Revisiting a sharpened version of Hadamard's determinant inequality

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Abstract

Hadamard's determinant inequality was refined and generalized by Zhang and Yang in [Acta Math. Appl. Sinica 20 (1997) 269-274]. Some special cases of the result were rediscovered recently by Rozanski, Witula and Hetmaniok in [Linear Algebra Appl. 532 (2017) 500-511]. We revisit the result in the case of positive semidefinite matrices, giving a new proof in terms of majorization and a complete description of the conditions for equality in the positive definite case. We also mention a block extension, which makes use of a result of Thompson in the 1960s.

Keywords: Hadamard's determinant inequality, positive semidefinite matrix, majorization.

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1. Introduction

Perhaps the best known determinantal inequality in mathematical sciences is the Hadamard inequality (e.g., [3, p. 505]) which says that if $A = (a_{ij})$ is an $n \times n$ (Hermitian) positive definite matrix, then

$$\det A \leq a_{11} \cdots a_{nn}, \tag{1}$$

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and equality holds if and only if A is diagonal.

Two decades ago, Zhang and Yang obtained an elegant sharpening of the Hadamard inequality. They proved it for a more general class of matrices and included a term involving off-diagonal entries of the matrix.

Let A be an $n \times n$ complex matrix. For a non-empty proper subset G of $\{1, \ldots, n\}$, let $A[G]$ denote the principal submatrix of A formed by discarding the ith row and column of A, for each $i \notin G$. We say A is an F-matrix if for all such G, $\det(A[G]) \geq 0$ and $\det(A) \leq \det(A[G]) \det(A[G^C])$ That is, all principal minors of A are non-negative and they satisfy a Fischer-type inequality. Standard results show that the Hermitian F-matrices are exactly the positive semi-definite matrices.

Theorem 1.1. *(Zhang-Yang [11]) If $A = (a_{ij})$ is an F-matrix, then $a_{ij}a_{ji} \geq 0$ for all i, j and, if σ is a non-trivial permutation of $\{1, \ldots, n\}$, then*

$$\det(A) + \Big(\prod_{i=1}^{n} a_{i,\sigma(i)} a_{\sigma(i),i}\Big)^{1/2} \leq \prod_{i=1}^{n} a_{ii}. \tag{2}$$

Without noticing the work of Zhang and Yang in [11], which was written in Chinese, Rozanski, Witula and Hetmaniok recently rediscovered some special cases of (2) in [7]. For this reason, we think it worthwhile to bring the nice result of Theorem 1.1 to the attention of the linear algebra community.

Besides Theorem 1.1, [11] also contains necessary and sufficient conditions for an F-matrix A to satisfy the equation $\det(A) = a_{11} \ldots a_{nn}$. The same was done for the equation permanent$(A) = a_{11} \ldots a_{nn}$. However, Zhang and Yang did not give conditions for equality to hold in (2). Conditions for equality were included by Rozanski, et al. for the special cases considered in [7].

There are multiple known ways to prove the Hadamard inequality (1) in the literature (see, e.g., [2, 3, 5]). We recall that one insightful way of seeing the Hadamard inequality is via Schur's majorization inequality (see Lemma 1.3). For a quick summary of the intimate connection between majorization and determinant inequalities, we refer to [4].

Our initial motivation was that since the original Hadamard inequality (1) is immediate from majorization (see, e.g. [1, p. 44], [9, p. 67]), it would be nice if Theorem 1.1 could also be seen from that perspective. In this note, we give a new proof of Theorem 1.1 for positive semi-definite matrices using majorization techniques. Our results include necessary and sufficient conditions for equality to hold when the matrix A is positive definite.

Before proceeding, let us fix some notation. For a vector $x \in \mathbb{R}^n$, we denote by $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow}) \in \mathbb{R}^n$ the vector with the same components as x, but sorted in nonincreasing order. Given $x, y \in \mathbb{R}^n$, we say that x majorizes y (or y is majorized by x), written as $x \succ y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \geq \sum_{i=1}^{k} y_i^{\downarrow} \quad \text{for } k = 1, \ldots, n-1$$

and equality holds at $k = n$.

