

CALDERÓN-MITYAGIN COUPLES OF BANACH SPACES RELATED TO DECREASING FUNCTIONS

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ABSTRACT. A number of Calderón-Mityagin couples and relative Calderón-Mityagin pairs are identified among Banach function spaces defined in terms of the least decreasing majorant construction on the half line. The interpolation structure of such spaces is shown to closely parallel that of the rearrangement invariant spaces, and it is proved that a couple of these spaces is a Calderón-Mityagin couple if and only if the corresponding couple of rearrangement invariant spaces is a Calderón-Mityagin couple. Consequently, the class of all interpolation spaces for any couple of spaces of this type admits a complete description by the K -method if and only if the class of all interpolation spaces for the corresponding couple of rearrangement invariant spaces does. Analogous results are proved for spaces defined in terms of the level function construction. In the main, the conclusions for both types of spaces remain valid when Lebesgue measure on the half line is replaced by a general Borel measure on \mathbb{R} . However, for certain measures the class of interpolation spaces of these new spaces may be degenerate, reducing the “if and only if” of the main results to a single implication.

1. INTRODUCTION

The problem of characterizing the class of all interpolation spaces for a given couple of compatible Banach spaces is one of the fundamental problems in interpolation theory. Although there are many constructions used to generate interpolation spaces, it can be very difficult to show that all interpolation spaces have been generated. For the couple (L^1, L^∞) , however, the problem has been completely solved. Mityagin [14] in 1965 and, independently, Calderón [7] in 1966 gave characterizations of the class of all interpolation spaces with respect to (L^1, L^∞) . This discovery was an important point in the study of abstract interpolation spaces.

Couples for which a similar characterization exists are called Calderón-Mityagin couples. While the benefits are great, the problem of determining whether or not a given Banach couple is a Calderón-Mityagin couple is still a difficult one in general. Nonetheless, many Calderón-Mityagin couples have been discovered; we refer to papers [9] and [8] and references therein related to this topic.

We concern ourselves here with two couples closely related to (L^1, L^∞) . They are based on a pair of constructions on a function $f : (0, \infty) \rightarrow \mathbb{R}$: the least decreasing majorant \check{f} and the level function f° . The constructions are dual to one another

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in the following sense: If X is a rearrangement-invariant space, then the inequality

$$\int_0^\infty |fg| \leq \|f^o\|_X \|\tilde{g}\|_{X'}$$

holds and is sharp in f and g separately. See [18]. These constructions arise naturally when monotone functions are involved and have been used to solve a variety of problems in the theory of function spaces and norm inequalities. For example, they were applied to describe the dual of the classical Lorentz spaces in [10] and to give weight conditions for the boundedness of the Hardy operator in [15, 19]. They have also been used to study absolute convergence of Fourier series, by Beurling in [3] and more recently in [2]. See the references in [2] for additional applications.

Spaces defined in terms of the least decreasing majorant construction form a class of subspaces of the interpolation spaces for (L^1, L^∞) , while spaces defined in terms of the level function construction form a class of superspaces of the interpolation spaces for (L^1, L^∞) . We will show that the parallels between these three classes of function spaces are very strong. Specifically, we show that a couple of interpolation spaces from one class is a Calderón-Mityagin couple if and only if the corresponding couple in either of the other two classes is also a Calderón-Mityagin couple.

The *least decreasing majorant* of a Lebesgue measurable function f on $(0, \infty)$, is defined by,

$$\tilde{f}(x) = \text{ess sup}\{|f(y)| : y \geq x\}.$$

The map $f \mapsto \tilde{f}$ is sublinear, that is, $\widetilde{f+g} \leq \tilde{f} + \tilde{g}$. For any f , \tilde{f} is non-negative and decreasing, $|f| \leq \tilde{f}$ almost everywhere, and \tilde{f} is minimal among functions with these two properties. In consequence, $\tilde{f} = f$ whenever f is non-negative and decreasing; in particular, the map is a projection, that is, $\widetilde{\tilde{f}} = \tilde{f}$. If X is a Banach function space of Lebesgue measurable functions on $(0, \infty)$, then \tilde{X} is the space of functions f for which $\tilde{f} \in X$, equipped with the norm $\|f\|_{\tilde{X}} = \|\tilde{f}\|_X$. Sublinearity makes it routine to verify that \tilde{X} is a Banach function space.

The *level function* of f , denoted f^o , is a closely related projection defined (almost everywhere) to be the derivative of the least concave majorant of $\int_0^x |f|$, provided it is finite. Results of [17] show that that if $0 \leq f_n \uparrow f$ then $f_n^o \uparrow f^o$; this property is used to define f^o in the event that the least concave majorant of $\int_0^x |f|$ is infinite for some $x > 0$. For any f , f^o is non-negative and decreasing, $f^o = f$ whenever f is non-negative and decreasing, and $(f^o)^o = f^o$. If X is an exact interpolation space for (L^1, L^∞) , spaces of Lebesgue measurable functions on $(0, \infty)$, then X^o is the space of functions f for which $f^o \in X$, equipped with the norm $\|f\|_{X^o} = \|f^o\|_X$. Verification that X^o is a Banach function space is routine, except for the triangle inequality. If $f, g \in X^o$ then the function $x \mapsto \int_0^x (f^o + g^o)$ is a concave majorant of $x \mapsto \int_0^x |f+g|$ and hence $\int_0^x (f+g)^o \leq \int_0^x (f^o + g^o)$ for all $x > 0$. The hypothesis on X and Calderón's results from [7] show that

$$\|f+g\|_{X^o} = \|(f+g)^o\|_X \leq \|f^o + g^o\|_X \leq \|f^o\|_X + \|g^o\|_X = \|f\|_{X^o} + \|g\|_{X^o}.$$

It is immediate that $\widetilde{L^\infty} = L^\infty$ with identical norms and $(L^1)^o = L^1$ with identical norms, so in future we avoid writing $\widetilde{L^\infty}$ and $(L^1)^o$. On the other hand,

it is clear that $\|f\|_{L^1} \leq \|f\|_{\widetilde{L^1}}$ whenever $f \in \widetilde{L^1}$ and a calculation shows that,

$$(1.1) \quad \|f\|_{(L^\infty)^\circ} = \sup_{x>0} \frac{1}{x} \int_0^x |f| \leq \|f\|_{L^\infty}, \quad f \in L^\infty.$$

To complete this introduction, we recall some fundamental definitions, notation and two key known results that will be used in the paper. For notation and background for Banach function spaces we refer to [20] and for interpolation we refer the reader to [5] and [6]. A pair (X_0, X_1) of Banach spaces that can be continuously embedded into a common Hausdorff topological vector space is called a *compatible couple*. If (X_0, X_1) and (Y_0, Y_1) are compatible couples we write $T : (X_0, X_1) \rightarrow (Y_0, Y_1)$ to indicate that $T : X_0 + X_1 \rightarrow Y_0 + Y_1$ is a linear operator and the restriction of T to X_j is a bounded map into Y_j for $j = 0, 1$. The *norm* of such an operator is the maximum of the norms of these two maps. A subspace X of $X_0 + X_1$, containing $X_0 \cap X_1$, is called an *exact interpolation space* for (X_0, X_1) provided every linear operator $T : (X_0, X_1) \rightarrow (X_0, X_1)$ of norm one is also a bounded map from X to X of norm at most one. Let $\text{Int}_1(X_0, X_1)$ denote the collection of all exact interpolation spaces for (X_0, X_1) .

The *K-functional* of $x \in X_0 + X_1$ is defined by,

$$K(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x_0 \in X_0, x_1 \in X_1, x_0 + x_1 = x\}.$$

A pair of compatible couples $[(X_0, X_1), (Y_0, Y_1)]$ form a *c-uniform relative Calderón-Mityagin pair* if whenever $K(t, y; Y_0, Y_1) \leq K(t, x; X_0, X_1)$ for all $t > 0$, there exists a $T : (X_0, X_1) \rightarrow (Y_0, Y_1)$, of norm at most c , such that $Tx = y$. If $(X_0, X_1) = (Y_0, Y_1)$ we call (X_0, X_1) a *c-uniform Calderón-Mityagin couple*. If $c = 1$ we call the pair (or the couple) *exact*.

With these definitions, the famous description, due to Calderón [7], of all interpolation of spaces with respect to (L^1, L^∞) simply states: *The couple (L^1, L^∞) is an exact Calderón-Mityagin couple.*

A *parameter of the K-method* is a Banach function space Φ of functions on the measure space $((0, \infty), dt/t)$ that contains the function $t \mapsto \min(1, t)$. The *K-method* with parameter Φ is the interpolation functor that takes the couple (X_0, X_1) to the space $(X_0, X_1)_\Phi$ consisting of all $x \in X_0 + X_1$ for which the norm,

$$\|x\|_{(X_0, X_1)_\Phi} = \|K(\cdot, x; X_0, X_1)\|_\Phi,$$

is finite. The *K-divisibility constant* of a couple (X_0, X_1) is the smallest γ such that whenever $K(\cdot, x; X_0, X_1) \leq \sum_{j=1}^\infty \varphi_j$, with $0 \leq \varphi_j$ concave on $(0, \infty)$, there exist x_j such that $x = \sum_{j=1}^\infty x_j$ and $K(\cdot, x_j; X_0, X_1) \leq \gamma \varphi_j$ for each j . The *K-divisibility constant* of (L^1, L^∞) is known to be 1.

Brudnyĭ and Krugljak [6, Theorem 4.4.5] showed that $\gamma < 14$ for every couple, and used the *K-method* to give a complete description of all interpolation spaces for Calderón-Mityagin couples: *If (X_0, X_1) is a c-uniform Calderón-Mityagin couple and $X \in \text{Int}_1(X_0, X_1)$, then there exists a parameter of the K-method, Φ , such that $X = (X_0, X_1)_\Phi$ and*

$$\|x\|_{(X_0, X_1)_\Phi} \leq \|x\|_X \leq c\gamma \|x\|_{(X_0, X_1)_\Phi}, \quad x \in X.$$

Here γ is the *K-divisibility constant* of (X_0, X_1) .

In the special case that (X_0, X_1) is an exact Calderón-Mityagin couple with *K-divisibility constant* $\gamma = 1$, the conclusion is that $X = (X_0, X_1)_\Phi$ with identical norms. Our main results in Section 3 depend on this special case.

2. OPERATORS BETWEEN COUPLES CLOSE TO (L^1, L^∞)

In this section we recall or construct a number of linear operators that will be needed to work with Calderón-Mityagin couples and relative Calderón-Mityagin pairs. Since $\widetilde{L^\infty} = L^\infty$ and $(L^1)^\circ = L^1$ we write the couple $(\widetilde{L^1}, \widetilde{L^\infty})$ as $(\widetilde{L^1}, L^\infty)$ and the couple $((L^1)^\circ, (L^\infty)^\circ)$ as $(L^1, (L^\infty)^\circ)$. Their K -functionals have been calculated in [16, Theorem 1] and [13, Lemma 2.1].

