all functions for which the right hand side is finite, in a way that preserves the norm inequalities.

Also, the results of Theorem 2.1 may be seen to hold for a more general class of positive operators. Instead of supposing that the factors K_1 and K_2 are integral operators with non-negative kernels, it would suffice to assume that they map positive functions to positive functions and possess formal adjoints. See [8, Lemma 2.4] for properties of such operators and [7, Section 4] for connections with positive integral operators. One advantage of such an extension is that the identity operator is not an integral operator (in general) but it does have a formal adjoint.

Corollary 2.2. *Suppose $p_1, p_2 \in (1, 2]$. The bivariate Laplace transform $\mathscr{L}_2$, defined by*

$$\mathscr{L}_2 f(x_1, x_2) = \int_0^\infty \int_0^\infty e^{-x_1 t_1 - x_2 t_2} f(t_1, t_2) \, dt_1 \, dt_2, \quad x_1 > 0, \, x_2 > 0,$$

satisfies the permuted mixed norm inequality

$$\|\mathscr{L}_2 f\|_{L^{(p_1', p_2')}} \leq C(p_1) C(p_2) \|f\|_{L^{(p_2, p_1)}}.$$

Here $1/p_1 + 1/p_1' = 1/p_2 + 1/p_2' = 1$, and $C(p)$ is the norm of the single-variable Laplace transform as a map from L^p to $L^{p'}$.

Proof. The bivariate Laplace transform is a product operator, in the above sense, whose factors are both the single-variable Laplace transform, $\mathscr{L}$, given by

$$\mathscr{L} f(x) = \int_0^\infty e^{-xt} f(t) \, dt, \quad x > 0.$$

It is well known that $\mathscr{L}$ is bounded as a map from L^p to $L^{p'}$, i.e. $C(p)$ is finite, when $1 < p \le 2$. Since $p_1' \ge 1$, $p_2' \ge 1$, and $\min(p_1, p_1') = p_1 \le 2 \le p_2' = \max(p_2, p_2')$ the second statement of Theorem 2.1 gives the conclusion. $\square$

Note that if $1 < p_2 < p_1 \le 2$, the permuted mixed norm inequality given above is stronger than either of the unpermuted ones given by Theorem 2.1(a). By Minkowski's inequality we have,

$$\|\mathscr{L}_2 f\|_{L^{(p_2',p_1')}} \le \|\mathscr{L}_2 f\|_{L^{(p_1',p_2')}} \le C(p_1)C(p_2)\|f\|_{L^{(p_2,p_1)}} \le C(p_1)C(p_2)\|f\|_{L^{(p_1,p_2)}}.$$

Using Theorem 2.1 it is easy to generate additional permuted mixed norm inequalities, by beginning with known single-variable inequalities. See, for example, [1, Corollary 1] for more single-variable inequalities involving the Laplace transform.

3. Young's inequality

Fix a positive integer n and let L^+ be the collection of non-negative Lebesgue measurable functions on $\mathbb{R}^n$. For $P = (p_1, \dots, p_n) \in [1, \infty)^n$ define

$$\|f\|_{L^P} = \left(\int \dots \left(\int \left(\int |f(t_1, \dots, t_n)|^{p_1} \, dt_1 \right)^{p_2/p_1} dt_2 \right)^{p_3/p_2} \dots dt_n \right)^{1/p_n}.$$

The convolution of two (real-valued) functions on $\mathbb{R}^n$ is defined by,

$$f * g(x) = \int_{\mathbb{R}^n} f(x-t) g(t) \, dt,$$

whenever the integral exists.

For a fixed function g the map $f \mapsto f * g$ has a kind of product structure. But, even in the two-variable case, it is not a product operator of the sort considered in the previous section unless g factors as $g(t_1, t_2) = g_1(t_1) g_2(t_2)$. This will keep us from establishing permuted mixed norm inequalities. Nevertheless, exploiting the existing product structure provides an elementary proof of a n-variable mixed norm Young's inequality.

Recall the single-variable Young's inequality: If $p, q, r \in [1, \infty]$ satisfy $1/p + 1/q = 1/r + 1$, $f \in L^p$ and $g \in L^q$, then $f * g$ is well defined and $\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}$.

Theorem 3.1. *Suppose* $P = (p_1, \dots, p_n)$, $Q = (q_1, \dots, q_n)$, *and* $R = (r_1, \dots r_n)$ *satisfy,*

$$\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r_j} + 1, \quad j = 1, \dots, n.$$

If $f \in L^P$ *and* $g \in L^Q$ *then* $f * g$ *is well defined, and*

$$\|f * g\|_{L^R} \le \|f\|_{L^P} \|g\|_{L^Q}.$$

Proof. We begin by proving the inequality when f and g are non-negative, so that $f * g$ is well-defined as a function taking values in $[0, \infty]$.