Three basic facts about majorization are given below. The first is a matrix characterization, the next is Schur's majorization inequality, and the last is a consequence for the elementary symmetric functions. A *doubly stochastic matrix* is square matrix with non-negative entries and all row and column sums equal to 1.

Lemma 1.2. *[3, p. 253] If x and y are real row vectors then x majorizes y if and only if there exists a doubly stochastic matrix S such that $y = xS$.*

Lemma 1.3. *[3, p. 249] The eigenvalues of a Hermitian matrix majorize its diagonal entries.*

Fix a positive integer n and let $e_k(x)$, $k = 1, 2, \ldots, n$, denote the kth elementary symmetric function in the n variables $x_1, \ldots, x_n$. See [6, p.114]. By convention, $e_0(x) = 1$.

Lemma 1.4. *[6, p.115] Let $x, y \in [0, \infty)^n$. If $n \geq 2$ and $x \succ y$, then $e_k(x) \leq e_k(y)$ for $k = 0, 1, \ldots, n$. If $k > 1$ and $x, y \in (0, \infty)^n$ then equality holds if and only if $x^{\downarrow} = y^{\downarrow}$.*

2. A new proof of Theorem 1.1 and more

In this section, Λ, V and B will be as follows: Let Λ be a diagonal matrix with non-negative diagonal entries $\lambda_1, \ldots, \lambda_n$. Let $V = (v_{ij})$ be an $n \times n$ matrix whose rows and columns are all unit vectors, that is, all diagonal entries of V^*V and VV^* are equal to 1. Set $B = (b_{ij}) = V^*\Lambda V$. It is important to point out that, in general, $\prod_{i=1}^n \lambda_i \neq \det(B)$, although they are equal when V is unitary.

Lemma 2.1. *Let $P(t) = \prod_{i=1}^n (\lambda_i - t)$ and $Q(t) = \prod_{i=1}^n (b_{ii} - t)$. Fix $s < t \leq \min(\lambda_1, \ldots, \lambda_n)$. Then $P(t) \leq Q(t)$ and $P(s) - P(t) \leq Q(s) - Q(t)$. If $n \geq 3$ and $P(s) - P(t) = Q(s) - Q(t)$ then $(b_{11}, \ldots, b_{nn})$ is a permutation of $(\lambda_1, \ldots, \lambda_n)$.*

Proof. The conditions on V show that the matrix $S = (|v_{ij}|^2)$ is doubly stochastic and a calculation shows that $(b_{11} - t, \ldots, b_{nn} - t) = (\lambda_1 - t, \ldots, \lambda_n - t)S$. By Lemma 1.2 and Lemma 1.4,

$$e_k(\lambda_1 - t, \ldots, \lambda_n - t) \leq e_k(b_{11} - t, \ldots, b_{nn} - t)$$

for $k = 0, \ldots, n$. In particular,

$$P(t) = e_n(\lambda_1 - t, \ldots, \lambda_n - t) \leq e_n(b_{11} - t, \ldots, b_{nn} - t) = Q(t).$$

Also,

$$\begin{aligned}
P(s) - P(t) &= \prod_{i=1}^n (\lambda_i - t + t - s) - \prod_{i=1}^n (\lambda_i - t) \\
&= \sum_{k=0}^{n-1} e_k(\lambda_1 - t, \ldots, \lambda_n - t)(t - s)^{n-k} \\
&\leq \sum_{k=0}^{n-1} e_k(b_{11} - t, \ldots, b_{nn} - t)(t - s)^{n-k} \\
&= \prod_{i=1}^n (b_{ii} - t + t - s) - \prod_{i=1}^n (b_{ii} - t) = Q(s) - Q(t).
\end{aligned}$$