Proposition 2.1. *If $f \in \widetilde{L^1} + L^\infty$ then $K(t, f; \widetilde{L^1}, L^\infty) = K(t, \tilde{f}; L^1, L^\infty)$ for $t > 0$. If $f \in L^1 + (L^\infty)^\circ$ then $K(t, f; L^1, (L^\infty)^\circ) = K(t, f^\circ; L^1, L^\infty)$.*

Taking $t = 1$ in this proposition shows that we have $(L^1 + L^\infty)^\sim = \widetilde{L^1} + L^\infty$ and $(L^1 + L^\infty)^\circ = L^1 + (L^\infty)^\circ$, with identical norms in both cases.

Our first operator does double duty, as a map from $(L^1, L^\infty) \rightarrow (\widetilde{L^1}, L^\infty)$ and a map from $(L^1, (L^\infty)^\circ)$ to (L^1, L^∞) .

Lemma 2.2. *There is a linear map $S : L^1 + (L^\infty)^\circ \rightarrow L^1 + (L^\infty)^\circ$, such that $h \leq Sh$ for any non-negative, decreasing h ; and*

- (a) $S : L^\infty \rightarrow L^\infty$ has norm at most 1,
- (b) $S : L^1 \rightarrow L^1$ has norm at most 2,
- (c) $S : L^1 \rightarrow \widetilde{L^1}$ has norm at most 4, and
- (d) $S : (L^\infty)^\circ \rightarrow L^\infty$ has norm at most 2.

Proof. For $h \in L^1 + (L^\infty)^\circ$ let

$$Sh = \sum_{j \in \mathbb{Z}} \frac{\chi_{[2^j, 2^{j+1})}}{2^j - 2^{j-1}} \int_{2^{j-1}}^{2^j} h.$$

At each point the sum has at most one term, and h is integrable on each interval $(2^j, 2^{j+1})$, so Sh well-defined for each $h \in L^1 + (L^\infty)^\circ$. If h is non-negative and decreasing, and $x > 0$, then, with $j \in \mathbb{Z}$ chosen so that $x \in [2^j, 2^{j+1})$,

$$h(x) \leq h(2^j) \leq \frac{1}{2^j - 2^{j-1}} \int_{2^{j-1}}^{2^j} h = Sh(x).$$

The value of Sh at each point is an average of h over an interval so, if $h \in L^\infty$, we see $\|Sh\|_{L^\infty} \leq \|h\|_{L^\infty}$, which proves (a). If $h \in L^1$ then $\|Sh\|_{L^1} \leq 2\|h\|_{L^1}$, giving (b). Also, Sh is dominated by the decreasing function,

$$\sum_{j \in \mathbb{Z}} \frac{\chi_{(0, 2^{j+1})}}{2^j - 2^{j-1}} \int_{2^{j-1}}^{2^j} |h|$$

and so is its least decreasing majorant, \widetilde{Sh} . Therefore, if $h \in L^1$,

$$\|Sh\|_{\widetilde{L^1}} = \|\widetilde{Sh}\|_{L^1} \leq \sum_{j \in \mathbb{Z}} \frac{2^{j+1}}{2^j - 2^{j-1}} \int_{2^{j-1}}^{2^j} |h| = 4\|h\|_{L^1}.$$

This proves (c). If $h \in (L^\infty)^\circ$, then for each $j \in \mathbb{Z}$, the formula (1.1) shows that

$$\left| \frac{1}{2^j - 2^{j-1}} \int_{2^{j-1}}^{2^j} h \right| \leq \frac{2}{2^j} \int_0^{2^j} |h| \leq 2\|h\|_{(L^\infty)^\circ}.$$

Thus $\|Sh\|_{L^\infty} \leq 2\|h\|_{(L^\infty)^\circ}$, giving (d). \square

The remainder of the section is divided into two parts. Operators related to the couple $(\widetilde{L}^1, L^\infty)$ are considered first and operators related to $(L^1, (L^\infty)^\circ)$ follow.

Theorem 2.3. *Suppose $f, g \in \widetilde{L}^1 + L^\infty$, $h \in L^1 + L^\infty$, and*

$$K(t, g; \widetilde{L}^1, L^\infty) \leq K(t, h; L^1, L^\infty) \leq K(t, f; \widetilde{L}^1, L^\infty).$$

- (a) *There are linear operators $V_1, V_2 : (L^1, L^\infty) \rightarrow (L^1, L^\infty)$, each of norm at most 1, such that $V_1 \tilde{f} = h$ and $V_2 h = \tilde{g}$.*
- (b) *The identity map $I : (\widetilde{L}^1, L^\infty) \rightarrow (L^1, L^\infty)$ has norm at most 1.*
- (c) *There is a linear operator $\overline{M} : (L^1, L^\infty) \rightarrow (\widetilde{L}^1, L^\infty)$, of norm at most 4, such that $\overline{M} \tilde{g} = g$.*
- (d) *There are linear operators $W_1, W_2, W_3 : (\widetilde{L}^1, L^\infty) \rightarrow (\widetilde{L}^1, L^\infty)$, each of norm at most 1, such that $W_1 f = \tilde{f}$, $W_2 \tilde{f} = \tilde{g}$, and $W_3 \tilde{g} = g$.*

For reference, these operators may be represented schematically as follows.

$$(2.1) \quad \begin{array}{ccccc} (L^1, L^\infty) : & & \tilde{f} & \xrightarrow{V_1} & h & \xrightarrow{V_2} & \tilde{g} \\ & & \uparrow I & & & & \downarrow M \\ (\widetilde{L}^1, L^\infty) : & f & \xrightarrow{W_1} & \tilde{f} & \xrightarrow{W_2} & \tilde{g} & \xrightarrow{W_3} & g \end{array}$$

See Definition 2.4, below, for the relationship between M and \overline{M} . We do not assert that the diagram above is a commuting diagram. That is, although the operators MV_2V_1I and W_3W_2 both map \tilde{f} to g , the operators themselves may not be equal.

Proof. By Proposition 2.1, we have

$$K(t, \tilde{g}; L^1, L^\infty) \leq K(t, h; L^1, L^\infty) \leq K(t, \tilde{f}; L^1, L^\infty).$$

Thus Calderón's theorem provides V_1 and V_2 to prove (a). Since $|\psi| \leq \tilde{\psi}$, (b) is trivial. For (c) we define $\overline{M} : (L^1, L^\infty) \rightarrow (\widetilde{L}^1, L^\infty)$ by $\overline{M}\psi = (g/S\tilde{g})S\psi$, where S is the operator of Lemma 2.2. Since $|g| \leq \tilde{g} \leq S\tilde{g}$, it follows from Lemma 2.2, parts (c) and (a), that the norm of \overline{M} is at most 4. (If necessary, take $0/0 = 0$ in the definition of \overline{M} .)

Now we turn to (d). Let $W_3\psi = (g/\tilde{g})\psi$, which has norm at most 1 since $|g/\tilde{g}| \leq 1$. For W_2 we apply Theorem 5(b) of [4], which shows that there is a map $W_2 : (L^1, L^\infty) \rightarrow (L^1, L^\infty)$, of norm at most 1, such that $W_2\tilde{f} = \tilde{g}$ and which maps non-negative functions to non-negative functions and maps non-negative, decreasing functions to non-negative, decreasing functions. If $\psi \in \widetilde{L}^1$ then $\tilde{\psi} \in L^1$ and $\tilde{\psi} \pm \psi \geq 0$ so $\pm W_2\psi \leq W_2\tilde{\psi}$. It follows that $|W_2\psi| \leq W_2\tilde{\psi}$. But $\tilde{\psi}$ is non-negative and decreasing so $W_2\tilde{\psi}$ is a decreasing majorant of $|W_2\psi|$. Therefore, the least decreasing majorant $\widetilde{W_2\psi}$ of $|W_2\psi|$ is no greater than $W_2\tilde{\psi}$. We have,

$$\|W_2\psi\|_{\widetilde{L}^1} = \|\widetilde{W_2\psi}\|_{L^1} \leq \|W_2\tilde{\psi}\|_{L^1} \leq \|\tilde{\psi}\|_{L^1} = \|\psi\|_{\widetilde{L}^1}.$$

This shows that the norm of W_2 on \widetilde{L}^1 is at most 1 so W_2 may be viewed as a map from $(\widetilde{L}^1, L^\infty)$ to $(\widetilde{L}^1, L^\infty)$ of norm at most one.

The existence of W_1 follows from the Hahn-Banach-Kantorovich theorem, see Theorem 1.25 of [1]. On the one-dimensional subspace $\mathbb{R}f$ of $\widetilde{L}^1 + L^\infty$ the linear operator $W_1(\alpha f) = \alpha \tilde{f} \in \widetilde{L}^1 + L^\infty$ satisfies $W_1\psi \leq \tilde{\psi}$. But $\psi \mapsto \tilde{\psi}$ is a sublinear

map from $\widetilde{L}^1 + L^\infty$ to $\widetilde{L}^1 + L^\infty$ so W_1 extends to a linear operator from $\widetilde{L}^1 + L^\infty$ to $\widetilde{L}^1 + L^\infty$, preserving the bound $W_1\psi \leq \widetilde{\psi}$. Now, $-W_1\psi = W_1(-\psi) \leq -\widetilde{\psi} = \widetilde{\psi}$ so we have $|W_1\psi| \leq \widetilde{\psi}$. Therefore, $\|W_1\psi\|_{L^\infty} \leq \|\widetilde{\psi}\|_{L^\infty} \leq \|\psi\|_{L^\infty}$ and $\|W_1\psi\|_{\widetilde{L}^1} \leq \|\widetilde{\psi}\|_{\widetilde{L}^1} = \|\widetilde{\psi}\|_{L^1} = \|\psi\|_{L^1} = \|\psi\|_{\widetilde{L}^1}$. This shows W_1 has norm at most 1. \square

We do not claim that the operator \overline{M} , constructed above, has smallest possible norm. Since this norm will appear in subsequent results we make the following definition.

Definition 2.4. *Let m be a constant such that for each $g \in \widetilde{L}^1 + L^\infty$, there is a linear operator $M : (L^1, L^\infty) \rightarrow (\widetilde{L}^1, L^\infty)$, of norm at most m , satisfying $M\tilde{g} = g$.*

Theorem 2.3(c) shows that $m = 4$ is such a constant, with $M = \overline{M}$. Lemma 2.5, below, shows that no $m < 9/8$ can satisfy Definition 2.4.