As the first step in a recursive argument, let $f_1 = f$ and $g_1 = g$. Define $\hat{x}$ and $\hat{t}$ so that $(x_1, x_2, \dots, x_n) = (x_1, \hat{x})$ and $(t_1, t_2, \dots, t_n) = (t_1, \hat{t})$. Also, let $d\hat{t}$ denote

$dt_2, \ldots dt_n$. Then Young's inequality gives, for each $\hat{x}$ and $\hat{t}$,

$$\left(\int \left(\int f_1(x_1 - t_1, \hat{x} - \hat{t}) g_1(t_1, \hat{t})\, dt_1 \right)^{r_1} dx_1 \right)^{1/r_1}$$
$$\leq \left(\int f_1(y_1, \hat{x} - \hat{t})^{p_1}\, dy_1 \right)^{1/p_1} \left(\int g_1(t_1, \hat{t})^{q_1}\, dt_1 \right)^{1/q_1}.$$

So, applying Minkowski's inequality, we get

$$\begin{aligned}
&\left(\int f_1 * g_1(x_1, \hat{x})^{r_1}\, dx_1 \right)^{1/r_1} \\
&= \left(\int \left(\int_{\mathbb{R}^{n-1}} \left[\int f_1(x_1 - t_1, \hat{x} - \hat{t}) g_1(t_1, \hat{t})\, dt_1 \right] d\hat{t} \right)^{r_1} dx_1 \right)^{1/r_1} \\
&\leq \int_{\mathbb{R}^{n-1}} \left(\int \left[\int f_1(x_1 - t_1, \hat{x} - \hat{t}) g_1(t_1, \hat{t})\, dt_1 \right]^{r_1} dx_1 \right)^{1/r_1} d\hat{t} \\
&\leq \int_{\mathbb{R}^{n-1}} \left(\int f_1(y_1, \hat{x} - \hat{t})^{p_1}\, dy_1 \right)^{1/p_1} \left(\int g_1(t_1, \hat{t})^{q_1}\, dt_1 \right)^{1/q_1} d\hat{t}.
\end{aligned}$$

Now define $f_2, g_2 : \mathbb{R}^{n-1} \to [0, \infty]$ by

$$f_2(y_2, \ldots, y_n) = \left(\int f_1(y_1, y_2, \ldots, y_n)^{p_1}\, dy_1 \right)^{1/p_1}$$

and

$$g_2(t_2, \ldots, t_n) = \left(\int g_1(t_1, t_2, \ldots, t_n)^{q_1}\, dt_1 \right)^{1/q_1}.$$

We have just shown that,

$$\left(\int f_1 * g_1(x_1, x_2, \ldots, x_n)^{r_1}\, dx_1 \right)^{1/r_1} \leq f_2 * g_2(x_2, \ldots, x_n). \tag{3.1}$$

Applying the above argument to f_2 and g_2 gives

$$\left(\int f_2 * g_2(x_2, x_3, \ldots, x_n)^{r_2}\, dx_2 \right)^{1/r_2} \leq f_3 * g_3(x_3, \ldots, x_n), \tag{3.2}$$

where

$$f_3(y_3, \ldots, y_n) = \left(\int f_2(y_2, y_3, \ldots, y_n)^{p_2}\, dy_2 \right)^{1/p_2}$$

and

$$g_3(t_3, \ldots, t_n) = \left(\int g_2(t_2, t_3, \ldots, t_n)^{q_2}\, dt_2 \right)^{1/q_2}.$$

We continue in this way until reaching

$$\left(\int f_{n-1} * g_{n-1}(x_{n-1}, x_n)^{r_{n-1}}\, dx_{n-1} \right)^{1/r_{n-1}} \leq f_n * g_n(x_n), \tag{3.3}$$

where

$$f_n(y_n) = \left(\int f_{n-1}(y_{n-1}, y_n)^{p_{n-1}}\, dy_{n-1} \right)^{1/p_{n-1}}$$

and

$$g_n(t_n) = \left(\int g_{n-1}(t_{n-1}, t_n)^{q_{n-1}}\, dt_{n-1} \right)^{1/q_{n-1}}.$$

Then Young's inequality gives,

$$\left(\int f_n * g_n(x_n)^{r_n}\, dx_n \right)^{1/r_n} \le \left(\int f_n(t_n)^{p_n}\, dt_n \right)^{1/p_n} \left(\int g_n(t_n)^{q_n}\, dt_n \right)^{1/q_n}. \tag{3.4}$$

Recursively applying the definitions of f_j and g_j shows that the right hand side of (3.4) is just

$$\|f\|_{L^P} \|g\|_{L^Q}.$$

The convolution estimates (3.1),(3.2),...,(3.3) concatenate to give a lower bound for the left hand side of (3.4) and we have

$$\|f * g\|_{L^R} \le \|f\|_{L^P} \|g\|_{L^Q}.$$

Now we drop the assumption of positivity on $f \in L^P$ and $g \in L^Q$. For bounded, integrable functions the convolution exists and is finite everywhere. And it is routine to express f and g as pointwise limits of bounded, integrable functions f_k and g_k, respectively, satisfying $|f_k| \le |f|$ and $|g_k| \le |g|$. Since $|f| \in L^P$ and $|g| \in L^Q$ we have shown $|f| * |g| \in L^R$ and hence $|f| * |g|(x) < \infty$ almost everywhere. So for almost every x the dominated convergence theorem proves that $f * g(x)$ exists. Also,

$$\|f * g\|_{L^R} \le \||f| * |g|\|_{L^R} \le \||f|\|_{L^P} \||g|\|_{L^Q} = \|f\|_{L^P} \|g\|_{L^Q}.$$

This completes the proof. $\square$

Once again, the statement of the theorem remains valid when some indices are infinite, but the proof would have to be modified to accommodate occurrences of the supremum norm.

The same straightforward procedure may be applied to prove a mixed norm Young's inequality over any finite product of locally compact unimodular groups.

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