If $n \geq 3$ and $P(s) - P(t) = Q(s) - Q(t)$, then the above estimate reduces to equality throughout, which implies $e_2(\lambda_1 - t, \ldots, \lambda_n - t) = e_2(b_{11} - t, \ldots, b_{nn} - t)$. But e_2 is strictly Schur concave on all of $\mathbb{R}^n$. (See [6, A.4] to prove concavity and then [6, A.3.a] to prove strict concavity.) Thus $(b_{11}-t,\dots,b_{nn}-t)$ is a permutation of $(\lambda_1-t,\dots,\lambda_n-t)$ and therefore $(b_{11},\dots,b_{nn})$ is a permutation of $(\lambda_1,\dots,\lambda_n)$. □

Theorem 2.2. *If $\lambda_1,\dots,\lambda_n$ are non-negative, then*

$$\prod_{i=1}^{n}\lambda_i \le \prod_{i=1}^{n} b_{ii}. \tag{3}$$

If $n \ge 3$ and $\lambda_1,\dots,\lambda_n$ are strictly positive and distinct then equality holds if and only if V has exactly one non-zero entry in each row and column.

Proof. Lemma 2.1 shows that $P(0) \le Q(0)$, which is (3). Note that V has exactly one non-zero entry in each row and column if and only if $S = (|v_{ij}|^2)$ is a permutation matrix. In that case, since $(b_{11},\dots,b_{nn}) = (\lambda_1,\dots,\lambda_n)S$, (3) holds with equality.

Now suppose $\lambda_1,\dots,\lambda_n$ are strictly positive and distinct. If equality holds in (3), that is, if $e_n(\lambda_1,\dots,\lambda_n) = e_n(b_{11},\dots,b_{nn})$, then $b_{11},\dots,b_{nn}$ are also strictly positive. Therefore Lemma 1.4 shows $(b_{11},\dots,b_{nn})$ is a permutation of $(\lambda_1,\dots,\lambda_n)$. It follows that there are permutation matrices R and R' such that $(\lambda_1^{\downarrow},\dots,\lambda_n^{\downarrow}) = (\lambda_1^{\downarrow},\dots,\lambda_n^{\downarrow})R'SR$. Clearly, $R'SR = (t_{ij})$ is also a doubly stochastic matrix.

Assume that for some i, there exists an $m < i$ such that $t_{im} \ne 0$. For this i choose the largest such m. Then $t_{ij}(\lambda_j^{\downarrow}-\lambda_i^{\downarrow}) = 0$ for $j > m$, $t_{ij}(\lambda_j^{\downarrow}-\lambda_i^{\downarrow}) \ge 0$ for $j < m$, and $t_{im}(\lambda_m^{\downarrow}-\lambda_i^{\downarrow}) > 0$. This shows that $\sum_{j=1}^{n} t_{ij}(\lambda_j^{\downarrow}-\lambda_i^{\downarrow}) > 0$, which contradicts $(\lambda_1^{\downarrow},\dots,\lambda_n^{\downarrow}) = (\lambda_1^{\downarrow},\dots,\lambda_n^{\downarrow})R'SR$. We conclude that $R'PR$ is upper triangular. It is easy to see that the only upper triangular, doubly stochastic matrix is I. Thus, S is a permutation matrix. □

The following examples show that the conditions for equality may change in the non-generic cases where the $\lambda_1,\dots,\lambda_n$ are not distinct or include one or more zeros.

Example 2.3. *Let $\lambda_1 = \lambda_2 = 1$ and $V = \begin{pmatrix} i\cos\theta & i\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$ for some θ. Then $B = V^*V = \begin{pmatrix} 1 & \sin 2\theta & 0 \\ \sin 2\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, but $VV^* = \begin{pmatrix} 1 & i\sin 2\theta & 0 \\ -i\sin 2\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. So we have equality* *in (3) but V does not have only one non-zero entry in each row and column. Significantly, the doubly stochastic matrix* $S = \begin{pmatrix} \cos^2\theta & \sin^2\theta & 0 \\ \sin^2\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *is not uniquely determined; it varies with θ.*

Example 2.4. *With Λ and V as defined above, let $\Lambda' = \left(\begin{smallmatrix} \Lambda & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $V' = \left(\begin{smallmatrix} V & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then we have equality in (3) because both sides are zero, but V' does not have one non-zero entry in each row and column unless V does.*

The conditions imposed on V, above, are satisfied by any unitary matrix but the unitaries are only a small subclass of the possible matrices V. The next example gives large class of matrices V that are not unitary.