Lemma 2.5. *There exists a function $g \in \widetilde{L}^1 \cap L^\infty$ such that any linear operator $M : (L^1, L^\infty) \rightarrow (\widetilde{L}^1, L^\infty)$ satisfying $M\tilde{g} = g$ has norm at least $9/8$.*

Proof. Let $a = \chi_{[0,1]}$, $b = \chi_{[1,3]}$, and set $g = 2a + b$. Clearly $g \in \widetilde{L}^1 \cap L^\infty$. Suppose $M : (L^1, L^\infty) \rightarrow (\widetilde{L}^1, L^\infty)$ has norm C and satisfies $M\tilde{g} = g$. Since g is decreasing, $g = \tilde{g}$ so $Mg = g$. Now $g = 2$ on $(0, 1)$, so

$$2 - \int_0^1 Ma = \int_0^1 (Mg - Ma) = \int_0^1 M(a+b) \leq \|M(a+b)\|_{L^\infty} \leq C\|a+b\|_{L^\infty} = C.$$

But $g = 1$ on $(1, 3)$, $Mb \leq \widetilde{Mb}$, and \widetilde{Mb} is non-negative and decreasing, so

$$1 - \int_1^3 Ma = \frac{1}{2} \int_1^3 (Mg - 2Ma) \leq \frac{1}{2} \int_1^3 \widetilde{Mb} \leq \frac{1}{3} \int_0^3 \widetilde{Mb} \leq \frac{C}{3} \int_0^\infty b = \frac{2}{3}C.$$

It follows that

$$3 - \frac{5}{3}C \leq \int_0^1 Ma + \int_1^3 Ma = \int_0^3 Ma \leq \int_0^3 \widetilde{Ma} \leq C \int_0^\infty a = C.$$

Thus, $C \geq 9/8$. \square

A similar collection of operators can be found for the couple $(L^1, (L^\infty)^\circ)$.

Theorem 2.6. *Suppose $f, g \in L^1 + (L^\infty)^\circ$, $h \in L^1 + L^\infty$, and*

$$K(t, g; L^1, (L^\infty)^\circ) \leq K(t, h; L^1, L^\infty) \leq K(t, f; L^1, (L^\infty)^\circ).$$

- There are linear operators $V_3, V_4 : (L^1, L^\infty) \rightarrow (L^1, L^\infty)$, each of norm at most 1, such that $V_3f^\circ = h$ and $V_4h = g^\circ$.*
- The identity map $I : (L^1, L^\infty) \rightarrow (L^1, (L^\infty)^\circ)$, has norm at most 1.*
- There is a linear operator $\overline{N} : (L^1, (L^\infty)^\circ) \rightarrow (L^1, L^\infty)$, of norm at most 2, such that $\overline{N}f = f^\circ$.*
- There are linear operators $U_1, U_2, U_3 : (L^1, (L^\infty)^\circ) \rightarrow (L^1, (L^\infty)^\circ)$, each of norm at most 1, such that $U_1f = f^\circ$, $U_2f^\circ = g^\circ$, and $U_3g^\circ = g$.*

For reference, these operators may be represented schematically as follows.

$$(2.2) \quad \begin{array}{ccccc} (L^1, (L^\infty)^\circ) : & f & \xrightarrow{U_1} & f^\circ & \xrightarrow{U_2} & g^\circ & \xrightarrow{U_3} & g \\ & \downarrow N & & & & & & \uparrow I \\ (L^1, L^\infty) : & f^\circ & \xrightarrow{V_3} & h & \xrightarrow{V_4} & g^\circ & & \end{array}$$

See Definition 2.7, below, for the relationship between N and \bar{N} . We do not assert that the diagram above is a commuting diagram. That is, although the operators IV_4V_3N and U_2U_1 both map f to g° , the operators themselves may not be equal.

Proof. By Proposition 2.1, we have

$$K(t, g^\circ; L^1, L^\infty) \leq K(t, h; L^1, L^\infty) \leq K(t, f^\circ; L^1, L^\infty).$$

So Calderón's theorem provides V_3 and V_4 to prove (a). Part (b) follows from (1.1). The operators U_1 and U_3 were constructed in Theorems 3.5 and 3.6 of [13], respectively. Since $(f^\circ)^\circ = f^\circ$ and $(g^\circ)^\circ = g^\circ$ we have,

$$K(t, g^\circ; L^1, (L^\infty)^\circ) \leq K(t, f^\circ; L^1, (L^\infty)^\circ),$$

so Theorem 3.8 of [13] gives the operator U_2 .

For (c) we define $\bar{N} : (L^1, (L^\infty)^\circ) \rightarrow (L^1, L^\infty)$ by $\bar{N}\psi = (f^\circ/Sf^\circ)SU_1\psi$, where S is the operator of Lemma 2.2. Since $f^\circ \leq Sf^\circ$ and U_1 has norm at most 1, it follows from Lemma 2.2, parts (b) and (d), that the norm of \bar{N} is at most 2. (If necessary, take $0/0 = 0$ in the definition of \bar{N} .) \square

We do not claim that the operator \bar{N} , constructed above, has smallest possible norm. Since this norm will affect our subsequent results we make the following definition.

Definition 2.7. *Let n be a constant such that for each $f \in L^1 + (L^\infty)^\circ$, there is a linear operator $N : (L^1, (L^\infty)^\circ) \rightarrow (L^1, L^\infty)$, of norm at most n , satisfying $Nf = f^\circ$.*

Theorem 2.6(c) shows that $n = 2$ is such a constant, with $N = \bar{N}$. Lemma 2.8, below, shows that no $n < 9/8$ satisfies Definition 2.7.

Lemma 2.8. *There exists a function $f \in L^1 \cap L^\infty$ such that any linear operator $N : (L^1, (L^\infty)^\circ) \rightarrow (L^1, L^\infty)$ satisfying $Nf = f^\circ$ has norm at least $9/8$.*

Proof. Let $a = \chi_{[0,1]}$, $b = \chi_{[1,3]}$, and set $f = 2a + b$. Clearly, $f \in L^1 \cap L^\infty$. Suppose $N : (L^1, (L^\infty)^\circ) \rightarrow (L^1, L^\infty)$ has norm C and satisfies $Nf = f^\circ$. Since f is decreasing, $f = f^\circ$ so $Nf = f$, that is, $N(2a + b) = 2a + b$. Now $\|a + b\|_{(L^\infty)^\circ} = 1$ and $\|a\|_{L^1} = 1$, so

$$\int_0^1 N(a + b) + \int_0^3 Na \leq \|N(a + b)\|_{L^\infty} + \|Na\|_{L^1} \leq C\|a + b\|_{(L^\infty)^\circ} + C\|a\|_{L^1} = 2C,$$

and we have

$$\int_1^3 Na = \int_0^1 N(a + b) + \int_0^3 Na - \int_0^1 N(2a + b) \leq 2C - 2.$$

But, $\|Nb\|_{L^\infty} \leq C\|b\|_{(L^\infty)^\circ} = (2/3)C$, by (1.1), so

$$2 = \int_1^3 N(2a + b) = 2 \int_1^3 Na + \int_1^3 Nb \leq 2(2C - 2) + 2(2/3)C = (16/3)C - 4.$$

Thus, $C \geq 9/8$. \square

3. CONCERNING CALDERÓN-MITYAGIN COUPLES AND RELATIVE CALDERÓN-MITYAGIN PAIRS

The authors showed, in [13], that $(L^1, (L^\infty)^o)$ is an exact Calderón-Mityagin couple and, more recently, Leśnik, in [11], showed that $(\widetilde{L}^1, L^\infty)$ is also a Calderón-Mityagin couple. The first part of Theorem 3.1 provides an alternative proof of Leśnik's result, improving his $1 + \varepsilon$ constant to give exactness. More importantly, the pairs $[(\widetilde{L}^1, L^\infty), (L^1, L^\infty)]$, $[(L^1, L^\infty), (\widetilde{L}^1, L^\infty)]$, $[(L^1, (L^\infty)^o), (L^1, L^\infty)]$, and $[(L^1, L^\infty), (L^1, (L^\infty)^o)]$ are shown to be relative Calderón-Mityagin pairs. See Theorems 3.1 and 3.6.

Theorems 3.3 and 3.7 set out the close correspondence between the class of all interpolation spaces for the couples (L^1, L^∞) , $(\widetilde{L}^1, L^\infty)$, and $(L^1, (L^\infty)^o)$. Theorems 3.5 and 3.9 take this correspondence further, showing that the relative Calderón-Mityagin pairs within $\text{Int}_1(\widetilde{L}^1, L^\infty)$ correspond exactly to the relative Calderón-Mityagin pairs within $\text{Int}_1(\widetilde{L}^1, L^\infty)$ and also to those within $\text{Int}_1(L^1, (L^\infty)^o)$.

Results for the couple $(\widetilde{L}^1, L^\infty)$ will be given first. Analogous results for the couple $(L^1, (L^\infty)^o)$ follow.

Theorem 3.1. *Let m be a constant satisfying Definition 2.4. Then,*

- (a) $(\widetilde{L}^1, L^\infty)$ is an exact Calderón-Mityagin couple.
- (b) $(\widetilde{L}^1, L^\infty)$ and (L^1, L^∞) form an exact relative Calderón-Mityagin pair.
- (c) (L^1, L^∞) and $(\widetilde{L}^1, L^\infty)$ form an m -uniform relative Calderón-Mityagin pair.
- (d) (L^1, L^∞) and $(\widetilde{L}^1, L^\infty)$ do not form an exact relative Calderón-Mityagin pair.

Proof. Operator names refer to those constructed in Theorem 2.3. Refer to the diagram (2.1).

To prove (a), suppose $f, g \in \widetilde{L}^1 + L^\infty$ satisfy $K(t, g; \widetilde{L}^1, L^\infty) \leq K(t, f; \widetilde{L}^1, L^\infty)$ for all $t > 0$, and let $T = W_3W_2W_1$. All three have norm at most 1 so $T : (\widetilde{L}^1, L^\infty) \rightarrow (\widetilde{L}^1, L^\infty)$ has norm at most 1 as well. Also $Tf = W_3W_2W_1f = W_3W_2\tilde{f} = W_3\tilde{g} = g$.

For (b), suppose $f \in \widetilde{L}^1 + L^\infty$ and $h \in L^1 + L^\infty$ satisfy $K(t, h; L^1, L^\infty) \leq K(t, f; \widetilde{L}^1, L^\infty)$ for all $t > 0$, and let $T = V_1IW_1$. All three have norm at most 1 so $T : (\widetilde{L}^1, L^\infty) \rightarrow (L^1, L^\infty)$ has norm at most 1 as well. Also, $Tf = V_1IW_1f = V_1I\tilde{f} = V_1\tilde{f} = h$.