Example 2.5. *Suppose $T = (t_{ij})$ is an $n \times n$ Hermitian matrix with spectral norm $\|T\| \leq 1$ satisfying $t_{ii} = 0$ for all i. Since $\|T\| \leq 1$, $I + T$ is positive semi-definite and therefore has a positive semi-definite square root $V = (I + T)^{1/2}$. Note that $V^*V = VV^* = I + T$, a matrix with ones on the diagonal. If T is not zero the matrix V is not unitary.*

On the other hand, if V is unitary, the conditions on V are satisfied automatically, the product $\lambda_1 \cdots \lambda_n$ is the determinant of B, and B could be any positive semi-definite matrix.

Let S_n be the group of permutations of $\{1, \ldots, n\}$ and let D_n denote the collection of permutations that have no fixed point. These are the so-called derangements of $\{1, \ldots, n\}$. We also let e_i be the ith column of the $n \times n$ identity matrix so that Ae_i extracts the ith column of A. We remind the reader not to confuse e_i with $e_i(x)$.

Theorem 2.6. *Let $n \geq 3$. If $A = (a_{ij})$ is positive semi-definite and $\tau \in D_n$, then*

$$\det(A) + \prod_{i=1}^{n} |a_{i,\tau(i)}| \leq \prod_{i=1}^{n} a_{ii}. \tag{4}$$

Equality holds if and only if A is diagonal or the two vectors Ae_i and $Ae_{\tau(i)}$ are collinear for each i.

Proof. Choose a unitary V such that $A = V^*\Lambda V$, where $\lambda_1, \ldots, \lambda_n$ are the (necessarily non-negative) eigenvalues of A. This makes $B = A$. Then set $t =$

$\min(\lambda_1, \ldots, \lambda_n)$. Lemma 2.1 shows that $P(0) - P(t) \le Q(0) - Q(t)$. The choice of t ensures that $P(t) = 0$. Also $P(0) = \det(A)$ and $Q(0) = \prod_{i=1}^n a_{ii}$. To prove (4) we use the Cauchy-Schwarz inequality to show that $Q(t) \ge \prod_{i=1}^n |a_{i,\tau(i)}|$ for each $\tau \in D_n$. Fix $\tau \in D_n$ and observe that

$$\begin{aligned} |a_{i,\tau(i)}|^2 &= \left| \sum_{j=1}^n \bar{v}_{ji} \lambda_j v_{j,\tau(i)} \right|^2 \\ &= \left| \sum_{j=1}^n \bar{v}_{ji} (\lambda_j - t) v_{j,\tau(i)} \right|^2 \\ &\le \sum_{j=1}^n |v_{ji}|^2 (\lambda_j - t) \sum_{j=1}^n |v_{j,\tau(i)}|^2 (\lambda_j - t) \\ &= (a_{ii} - t)(a_{\tau(i),\tau(i)} - t). \end{aligned} \tag{5}$$

Thus,

$$\prod_{i=1}^n |a_{i,\tau(i)}| \le \left(\prod_{i=1}^n (a_{ii} - t) \prod_{i=1}^n (a_{\tau(i),\tau(i)} - t) \right)^{1/2} = \prod_{i=1}^n (a_{ii} - t) = Q(t). \tag{6}$$

Next we consider conditions for equality. If A is diagonal it is clear that (4) holds with equality. Suppose the two vectors Ae_i and $Ae_{\tau(i)}$ are collinear for each i. Then A is singular, $\det(A) = 0$, and $t = 0$. Fix i and choose a and b, not both zero, such that $aAe_i = bAe_{\tau(i)}$. Then for each j, $e_j^* V A(ae_i - be_{\tau(i)}) = 0$. But $AV^* e_j = V^* \Lambda e_j = \lambda_j V^* e_j$ so $\lambda_j e_j^* V(ae_i - be_{\tau(i)}) = 0$. Therefore $av_{ji} = bv_{j,\tau(i)}$ for all j such that $\lambda_j \ne 0$. This gives equality in (5). Since this holds for all i, we have equality in (6) as well. Since $\det(A) = 0$ we also have equality in (4).

Conversely, suppose equality holds in (4). The above proof shows that we must have $P(0) - P(t) = Q(0) - Q(t)$ and equality in (5) for all i. If $t > 0$, Lemma 2.1 shows that $(a_{11}, \ldots, a_{nn})$ is a permutation of $(\lambda_1, \ldots, \lambda_n)$ giving equality in Hadamard's inequality. Therefore A is diagonal. If $t = 0$ then equality in (5) implies that for all i there exist constants a and b, not both zero, such that $av_{ji} = bv_{j,\tau(i)}$ for all j such that $\lambda_j \ne 0$. Thus, for all j, $\lambda_j e_j^* V(ae_i - be_{\tau(i)}) = 0$ and hence, as above, $e_j^* V A(ae_i - be_{\tau(i)}) = 0$. This holds for all j, and V is invertible, so we conclude that $aAe_i = bAe_{\tau(i)}$, that is, Ae_i and $Ae_{\tau(i)}$ are collinear. □

Remark 2.7. *The collinearity condition for equality above may be expressed in terms of matrix rank: For $J \subseteq \{1, \ldots, n\}$, let P_J be the orthogonal projection onto* $\mathrm{span}\{e_j : j \in J\}$. *Let $J_1^\tau, \ldots, J_{n_\tau}^\tau$ be the orbits of $\tau \in D_n$. Then $I = \sum_{k=1}^{n_\tau} P_{J_k}$ so $A = \sum_{k=1}^{n_\tau} AP_{J_k}$. The condition that the two vectors Ae_i and $Ae_{\tau(i)}$ are collinear for each i is equivalent to saying that the rank of AP_{J_k} is at most 1 for $k = 1, \ldots, n_\tau$.*

The next result follows from Lemma 2.1 and the Cauchy-Schwarz estimates of Theorem 2.6. We state it without proof.

Theorem 2.8. *Let $\tau \in D_n$. If for all i, the $(i, \tau(i))$ entry of V^*V is zero then*

$$\prod_{i=1}^{n} \lambda_i + \prod_{i=1}^{n} |b_{i,\tau(i)}| \leq \prod_{i=1}^{n} b_{ii}.$$

Next is our proof of the motivating result: Theorem 1.1 for positive semi-definite matrices, including conditions for equality.

Observe that if $n = 2$, (4) reduces to equality for every A.

Corollary 2.9. *[11] Let $A = (a_{ij})$ be positive semi-definite matrix, and $\sigma \in S_n$. If σ is not the identity permutation, then*

$$\det(A) + \prod_{i=1}^{n} |a_{i,\sigma(i)}| \leq \prod_{i=1}^{n} a_{ii}. \tag{7}$$

Equality holds if A is diagonal, or if for each i either: $\sigma(i) = i$ and Ae_i, e_i are collinear; $\sigma(i) \neq i$ and σ is a transposition; or $\sigma(i) \neq i$ and Ae_i, $Ae_{\sigma(i)}$ are collinear. If A is positive definite, these conditions are also necessary for equality.