For (c), suppose $h \in L^1 + L^\infty$ and $g \in \widetilde{L}^1 + L^\infty$ satisfy $K(t, g; \widetilde{L}^1, L^\infty) \leq K(t, h; L^1, L^\infty)$ for all $t > 0$, and let $T = MV_2$, where M comes from Definition 2.4. The operator V_2 has norm at most one and M has norm at most m . Thus $T : (L^1, L^\infty) \rightarrow (\widetilde{L}^1, L^\infty)$ has norm at most m . Also, $Th = MV_2h = M\tilde{g} = g$.

Lemma 2.5 proves (d). □

The next lemma will give control over the constants in subsequent results.

Lemma 3.2. *The K -divisibility constant for the couple $(\widetilde{L}^1, L^\infty)$ is 1.*

Proof. Suppose $g \in \widetilde{L}^1 + L^\infty$, $\{\varphi_j\}$ is a sequence of non-negative, concave functions such that $\sum_{j=0}^{\infty} \varphi_j(1) < \infty$, and

$$K(t, g; \widetilde{L}^1, L^\infty) \leq \sum_{j=1}^{\infty} \varphi_j(t), \quad t > 0.$$

Since $\int_0^t \tilde{g} = K(t, g; \widetilde{L}^1, L^\infty) < \infty$, the convex function $K = K(\cdot, g; \widetilde{L}^1, L^\infty)$ satisfies $K(0+) = 0$. Replacing φ_j by $\min(K, \varphi_j)$ we may assume with no loss of generality that $\varphi_j(0+) = 0$. Now, taking f_j to be the derivative of φ_j (which exists almost everywhere) we have $\varphi_j(t) = \int_0^t f_j$ for $t > 0$. Note that a non-negative, concave function on $(0, \infty)$ is necessarily increasing, so each f_j is non-negative and decreasing. Set $f = \sum_{j=0}^\infty f_j$ and observe that for each J ,

$$f - \sum_{j=0}^J f_j = \sum_{j=J+1}^\infty f_j$$

is a non-negative, decreasing function dominated by f . But $\int_0^1 f \leq \sum_{j=1}^\infty \varphi_j(1) < \infty$ so the dominated convergence theorem shows that,

$$\left\| f - \sum_{j=0}^J f_j \right\|_{\widetilde{L}^1 + L^\infty} = \int_0^1 \left(f - \sum_{j=0}^J f_j \right) = \int_0^1 \left(f - \sum_{j=0}^J f_j \right) \rightarrow 0$$

as $J \rightarrow \infty$. Thus $\sum_{j=0}^\infty f_j$ converges to f in $\widetilde{L}^1 + L^\infty$.

Since f is non-negative and decreasing,

$$K(t, g; \widetilde{L}^1, L^\infty) \leq \sum_{j=1}^\infty \varphi_j(t) = \int_0^t f = K(t, f; \widetilde{L}^1, L^\infty).$$

Theorem 2.3(d) provides operators W_2 and W_3 on $\widetilde{L}^1 + L^\infty$, both of norm at most 1, such that $g = W_3 W_2 \tilde{f} = W_3 W_2 f$. Let $g_j = W_3 W_2 f_j$. The boundedness of $W_3 W_2$ implies that $\sum_{j=1}^\infty g_j$ converges to g in $\widetilde{L}^1 + L^\infty$. Because the norm of $W_3 W_2$ is at most 1,

$$K(t, g_j, \widetilde{L}^1, L^\infty) \leq K(t, f_j, \widetilde{L}^1, L^\infty) = \int_0^t \tilde{f}_j = \int_0^t f_j = \varphi_j(t). \quad \square$$

Theorem 3.3. *If X is in $\text{Int}_1(L^1, L^\infty)$ then \tilde{X} is in $\text{Int}_1(\widetilde{L}^1, L^\infty)$. Conversely, if Y is in $\text{Int}_1(\widetilde{L}^1, L^\infty)$ then $Y = \tilde{X}$, with identical norms, for some space X in $\text{Int}_1(L^1, L^\infty)$.*

Proof. If $X \in \text{Int}_1(L^1, L^\infty)$ we apply the theorems of Calderón and Brudnyĭ-Krugljak to get a parameter Φ of the K -method such that $X = (L^1, L^\infty)_\Phi$, with identical norms. By Proposition 2.1, for any $f \in \widetilde{L}^1 + L^\infty$, all five of the statements: $f \in \tilde{X}$, $\tilde{f} \in X$, $K(\cdot, \tilde{f}; L^1, L^\infty) \in \Phi$, $K(\cdot, f; \widetilde{L}^1, L^\infty) \in \Phi$ and $f \in (\widetilde{L}^1, L^\infty)_\Phi$ are equivalent. Moreover,

$$\|f\|_{\tilde{X}} = \|\tilde{f}\|_X = \|K(\cdot, \tilde{f}; L^1, L^\infty)\|_\Phi = \|K(\cdot, f; \widetilde{L}^1, L^\infty)\|_\Phi = \|f\|_{(\widetilde{L}^1, L^\infty)_\Phi}.$$

This shows $\tilde{X} = (\widetilde{L}^1, L^\infty)_\Phi$, with identical norms. In particular, $\tilde{X} \in \text{Int}_1(\widetilde{L}^1, L^\infty)$.

On the other hand, if $Y \in \text{Int}_1(\widetilde{L}^1, L^\infty)$ then by Theorem 3.1(a), the Brudnyĭ-Krugljak theorem, and Lemma 3.2, there exists a parameter Φ of the K -method such that $Y = (\widetilde{L}^1, L^\infty)_\Phi$, with identical norms. Let $X = (L^1, L^\infty)_\Phi$. Then $X \in \text{Int}_1(L^1, L^\infty)$. Arguing as above we see that $\tilde{X} = (\widetilde{L}^1, L^\infty)_\Phi = Y$, with identical norms. \square

Lemma 3.4. *Let m be a constant satisfying Definition 2.4. Suppose X_0 and X_1 are in $\text{Int}_1(L^1, L^\infty)$ and \mathcal{F} is an exact interpolation functor. Set $X = \mathcal{F}(X_0, X_1)$.*

- (a) *There exist Φ_0 and Φ_1 , parameters of the K -method, such that $X_j = (L^1, L^\infty)_{\Phi_j}$ and $\widetilde{X}_j = (\widetilde{L}^1, L^\infty)_{\Phi_j}$, in each case with identical norms, for $j = 0, 1$.*
- (b) *$\mathcal{F}(\widetilde{X}_0, \widetilde{X}_1) \hookrightarrow \widetilde{X}$ with norm at most 1.*
- (c) *$\widetilde{X} \hookrightarrow \mathcal{F}(\widetilde{X}_0, \widetilde{X}_1)$ with norm at most m .*
- (d) *$\widetilde{X}_0 + \widetilde{X}_1 = (X_0 + X_1)^\sim$, and for all f in this space,*

$$K(t, \tilde{f}; X_0, X_1) \leq K(t, f; \widetilde{X}_0, \widetilde{X}_1) \leq mK(t, \tilde{f}; X_0, X_1), \quad t > 0.$$

Proof. The proof of part (a) is contained in the proof of Theorem 3.3. For part (b), fix $f \in \mathcal{F}(\widetilde{X}_0, \widetilde{X}_1)$. Then $f \in \widetilde{X}_0 + \widetilde{X}_1 \subseteq \widetilde{L}^1 + L^\infty$ so Theorem 2.3 provides operators I and W_1 such that $IW_1 : (\widetilde{L}^1, L^\infty) \rightarrow (L^1, L^\infty)$ has norm at most 1, and $IW_1 f = \tilde{f}$. Real interpolation with parameters Φ_0 and Φ_1 shows that $IW_1 : (\widetilde{X}_0, \widetilde{X}_1) \rightarrow (X_0, X_1)$, with norm at most 1, and we may apply the exact functor \mathcal{F} to obtain $IW_1 : \mathcal{F}(\widetilde{X}_0, \widetilde{X}_1) \rightarrow X$, with norm at most 1. We conclude that $\tilde{f} = IW_1 f \in X$ and $\|\tilde{f}\|_{\widetilde{X}} = \|f\|_X \leq \|f\|_{\mathcal{F}(\widetilde{X}_0, \widetilde{X}_1)}$. Thus, each $f \in \mathcal{F}(\widetilde{X}_0, \widetilde{X}_1)$ is in \widetilde{X} and the norm of the embedding is at most 1.

For part (c), fix $g \in \widetilde{X}$. Since $\tilde{g} \in X = \mathcal{F}(X_0, X_1) \subseteq L^1 + L^\infty$, $g \in \widetilde{L}^1 + L^\infty$. By Definition 2.4 there exists a linear operator $M : (L^1, L^\infty) \rightarrow (\widetilde{L}^1, L^\infty)$, of norm at most m , such that $M\tilde{g} = g$. It follows by real interpolation with parameter Φ_j , that $M : X_j \rightarrow \widetilde{X}_j$ with norm at most m , for $j = 0, 1$. Applying the exact functor \mathcal{F} shows $M : X \rightarrow \mathcal{F}(\widetilde{X}_0, \widetilde{X}_1)$, with norm at most m . Since $\tilde{g} \in X$, $g = M\tilde{g} \in \mathcal{F}(\widetilde{X}_0, \widetilde{X}_1)$ and $\|g\|_{\mathcal{F}(\widetilde{X}_0, \widetilde{X}_1)} = \|M\tilde{g}\|_{\mathcal{F}(\widetilde{X}_0, \widetilde{X}_1)} \leq m\|\tilde{g}\|_X = m\|g\|_{\widetilde{X}}$. Thus, each $g \in \widetilde{X}$ is in $\mathcal{F}(\widetilde{X}_0, \widetilde{X}_1)$ and the norm of the embedding is at most m .

For part (d) we apply (b) and (c) to a family of exact functors. Fix $t > 0$. If (Z_0, Z_1) is a compatible couple of Banach spaces let $\Sigma_t(Z_0, Z_1)$ be the space $Z_0 + Z_1$, equipped with the norm $\|f\|_{\Sigma_t(Z_0, Z_1)} = K(t, f; Z_0, Z_1)$. Then the embeddings from (b) and (c) show that $\widetilde{X}_0 + \widetilde{X}_1 = (X_0 + X_1)^\sim$, and for all f in this space, the estimates of the constants in (b) and (c) show

$$K(t, \tilde{f}; X_0, X_1) = \|\tilde{f}\|_{\Sigma_t(X_0, X_1)} \leq \|f\|_{\Sigma_t(\widetilde{X}_0, \widetilde{X}_1)} = K(t, f; \widetilde{X}_0, \widetilde{X}_1),$$

and

$$K(t, f; \widetilde{X}_0, \widetilde{X}_1) = \|f\|_{\Sigma_t(\widetilde{X}_0, \widetilde{X}_1)} \leq m\|\tilde{f}\|_{\Sigma_t(X_0, X_1)} = mK(t, \tilde{f}; X_0, X_1). \quad \square$$

Note that if \mathcal{F} is a positive exact interpolation functor, then the assumption $X_0, X_1 \in \text{Int}_1(L^1, L^\infty)$ is not needed to prove (b) above. Because $f \mapsto \tilde{f}$ is sub-linear, the embedding $\mathcal{F}(\widetilde{X}_0, \widetilde{X}_1) \hookrightarrow \widetilde{X}$, with norm at most 1, is a consequence of Theorem 2.1 of [12].

The next theorem gives the correspondence between relative Calderón-Mityagin pairs in $\text{Int}_1(L^1, L^\infty)$ and those in $\text{Int}_1(\widetilde{L}^1, L^\infty)$. Note that the converse below is stated only for couples in $\text{Int}_1(\widetilde{L}^1, L^\infty)$ of the form $(\widetilde{X}_0, \widetilde{X}_1)$ with $X_0, X_1 \in \text{Int}_1(L^1, L^\infty)$. However, in view of Theorem 3.3 there is no loss of generality in this formulation.

Theorem 3.5. *Let m be a constant satisfying Definition 2.4 and let $X_0, X_1, Y_0, Y_1 \in \text{Int}_1(L^1, L^\infty)$. If (X_0, X_1) and (Y_0, Y_1) form a c -uniform relative Calderón-Mityagin pair then $(\widetilde{X}_0, \widetilde{X}_1)$ and $(\widetilde{Y}_0, \widetilde{Y}_1)$ form an (m^2c) -uniform relative Calderón-Mityagin*

pair. Conversely, if $(\widetilde{X}_0, \widetilde{X}_1)$ and $(\widetilde{Y}_0, \widetilde{Y}_1)$ form a c -uniform relative Calderón-Mityagin pair then (X_0, X_1) and (Y_0, Y_1) form an (m^2c) -uniform relative Calderón-Mityagin pair.

Proof. Suppose (X_0, X_1) and (Y_0, Y_1) form a c -uniform relative Calderón-Mityagin pair, $f \in \widetilde{X}_0 + \widetilde{X}_1$, $g \in \widetilde{Y}_0 + \widetilde{Y}_1$, and $K(t, g; \widetilde{Y}_0, \widetilde{Y}_1) \leq K(t, f; \widetilde{X}_0, \widetilde{X}_1)$ for $t > 0$. By Lemma 3.4(d), applied first to the couple (Y_0, Y_1) and then to the couple (X_0, X_1) , and by the homogeneity of the K -functional,

$$K(t, \tilde{g}; Y_0, Y_1) \leq K(t, g; \widetilde{Y}_0, \widetilde{Y}_1) \leq K(t, f; \widetilde{X}_0, \widetilde{X}_1) \leq K(t, m\tilde{f}; X_0, X_1).$$

By hypothesis, there exists a linear operator $T : (X_0, X_1) \rightarrow (Y_0, Y_1)$, of norm at most c , such that $T(m\tilde{f}) = \tilde{g}$. Since $f \in \widetilde{X}_0 + \widetilde{X}_1$, $\tilde{f} \in X_0 + X_1 \subseteq L^1 + L^\infty$. The product $IW_1 : (\widetilde{L}^1, \widetilde{L}^\infty) \rightarrow (L^1, L^\infty)$, from Theorem 2.3, has norm at most 1 and takes f to \tilde{f} . The operator $M : (L^1, L^\infty) \rightarrow (\widetilde{L}^1, \widetilde{L}^\infty)$, from Definition 2.4, has norm at most m and takes \tilde{g} to g . By Lemma 3.4(a), $IW_1 : (\widetilde{X}_0, \widetilde{X}_1) \rightarrow (X_0, X_1)$ has norm at most 1, and $M : (Y_0, Y_1) \rightarrow (\widetilde{Y}_0, \widetilde{Y}_1)$ has norm at most m . It follows that

$$\begin{array}{ccccccc} (\widetilde{X}_0, \widetilde{X}_1) & \xrightarrow{IW_1} & (X_0, X_1) & \xrightarrow{m} & (X_0, X_1) & \xrightarrow{T} & (Y_0, Y_1) & \xrightarrow{M} & (\widetilde{Y}_0, \widetilde{Y}_1) \\ f & \longmapsto & \tilde{f} & \longmapsto & m\tilde{f} & \longmapsto & \tilde{g} & \longmapsto & g \end{array}$$

where m is used to denote multiplication by m . That is, the operator $\bar{T} = MTmIW_1$ maps $(\widetilde{X}_0, \widetilde{X}_1)$ to $(\widetilde{Y}_0, \widetilde{Y}_1)$, with norm at most m^2c , and $\bar{T}f = g$. We conclude that $(\widetilde{X}_0, \widetilde{X}_1)$ and $(\widetilde{Y}_0, \widetilde{Y}_1)$ form an (m^2c) -uniform relative Calderón-Mityagin pair.

Conversely, suppose $(\widetilde{X}_0, \widetilde{X}_1)$ and $(\widetilde{Y}_0, \widetilde{Y}_1)$ form a c -uniform relative Calderón-Mityagin pair, $f \in X_0 + X_1$, $g \in Y_0 + Y_1$, and $K(t, g; Y_0, Y_1) \leq K(t, f; X_0, X_1)$ for $t > 0$. Since g and g^* are equimeasurable, and f and f^* are equimeasurable,

$$K(t, g^*; Y_0, Y_1) = K(t, g; Y_0, Y_1) \leq K(t, f; X_0, X_1) = K(t, f^*; X_0, X_1).$$

But $g^* = \widetilde{g}^*$ and $f^* = \widetilde{f}^*$, so by Lemma 3.4(d) and the homogeneity of the K -functional,

$$K(t, g^*; \widetilde{Y}_0, \widetilde{Y}_1) \leq mK(t, \widetilde{g}^*; Y_0, Y_1) \leq mK(t, \widetilde{f}^*; X_0, X_1) \leq K(t, mf^*; \widetilde{X}_0, \widetilde{X}_1).$$

By the hypothesis, there exists a $T : (\widetilde{X}_0, \widetilde{X}_1) \rightarrow (\widetilde{Y}_0, \widetilde{Y}_1)$, with norm at most c such that $T(mf^*) = g^*$.

Since $K(t, f^*; L^1, L^\infty) \leq K(t, f; L^1, L^\infty)$ and $K(t, g; L^1, L^\infty) \leq K(t, g^*; L^1, L^\infty)$, Calderón's theorem provides $V_f, V_g : (L^1, L^\infty) \rightarrow (L^1, L^\infty)$, each with norm at most 1, such that $V_f f = f^*$ and $V_g g^* = g$. Exact interpolation shows that $V_f : (X_0, X_1) \rightarrow (X_0, X_1)$ and $V_g : (Y_0, Y_1) \rightarrow (Y_0, Y_1)$, each with norm at most 1.

By Definition 2.4 there exists an $M : (L^1, L^\infty) \rightarrow (\widetilde{L}^1, \widetilde{L}^\infty)$, with norm at most m , such that $M\widetilde{f}^* = f^*$, that is, $Mf^* = f^*$. In Theorem 2.3(b) it was observed that the identity operator I maps $(\widetilde{L}^1, \widetilde{L}^\infty)$ to (L^1, L^∞) , with norm at most 1. By Lemma 3.4(a), $M : (X_0, X_1) \rightarrow (\widetilde{X}_0, \widetilde{X}_1)$, with norm at most m , and $I : (\widetilde{Y}_0, \widetilde{Y}_1) \rightarrow (Y_0, Y_1)$, with norm at most 1. Now,

$$\begin{array}{ccccccccccc} (X_0, X_1) & \xrightarrow{V_f} & (X_0, X_1) & \xrightarrow{M} & (\widetilde{X}_0, \widetilde{X}_1) & \xrightarrow{M} & (\widetilde{X}_0, \widetilde{X}_1) & \xrightarrow{T} & (\widetilde{Y}_0, \widetilde{Y}_1) & \xrightarrow{V_g} & (Y_0, Y_1) & \xrightarrow{V_g} & (Y_0, Y_1) \\ f & \longmapsto & f^* & \longmapsto & f^* & \longmapsto & mf^* & \longmapsto & g^* & \longmapsto & g^* & \longmapsto & g \end{array}$$

That is, with $\bar{T} = V_g I T m M V_f$, $\bar{T} : (X_0, X_1) \rightarrow (Y_0, Y_1)$, with norm at most $m^2 c$, and $\bar{T}f = g$. This shows that the couples (X_0, X_1) and (Y_0, Y_1) form an $(m^2 c)$ -uniform relative Calderón-Mityagin pair and completes the proof. \square

Next we establish similar results for $(L^1, (L^\infty)^\circ)$.

Theorem 3.6. *Let n be a constant satisfying Definition 2.7. Then,*

- (a) $(L^1, (L^\infty)^\circ)$ is an exact Calderón-Mityagin couple.
- (b) (L^1, L^∞) and $(L^1, (L^\infty)^\circ)$ form an exact relative Calderón-Mityagin pair.
- (c) $(L^1, (L^\infty)^\circ)$ and (L^1, L^∞) form an n -uniform relative Calderón-Mityagin pair.
- (d) $(L^1, (L^\infty)^\circ)$ and (L^1, L^∞) do not form an exact relative Calderón-Mityagin pair.

Proof. Statement (a) is Theorem 3.8 of [13]. For (b) and (c), operator names refer to those constructed in Theorem 2.6. Refer to the diagram (2.2).

To prove (b), suppose $h \in L^1 + L^\infty$ and $g \in L^1 + (L^\infty)^\circ$ satisfy the inequality $K(t, g; L^1, (L^\infty)^\circ) \leq K(t, h; L^1, L^\infty)$ for $t > 0$, and let $T = U_3 I V_4$. All three have norm at most 1 so $T : (L^1, L^\infty) \rightarrow (L^1, (L^\infty)^\circ)$ has norm at most 1 as well. Also, $Th = U_3 I V_4 h = U_3 I g^\circ = U_3 g^\circ = g$.

To prove (c), suppose $f \in L^1 + (L^\infty)^\circ$ and $h \in L^1 + L^\infty$ satisfy $K(t, h; L^1, L^\infty) \leq K(t, f; L^1, (L^\infty)^\circ)$ for $t > 0$, and let $T = V_3 N$, where N comes from Definition 2.7. The operator V_3 has norm at most one and N has norm at most n . Thus $T : (L^1, (L^\infty)^\circ) \rightarrow (L^1, L^\infty)$ has norm at most n . Also, $Tf = V_3 N f = V_3 f^\circ = h$.