Proof. Let F be the set of fixed points of σ, a proper subset of $\{1, \ldots, n\}$, and let G be its complement. If F is empty, the result follows from Theorem 2.6. Note that both $A[F]$ and $A[G]$ are positive semi-definite. Fischer's inequality (see [3, Theorem 7.8.5]), followed by Hadamard's inequality, gives

$$\det(A) \leq \det(A[G]) \det(A[F]) \leq \det(A[G]) \prod_{i \in F} a_{ii}. \tag{8}$$

Note that the restriction of σ to G is a permutation of G with no fixed point. By Theorem 2.6, we have

$$\det(A[G]) + \prod_{i\in G} |a_{i,\sigma(i)}| \leq \prod_{i\in G} a_{ii}, \tag{9}$$

provided G has at least three elements. We can remove that restriction, however, because (9) becomes equality when G has exactly two elements, it is impossible for G to have exactly one element, and we have excluded the case that G is empty. Combining the last two inequalities gives (7).

If A is diagonal then we clearly have equality in (7). Now suppose that for each i either: $\sigma(i) = i$ and Ae_i, e_i are collinear; or $\sigma(i) \neq i$ and, if σ is not just a transposition, then Ae_i, $Ae_{\sigma(i)}$ are collinear. This implies that $A[F]$ is diagonal, and A has (up to reordering of the standard basis) a block diagonal decomposition with blocks $A[F]$ and $A[G]$. Thus we have equality in (8). The conditions for equality in Theorem 2.6 give equality in (9) when $n \geq 3$ and equality is trivial when $n = 2$ so we have equality in (7).

Now suppose that A is positive definite and equality holds in (7). Since $\det(A) > 0$, the Fischer inequality shows that $\det(A[F]) > 0$ and $\det(A[G]) > 0$. Therefore we have equality in both (8) and (9).

Equality in the Fischer inequality from (8) implies that A has (up to reordering of the standard basis) a block diagonal decomposition with blocks $A[F]$ and $A[G]$ (see [10, p. 217]). Equality in the Hadamard inequality from (8) implies that $A[F]$ is diagonal. Together, these show that if $\sigma(i) = i$, then Ae_i and e_i are collinear. Equality in (9) implies, via Theorem 2.6, that if G has at least three elements and $\sigma(i) \neq i$, then Ae_i, $Ae_{\sigma(i)}$ are collinear. If G has fewer than three elements then σ can only be a transposition. This completes the proof. □

Recall that the Hadamard product "$\circ$" is the entrywise product of matrices. So the Hadamard inequality may be written as $\det A \leq \det(A \circ I)$ for a positive semi-definite A. Theorem 2.6 enables us to state the following result.

Theorem 2.10. *Let $\tau \in D_n$ be a derangement and $P = (p_{ij})$ where p_{ij} is* 1 *when $j = \tau(i)$ and zero otherwise. For a positive semi-definite matrix $A = (a_{ij})$,*

$$\det(A \circ I) \geq \det A + |\det(A \circ P)|. \tag{10}$$

Equality holds if and only if the matrix A is a diagonal matrix or the two vectors Ae_i and $Ae_{\tau(i)}$ are collinear for each i.

Remark 2.11. *We expect that Theorem 2.10 will stimulate further investigation of Oppenheim-Schur inequalities (see [3, p. 509]).*

In 1961, Thompson [8] published a remarkable determinant inequality.

Theorem 2.12. *If $A = (A_{ij})$ is positive definite with each block A_{ij} square, then*

$$\det A \leq \det(\det A_{ij}). \tag{11}$$

Equality holds if and only if A is block diagonal.

We point out an extension of Theorem 2.6 to the block matrix case.

Theorem 2.13. *If $A = (A_{ij})$ is an $n \times n$ block positive definite matrix with each block A_{ij} square, then for any derangement $\tau \in D_n$,*

$$\det A + \prod_{i=1}^{n} |\det A_{i,\tau(i)}| \leq \prod_{i=1}^{n} \det A_{ii}.$$

Equality holds if and only if A is block diagonal.

Proof. By (11), it suffices to work with the $n \times n$ positive definite matrix $(\det A_{ij})$. The conclusion then follows by Theorem 2.10. □

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