Lemma 2.8 proves (d). \square

Theorem 3.7. *If $X \in \text{Int}_1(L^1, L^\infty)$ then $X^\circ \in \text{Int}_1(L^1, (L^\infty)^\circ)$. Conversely, if Y is in $\text{Int}_1(L^1, (L^\infty)^\circ)$ then $Y = X^\circ$, with identical norms, for some space X in $\text{Int}_1(L^1, L^\infty)$.*

Proof. If $X \in \text{Int}_1(L^1, L^\infty)$ we apply the theorems of Calderón and Brudnyĭ-Krugljak to get a parameter Φ of the K -method such that $X = (L^1, L^\infty)_\Phi$, with identical norms. By Proposition 2.1, for any $f \in L^1 + (L^\infty)^\circ$, the five statements: $f \in X^\circ$, $f^\circ \in X$, $K(\cdot, f^\circ; L^1, L^\infty) \in \Phi$, $K(\cdot, f; L^1, (L^\infty)^\circ) \in \Phi$ and $f \in (L^1, (L^\infty)^\circ)_\Phi$ are all equivalent. Moreover,

$$\|f\|_{X^\circ} = \|f^\circ\|_X = \|K(\cdot, f^\circ; L^1, L^\infty)\|_\Phi = \|K(\cdot, f; L^1, (L^\infty)^\circ)\|_\Phi = \|f\|_{(L^1, (L^\infty)^\circ)_\Phi}.$$

This shows $X^\circ = (L^1, (L^\infty)^\circ)_\Phi$, with identical norms, so $X^\circ \in \text{Int}_1(L^1, (L^\infty)^\circ)$.

On the other hand, if $Y \in \text{Int}_1(L^1, (L^\infty)^\circ)$ then by Theorem 3.6(a), the Brudnyĭ-Krugljak theorem, and [13, Corollary 3.9] (which shows that the K -divisibility constant of $(L^1, (L^\infty)^\circ)$ is 1) there exists a parameter Φ of the K -method such that $Y = (L^1, (L^\infty)^\circ)_\Phi$, with identical norms. Let $X = (L^1, L^\infty)_\Phi \in \text{Int}_1(L^1, L^\infty)$. Arguing as above we see that $X^\circ = (L^1, (L^\infty)^\circ)_\Phi = Y$, with identical norms. \square

Lemma 3.8. *Let n be a constant satisfying Definition 2.7. Suppose X_0 and X_1 are in $\text{Int}_1(L^1, L^\infty)$ and \mathcal{F} is an exact interpolation functor. Set $X = \mathcal{F}(X_0, X_1)$. Then*

- (a) *There exist Φ_0 and Φ_1 , parameters of the K -method, such that $X_j = (L^1, L^\infty)_{\Phi_j}$ and $X_j^\circ = (L^1, (L^\infty)^\circ)_{\Phi_j}$, in each case with identical norms, for $j = 0, 1$.*
- (b) *$\mathcal{F}(X_0^\circ, X_1^\circ) \hookrightarrow X^\circ$ with norm at most n .*
- (c) *$X^\circ \hookrightarrow \mathcal{F}(X_0^\circ, X_1^\circ)$ with norm at most 1.*

(d) $X_0^\circ + X_1^\circ = (X_0 + X_1)^\circ$, and for all f in this space,

$$(1/n)K(t, f^\circ; X_0, X_1) \leq K(t, f; X_0^\circ, X_1^\circ) \leq K(t, f^\circ; X_0, X_1), \quad t > 0.$$

Proof. The proof of part (a) is contained in the proof of Theorem 3.7. As a consequence we have $X_j^\circ \subseteq L^1 + (L^\infty)^\circ$ for $j = 0, 1$. It follows that $F(X_0^\circ, X_1^\circ) \subseteq L^1 + (L^\infty)^\circ$. Fix $f \in F(X_0^\circ, X_1^\circ)$. By Definition 2.7 there exists a linear operator $N : (L^1, (L^\infty)^\circ) \rightarrow (L^1, L^\infty)$, of norm at most n , such that $Nf = f^\circ$. By real interpolation with parameter Φ_j , $N : X_j^\circ \rightarrow X_j$ with norm at most n , for $j = 0, 1$. Applying the exact functor \mathcal{F} shows $N : \mathcal{F}(X_0^\circ, X_1^\circ) \rightarrow X$, with norm at most n . Since $f^\circ = Nf \in X$ we have $f \in X^\circ$ and $\|f\|_{X^\circ} = \|f^\circ\|_X = \|Nf\|_X \leq n\|f\|_{\mathcal{F}(X_0^\circ, X_1^\circ)}$. This proves (b).

Now let $g \in X^\circ$. Then $g^\circ \in X = \mathcal{F}(X_0, X_1) \subseteq L^1 + L^\infty$ so $g \in L^1 + (L^\infty)^\circ$. Proposition 2.1 shows that $K(t, g^\circ; L^1, L^\infty) = K(t, g; L^1, (L^\infty)^\circ)$ so Theorem 2.6 provides operators I and U_3 such that $U_3I : (L^1, L^\infty) \rightarrow (L^1, (L^\infty)^\circ)$, with norm at most 1, and $U_3Ig^\circ = g$. By real interpolation, $U_3I : X_j \rightarrow (L^1, (L^\infty)^\circ)_{\Phi_j} = X_j^\circ$, with norm at most 1, for $j = 0, 1$. Applying \mathcal{F} , we get, $U_3I : X \rightarrow \mathcal{F}(X_0^\circ, X_1^\circ)$, with norm at most 1. In particular, $\|g\|_{\mathcal{F}(X_0^\circ, X_1^\circ)} = \|U_3Ig^\circ\|_{\mathcal{F}(X_0^\circ, X_1^\circ)} \leq \|g^\circ\|_X = \|g\|_{X^\circ}$. This proves (c).

For part (d) we apply (b) and (c) to the family of exact functors used in Lemma 3.4; $\Sigma_t(Z_0, Z_1) = Z_0 + Z_1$ with norm $\|f\|_{\Sigma_t(Z_0, Z_1)} = K(t, f; Z_0, Z_1)$. The embeddings of (b) and (c) show that $X_0^\circ + X_1^\circ = (X_0 + X_1)^\circ$ and, for all f in this space, the estimates of the constants in (b) and (c) show

$$K(t, f^\circ; X_0, X_1) = \|f^\circ\|_{\Sigma_t(X_0, X_1)} \leq n\|f\|_{\Sigma_t(X_0, X_1^\circ)} = nK(t, f; X_0^\circ, X_1^\circ),$$

and

$$K(t, f; X_0^\circ, X_1^\circ) = \|f\|_{\Sigma_t(X_0^\circ, X_1^\circ)} \leq \|f^\circ\|_{\Sigma_t(X_0, X_1)} = K(t, f^\circ; X_0, X_1). \quad \square$$

When considering couples of spaces in $\text{Int}_1(L^1, (L^\infty)^\circ)$, as we do in the converse below, there is no loss in generality in considering only couples of the form (X_0°, X_1°) for $X_0, X_1 \in \text{Int}_1(L^1, L^\infty)$. This is a consequence of Theorem 3.7.

Theorem 3.9. *Let n be a constant satisfying Definition 2.7 and suppose that X_0, X_1, Y_0 , and Y_1 are all in $\text{Int}_1(L^1, L^\infty)$. If (X_0, X_1) and (Y_0, Y_1) form a c -uniform relative Calderón-Mityagin pair then (X_0°, X_1°) and (Y_0°, Y_1°) form an (n^2c) -uniform relative Calderón-Mityagin pair. Conversely, if (X_0°, X_1°) and (Y_0°, Y_1°) form a c -uniform relative Calderón-Mityagin pair then (X_0, X_1) and (Y_0, Y_1) form an (n^2c) -uniform relative Calderón-Mityagin pair.*

Proof. Suppose (X_0, X_1) and (Y_0, Y_1) form a c -uniform relative Calderón-Mityagin pair, $f \in X_0^\circ + X_1^\circ$, $g \in Y_0^\circ + Y_1^\circ$, and $K(t, g; Y_0^\circ, Y_1^\circ) \leq K(t, f; X_0^\circ, X_1^\circ)$ for $t > 0$. By Lemma 3.8(d) and the homogeneity of the K -functional,

$$K(t, g^\circ; Y_0, Y_1) \leq nK(t, g; Y_0^\circ, Y_1^\circ) \leq nK(t, f; X_0^\circ, X_1^\circ) \leq K(t, nf^\circ; X_0, X_1).$$

By hypothesis, there exists a linear operator $T : (X_0, X_1) \rightarrow (Y_0, Y_1)$, of norm at most c , such that $T(nf) = g^\circ$. Since $f \in X_0^\circ + X_1^\circ$, $f^\circ \in X_0 + X_1 \subseteq L^1 + L^\infty$. The product $U_3I : (L^1, L^\infty) \rightarrow (L^1, (L^\infty)^\circ)$, from Theorem 2.6, has norm at most 1 and takes g° to g . The operator $N : (L^1, (L^\infty)^\circ) \rightarrow (L^1, L^\infty)$, from Definition 2.7, has norm at most n and takes f to f° . By Lemma 3.8(a), $U_3I : (Y_0, Y_1) \rightarrow (Y_0^\circ, Y_1^\circ)$

has norm at most 1, and $N : (X_0^o, X_1^o) \rightarrow (X_0, X_1)$ has norm at most n . It follows that

$$\begin{array}{ccccccc} (X_0^o, X_1^o) & \xrightarrow{N} & (X_0, X_1) & \xrightarrow{n} & (X_0, X_1) & \xrightarrow{T} & (Y_0, Y_1) & \xrightarrow{U_3 I} & (Y_0^o, Y_1^o) \\ f & \longmapsto & f^o & \longmapsto & n f^o & \longmapsto & g^o & \longmapsto & g \end{array}$$

where n is used to denote multiplication by n . That is, the operator $\bar{T} = U_3 I T n N$ maps (X_0^o, X_1^o) to (Y_0^o, Y_1^o) , with norm at most $n^2 c$, and $\bar{T} f = g$. We conclude that (X_0^o, X_1^o) and (Y_0^o, Y_1^o) form an $(n^2 c)$ -uniform relative Calderón-Mityagin pair.

Conversely, suppose (X_0^o, X_1^o) and (Y_0^o, Y_1^o) form a c -uniform relative Calderón-Mityagin pair, $f \in X_0 + X_1$, $g \in Y_0 + Y_1$ and $K(t, g; Y_0, Y_1) \leq K(t, f; X_0, X_1)$. Since g and g^* are equimeasurable, and f and f^* are equimeasurable,

$$K(t, g^*; Y_0, Y_1) = K(t, g; Y_0, Y_1) \leq K(t, f; X_0, X_1) = K(t, f^*; X_0, X_1).$$

But $g^* = (g^*)^o$ and $f^* = (f^*)^o$, so by Lemma 3.8(d) and the homogeneity of the K -functional,

$$K(t, g^*; Y_0^o, Y_1^o) \leq K(t, (g^*)^o; Y_0, Y_1) \leq K(t, (f^*)^o; X_0, X_1) \leq K(t, n f^*; X_0^o, X_1^o).$$

By the hypothesis, there exists a $T : (X_0^o, X_1^o) \rightarrow (Y_0^o, Y_1^o)$, with norm at most c such that $T(n f^*) = g^*$.

Since $K(t, f^*; L^1, L^\infty) \leq K(t, f; L^1, L^\infty)$ and $K(t, g; L^1, L^\infty) \leq K(t, g^*; L^1, L^\infty)$, Calderón's theorem provides $V_f, V_g : (L^1, L^\infty) \rightarrow (L^1, L^\infty)$, each with norm at most 1, such that $V_f f = f^*$ and $V_g g^* = g$. Exact interpolation shows that $V_f : (X_0, X_1) \rightarrow (X_0, X_1)$ and $V_g : (Y_0, Y_1) \rightarrow (Y_0, Y_1)$, each with norm at most 1.

By Definition 2.7 there exists an $N : (L^1, (L^\infty)^o) \rightarrow (L^1, L^\infty)$, with norm at most n , such that $N g^* = (g^*)^o$, that is, $N g^* = g^*$. In Theorem 2.6(b) it was observed that the identity operator I maps (L^1, L^∞) to $(L^1, (L^\infty)^o)$, with norm at most 1. By exact interpolation, $I : (X_0, X_1) \rightarrow (X_0^o, X_1^o)$, with norm at most 1, and $N : (Y_0^o, Y_1^o) \rightarrow (Y_0, Y_1)$, with norm at most n . Now,

$$\begin{array}{ccccccc} (X_0, X_1) & \xrightarrow{V_f} & (X_0, X_1) & \xrightarrow{I} & (X_0^o, X_1^o) & \xrightarrow{N} & (X_0^o, X_1^o) & \xrightarrow{T} & (Y_0^o, Y_1^o) & \xrightarrow{N} & (Y_0, Y_1) & \xrightarrow{V_g} & (Y_0, Y_1) \\ f & \longmapsto & f^* & \longmapsto & f^* & \longmapsto & n f^* & \longmapsto & g^* & \longmapsto & g^* & \longmapsto & g \end{array}$$

That is, with $\bar{T} = V_g N T n I V_f$, $\bar{T} : (X_0, X_1) \rightarrow (Y_0, Y_1)$, with norm at most $n^2 c$, and $\bar{T} f = g$. This shows that (X_0, X_1) and (Y_0, Y_1) form an $(n^2 c)$ -uniform relative Calderón-Mityagin pair and completes the proof. \square

It is interesting to observe the special case $(X_0, X_1) = (Y_0, Y_1)$ of Theorems 3.5 and 3.9, stated here without careful tracking of constants.

Corollary 3.10. *For $X_0, X_1 \in \text{Int}_1(L^1, L^\infty)$ the following are equivalent:*

- (a) (X_0, X_1) is a Calderón-Mityagin couple.
- (b) $(\tilde{X}_0, \tilde{X}_1)$ is a Calderón-Mityagin couple.
- (c) (X_0^o, X_1^o) is a Calderón-Mityagin couple.

4. EXTENSION TO GENERAL MEASURES

There are natural analogues of the operations $f \mapsto \tilde{f}$ and $f \mapsto f^o$ in spaces of λ -measurable functions on \mathbb{R} , provided $\lambda(-\infty, x] < \infty$ for $x \in \mathbb{R}$. The corresponding spaces \tilde{X} and X^o enjoy the same close connection to rearrangement invariant spaces that they do in the case of Lebesgue measure on $(0, \infty)$. In particular, most of the

results of Sections 2 and 3 remain valid for functions on a finite interval, for sequence spaces.

Our approach to proving these results is to identify the spaces of λ -measurable functions with complemented subspaces of the corresponding spaces over $(0, \infty)$ and then apply the previous results.

To begin, let us introduce the analogues of the least decreasing majorant and the level function. Let λ be a measure on \mathbb{R} for which the Borel subsets are measurable and $\Lambda(x) \equiv \lambda(-\infty, x] < \infty$ for all $x \in \mathbb{R}$. The least decreasing majorant of a λ -measurable f is

$$\tilde{f}(x) = \text{ess sup}_\lambda \{|f(y)| : y \geq x\}.$$

If X is a Banach function space of λ -measurable functions, let \tilde{X} be the space of all λ -measurable functions such that $\tilde{f} \in X$, equipped with the norm $\|f\|_{\tilde{X}} = \|\tilde{f}\|_X$.

We say a non-negative, λ -measurable function F on \mathbb{R} is λ -concave provided

$$(\Lambda(b) - \Lambda(x))(F(x) - F(a)) \geq (F(b) - F(x))(\Lambda(x) - \Lambda(a))$$

whenever $a \leq x \leq b$. (Note that when λ is Lebesgue measure on $(0, \infty)$ this agrees with the usual notion of concavity.) The *level function* f° of f is the Radon-Nikodym derivative, with respect to λ , of the least λ -concave majorant of $\int_{(-\infty, x]} f d\lambda$. As in the case of Lebesgue measure on $(0, \infty)$, it may be necessary to extend this definition by monotonicity in case the λ -concave majorant fails to be finite. If $X \in \text{Int}_1(L_\lambda^1, L_\lambda^\infty)$, let X° be the space of all λ -measurable functions such that $f^\circ \in X$, equipped with the norm $\|f\|_{X^\circ} = \|f^\circ\|_X$.

Notice that the operations $f \mapsto \tilde{f}$ and $f \mapsto f^\circ$, as well as the spaces \tilde{X} and X° , depend on the measure λ . To avoid confusion with these “overloaded” operators, we will be careful to associate each function with a particular measure before speaking of its least decreasing majorant or its level function.

We again observe that $(L_\lambda^1)^\circ = L_\lambda^1$ and $(L_\lambda^\infty)^\circ = L_\lambda^\infty$ with identical norms. Also, $\|f\|_{L_\lambda^1} \leq \|f\|_{L_\lambda^\infty}$ and

$$\|f\|_{(L_\lambda^\infty)^\circ} = \sup_{x \in \mathbb{R}} \frac{1}{\Lambda(x)} \int_{(-\infty, x]} |f| d\lambda \leq \|f\|_{L_\lambda^\infty}.$$

The *method of retracts* (see pages 54f in [5]) embeds a σ -finite measure space into a non-atomic one and exploits the close connection between spaces of measurable functions with respect to these two measures. When the original measure λ is on \mathbb{R} and satisfies $\lambda(-\infty, x] < \infty$ for each x , this method takes a particularly simple and powerful form that, in particular, preserves decreasing functions. This construction, from Section 4 of [13], gives an order-preserving, measurable transformation from (\mathbb{R}, λ) into a subspace of $(0, \infty)$ with Lebesgue measure. Define Ω , φ , E_λ , and A_λ by

$$\begin{aligned} \Omega &= \{t > 0 : t \leq \Lambda(y) \text{ for some } y \in \mathbb{R}\}, \\ \varphi(t) &= \inf\{y : t \leq \Lambda(y)\}, \quad t \in \Omega, \\ E_\lambda f &= (f \circ \varphi)\chi_\Omega, \quad \text{and} \\ A_\lambda h(t) &= \begin{cases} \frac{\chi_\Omega(t)}{|I|} \int_I h, & t \in I \in \mathcal{I}_\lambda \\ \chi_\Omega(t)h(t), & t \notin \cup I \in \mathcal{I}_\lambda. \end{cases} \end{aligned}$$

Here \mathcal{I}_λ denotes the collection of all non-empty intervals $(\Lambda(y-), \Lambda(y)] \subseteq \Omega$ for $y \in \mathbb{R}$.

The map E_λ takes λ -measurable functions to Lebesgue measurable functions, respecting the level function and least decreasing majorant constructions. Theorem 5.3 of [13] shows that $(E_\lambda f)^\circ = E_\lambda f^\circ$ on Ω . Note that f° is the level function of f with respect to λ , while $(E_\lambda f)^\circ$ is the level function of $E_\lambda f$ with respect to Lebesgue measure. The corresponding result for the least decreasing majorant is $\widetilde{E_\lambda f} = E_\lambda \widetilde{f}$ on Ω . To see this, let $t \in \Omega$. Then

$$\widetilde{E_\lambda f}(t) = \text{ess sup}\{f(\varphi(s)) : t \leq s \in \Omega\} \leq \text{ess sup}_\lambda\{f(x) : \varphi(t) \leq x\} = E_\lambda \widetilde{f}(t)$$

and, since $\varphi(t) \leq x$ if and only if $t \leq \Lambda(x)$ and $x = \varphi(\Lambda(x))$ λ -almost everywhere (see (5) and Lemma 4.1 of [13]),

$$\begin{aligned} E_\lambda \widetilde{f}(t) &= \text{ess sup}_\lambda\{f(\varphi(\Lambda(x))) : t \leq \Lambda(x)\} \\ &\leq \text{ess sup}\{f(\varphi(s)) : t \leq s \in \Omega\} = \widetilde{E_\lambda f}(t). \end{aligned}$$

The maps E_λ and A_λ work together to identify function spaces of λ -measurable functions with subspaces of the corresponding spaces of Lebesgue measurable functions. This is because A_λ is a projection onto the range of E_λ .

Proposition 4.1. *The map A_λ is a projection defined on $L^1 + (L^\infty)^\circ$ and is a contraction on L^1 , $(L^\infty)^\circ$, $\widetilde{L^1}$, and L^∞ . Also, $A_\lambda E_\lambda = E_\lambda$, and the maps $E_\lambda : L_\lambda^1 \leftrightarrow A_\lambda(L^1)$, $E_\lambda : (L_\lambda^\infty)^\circ \leftrightarrow A_\lambda((L^\infty)^\circ)$, $E_\lambda : \widetilde{L_\lambda^1} \leftrightarrow A_\lambda(\widetilde{L^1})$, and $E_\lambda : L_\lambda^\infty \leftrightarrow A_\lambda(L^\infty)$, are bijective isometries.*

Proof. It is easy to see that A_λ is a projection defined on $L^1 + (L^\infty)^\circ$ and that it is a contraction on L^∞ . Corollary 3.2 of [13] shows that A_λ is a contraction on L^1 and $(L^\infty)^\circ$. Since the averages in A_λ are all taken over intervals, A_λ maps non-negative decreasing functions to non-negative decreasing functions. Therefore, $\widetilde{A_\lambda f} \leq A_\lambda \widetilde{f} = A_\lambda \widetilde{f}$, so $\|A_\lambda f\|_{\widetilde{L^1}} \leq \|A_\lambda \widetilde{f}\|_{L^1} \leq \|\widetilde{f}\|_{L^1} = \|f\|_{\widetilde{L^1}}$. Thus, A_λ is a contraction on $\widetilde{L^1}$.

Lemma 4.4 of [13] shows $A_\lambda E_\lambda = E_\lambda$ and that $E_\lambda : L_\lambda^1 \leftrightarrow A_\lambda(L^1)$ and $E_\lambda : (L_\lambda^\infty)^\circ \leftrightarrow A_\lambda((L^\infty)^\circ)$ are bijective isometries. Since $\widetilde{L_\lambda^1} \subseteq L_\lambda^1$ and $L_\lambda^\infty \subseteq (L_\lambda^\infty)^\circ$ $E_\lambda : \widetilde{L_\lambda^1} \leftrightarrow A_\lambda(\widetilde{L^1})$ and $E_\lambda : L_\lambda^\infty \leftrightarrow A_\lambda(L^\infty)$ are both bijections. It remains to show that they are isometric. For $f \in \widetilde{L_\lambda^1}$,

$$\|E_\lambda f\|_{\widetilde{L^1}} = \|\widetilde{E_\lambda f}\|_{L^1} = \|E_\lambda \widetilde{f}\|_{L^1} = \|\widetilde{f}\|_{L_\lambda^1} = \|f\|_{\widetilde{L_\lambda^1}}.$$

It is an easy consequence of Lemma 4.2 of [13] that f and $E_\lambda f$ are equimeasurable. Therefore $\|E_\lambda f\|_{L^\infty} = \|f\|_{L_\lambda^\infty}$. \square

Corollary 4.2. (a) *If $f \in L_\lambda^1 + L_\lambda^\infty$ then $K(t, f; L_\lambda^1, L_\lambda^\infty) = K(t, E_\lambda f; L^1, L^\infty)$.*

(b) *If $f \in \widetilde{L_\lambda^1} + L_\lambda^\infty$ then*

$$K(t, f; \widetilde{L_\lambda^1}, L_\lambda^\infty) = K(t, E_\lambda f; \widetilde{L^1}, L^\infty) = K(t, \widetilde{f}; L_\lambda^1, L_\lambda^\infty).$$

(c) *If $f \in L_\lambda^1 + (L_\lambda^\infty)^\circ$ then*

$$K(t, f; L_\lambda^1, (L_\lambda^\infty)^\circ) = K(t, E_\lambda f; L^1, (L^\infty)^\circ) = K(t, f^\circ; L_\lambda^1, L_\lambda^\infty).$$

Proof. Fix $f \in L_\lambda^1 + L_\lambda^\infty$. If $f = f_0 + f_1$ with $f_0 \in L_\lambda^1$ and $f_1 \in L_\lambda^\infty$, then $E_\lambda f = E_\lambda f_0 + E_\lambda f_1$ so

$$K(t, E_\lambda f; L^1, L^\infty) \leq \|E_\lambda f_0\|_{L^1} + t \|E_\lambda f_1\|_{L^\infty} = \|f_0\|_{L_\lambda^1} + t \|f_1\|_{L_\lambda^\infty}.$$

Take the infimum over all such f_0 and f_1 to get “ \geq ” in (a).

For the reverse inequality, suppose $E_\lambda f = h_0 + h_1$ with $h_0 \in L^1$ and $h_1 \in L^\infty$, then

$$f = E_\lambda^{-1} A_\lambda E_\lambda f = E_\lambda^{-1} A_\lambda h_0 + E_\lambda^{-1} A_\lambda h_1.$$

But $E_\lambda^{-1} A_\lambda$ has norm at most 1 on both L^1 and L^∞ , so

$$K(t, f; L_\lambda^1, L_\lambda^\infty) \leq \|E_\lambda^{-1} A_\lambda h_0\|_{L_\lambda^1} + t \|E_\lambda^{-1} A_\lambda h_1\|_{L_\lambda^\infty} \leq \|h_0\|_{L^1} + t \|h_1\|_{L^\infty}.$$

Take the infimum over all such h_0 and h_1 to get “ \leq ” in (a).

With L_λ^1 and L^1 replaced by \widetilde{L}_λ^1 and \widetilde{L}^1 , the same argument proves the first equation in (b). The other follows from Proposition 2.1 and part (a).

With L_λ^∞ and L^∞ replaced by $(L_\lambda^\infty)^\circ$ and $(L^\infty)^\circ$, the same argument also proves the first equation in (c). The other follows from Proposition 2.1 and part (a). \square

Here is our extension of Theorems 3.1 and 3.6 to general measures. Interestingly, they generalize to the case of two measures as easily as to one.

Theorem 4.3. *Let m and n be constants satisfying Definitions 2.4 and 2.7, respectively. Then,*

- (a) $(\widetilde{L}_\mu^1, L_\mu^\infty)$ is an exact Calderón-Mityagin couple.
- (b) $(\widetilde{L}_\mu^1, L_\mu^\infty)$ and $(\widetilde{L}_\nu^1, L_\nu^\infty)$ form an exact relative Calderón-Mityagin pair.
- (c) $(\widetilde{L}_\mu^1, L_\mu^\infty)$ and (L_ν^1, L_ν^∞) form an exact relative Calderón-Mityagin pair.
- (d) (L_μ^1, L_μ^∞) and $(\widetilde{L}_\nu^1, L_\nu^\infty)$ form an m -uniform relative Calderón-Mityagin pair.
- (e) $(L_\mu^1, (L_\mu^\infty)^\circ)$ is an exact Calderón-Mityagin couple.
- (f) $(L_\mu^1, (L_\mu^\infty)^\circ)$ and $(L_\nu^1, (L_\nu^\infty)^\circ)$ form an exact relative Calderón-Mityagin pair.
- (g) (L_μ^1, L_μ^∞) and $(L_\nu^1, (L_\nu^\infty)^\circ)$ form an exact relative Calderón-Mityagin pair.
- (h) $(L_\mu^1, (L_\mu^\infty)^\circ)$ and (L_ν^1, L_ν^∞) form an n -uniform relative Calderón-Mityagin pair.

Proof. First note that (a) and (e) are just the case $\nu = \mu$ of (b) and (f), respectively. We prove part (d); the remaining parts may be proved in a similar fashion.

Suppose $f \in L_\mu^1 + L_\mu^\infty$ and $g \in \widetilde{L}_\nu^1 + L_\nu^\infty$ satisfy $K(t, g; \widetilde{L}_\nu^1, L_\nu^\infty) \leq K(t, f; L_\mu^1, L_\mu^\infty)$. Then $K(t, E_\nu g; \widetilde{L}^1, L^\infty) \leq K(t, E_\mu f; L^1, L^\infty)$ so, by Theorem 3.1(c), there exists a linear operator $T : (L^1, L^\infty) \rightarrow (\widetilde{L}^1, L^\infty)$, of norm at most m , such that $T(E_\mu f) = E_\nu g$. The operator $\bar{T} = E_\nu^{-1} A_\nu T E_\mu : (L_\mu^1, L_\mu^\infty) \rightarrow (\widetilde{L}_\nu^1, L_\nu^\infty)$ has norm at most m and

$$\bar{T}f = E_\nu^{-1} A_\nu T E_\mu f = E_\nu^{-1} A_\nu E_\nu g = g. \quad \square$$

We leave it to the reader to verify that Lemmas 3.2, 3.4 and 3.8 and Theorems 2.3, 2.6, 3.3 and 3.7 hold with L^1 , L^∞ , \widetilde{L}^1 , and $(L^\infty)^\circ$ replaced by L_λ^1 , L_λ^∞ , \widetilde{L}_λ^1 , and $(L_\lambda^\infty)^\circ$, respectively. Essentially one replaces each operator O by $E_\lambda^{-1} A_\lambda O E_\lambda$, applying Corollary 4.2 when needed. Note that Corollary 3.9 of [13] was extended to general measures in Corollary 4.7 of the same paper.

The first statement in each of Theorems 3.5 and 3.9 may be extended to the general measure case by the same procedure (and this includes extensions of the implications (a) \implies (b) and (a) \implies (c) of Corollary 3.10). However, it appears likely that the second statement of each will fail for certain measures. The example below shows that the couple $(L_\lambda^1, L_\lambda^\infty)$ may have a rich collection of exact interpolation spaces, while both $(\widetilde{L}_\lambda^1, L_\lambda^\infty)$ and $(L_\lambda^1, (L_\lambda^\infty)^\circ)$ become trivial.

Example 4.4. Let λ be the probability measure on \mathbb{R} consisting of atoms of weight 2^{-k} at k , for $k \in \mathbb{N}$. If $X \in \text{Int}_1(L_\lambda^1, L_\lambda^\infty)$ then $\tilde{X} = L_\lambda^\infty$ with equivalent norms and $X^\circ = L_\lambda^1$ with equivalent norms.

Proof. The constant function 1 is in $L_\lambda^1 \cap L_\lambda^\infty$ and hence in X . Thus $0 < \|1\|_X < \infty$. A calculation shows that

$$1^{**} = \min(1, \frac{1}{x}) \leq 2 \min(1, \frac{1}{2x}) = 2\chi_{\{1\}}^{**}.$$

Since $X \in \text{Int}_1(L_\lambda^1, L_\lambda^\infty)$, $\|1\|_X \leq 2\|\chi_{\{1\}}\|_X$. For any λ -measurable f , $\|f\|_{L_\lambda^\infty} \chi_{\{1\}} \leq \tilde{f} \leq \|f\|_{L_\lambda^\infty}$ and therefore

$$\|f\|_{\tilde{X}} = \|\tilde{f}\|_X \leq \|f\|_{L_\lambda^\infty} \|1\|_X \leq 2\|f\|_{L_\lambda^\infty} \|\chi_{\{1\}}\|_X \leq 2\|\tilde{f}\|_X = 2\|f\|_{\tilde{X}}.$$

This proves the first statement. Since $f^\circ(1)\chi_{\{1\}} \leq f^\circ \leq f^\circ(1)$,

$$\|f\|_{X^\circ} = \|f^\circ\|_X \leq f^\circ(1)\|1\|_X \leq 2f^\circ(1)\|\chi_{\{1\}}\|_X \leq 2\|f^\circ\|_X = 2\|f\|_{X^\circ}.$$

Combining this with

$$\|f\|_{L_\lambda^1} = \int_0^\infty f^\circ d\lambda \leq f^\circ(1) = 2f^\circ(1)\lambda\{1\} \leq 2 \int_0^\infty f^\circ d\lambda = 2\|f\|_{L_\lambda^1}$$

proves the second statement. \square

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