MONOTONICITY IN BANACH FUNCTION SPACES

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November 19, 2006

Abstract. This paper is an informal presentation of material from [28–34]. The monotone envelopes of a function, including the level function, are introduced and their properties are studied. Applications to norm inequalities are given. The down space of a Banach function space is defined and connections are made between monotone envelopes and the norms of the down space and its dual. The connection is shown to be particularly close in the case of universally rearrangement invariant spaces. Next, two equivalent norms are given for the down spaces and these are applied to advance a factorization theory for Hardy inequalities and to characterize embeddings of the classes of generalized quasiconcave functions between Lebesgue spaces. This embedding theory is, in turn, applied to find an expression for the dual space of Lorentz Γ-space and to find necessary and sufficient conditions for the boundedness of the Fourier transform, acting as a map between Lorentz spaces. A new Lorentz space, the Θ-space, is introduced and shown to be the key to extending the characterization of Fourier inequalities to a greater range of Lorentz spaces. Finally, the scale of down spaces of universally rearrangement invariant spaces is completely characterized by means of interpolation theory, when it is shown that the down spaces of L^1 and L^∞ (with general measures) form a Calderón couple.

1. Introduction

Monotone functions seem almost too simple to study seriously. What hidden structure could there be in such straightforward, concrete objects that would warrant an abstract treatment, a *theory* of monotone functions? In these lectures I hope to convince you that a theory of increasing or decreasing functions need not be trivial, that it is worth developing because it does reveal a rich structure, and

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2000 *Mathematics Subject Classification.* Primary 26D15 Secondary 46E30, 46B70.

Key words and phrases. Monotone envelope, level function, pushing mass, down space, Hardy inequality, Lorentz pace, rearrangement invariant space, quasiconcave function, Fourier inequality, interpolation, Calderón couple.

Support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged

that it can shed light on many seemingly unrelated problems, both old and new, solved and unsolved.

The first observation that we make on our way toward an abstract treatment of monotone functions (to fix ideas we focus on decreasing functions) is that considerable insight can be gained by investigating the functionals represented by such functions. In particular this point of view will be crucial when we introduce partial orders and their associated monotone envelopes.

Next we take a tried and true step in the process of abstraction by passing from individual objects to collections of these objects. We define a class of Banach Function Spaces, called *Down Spaces*, whose structure is determined by the decreasing elements in the base spaces. As we will see, the functional approach makes it simple to exhibit the close connection between down spaces and monotone envelopes. Using down spaces we can apply the whole theory of function spaces to the collection of decreasing functions in a given space.

Another way to increase the level of abstraction is work with generalized quasiconcave functions, functions satisfying two monotonicity conditions, instead of simple decreasing functions. Our abstract approach simplifies many of the technical problems that have arisen in work on quasiconcave functions in the past. As is often the case, simpler methods enable us to push results farther and we benefit in that way here when we look at embeddings of quasiconcave functions from one Lebesgue space to another. Other applications of this approach to quasiconcave functions include a formula for the dual of the Lorentz Γ-space and a characterization of Fourier inequalities involving Lorentz space norms.

In the last lecture we raise the level of abstraction to the next level when we investigate the scale of down spaces associated to universally rearrangement invariant spaces and also the scale of their dual spaces. These are completely described using powerful results from interpolation theory in a series of results that connect monotone envelopes and K-functionals, construct a large class of operators on down spaces, and establish a simple correspondence between a universally rearrangement invariant space and its down space.

2. Monotone Envelopes

Three Partial Orders.

Fix a measure λ on $\mathbf{R}$ satisfying $\lambda(-\infty, x] < \infty$ for all $x \in \mathbf{R}$. A λ-measurable function f is determined by its values $f(x)$ for λ-almost every point x. It is equally well determined by the values, $\int fh\,d\lambda$ for a λ-measurable function h, of the functional it determines. (Integrals written without limits are understood to be taken over $\mathbf{R}$.)

It will be useful to consider the partial orders defined below both in terms of the pointwise description of the functions f and g and in terms of their functional description. With

$$If(x) = \int_{(-\infty,x]} f\,d\lambda \quad \text{and} \quad I^*f(x) = \int_{[x,\infty)} f\,d\lambda$$

we have three partial orders defined on non-negative functions.

$$f \le g: \quad f(x) \le g(x) \quad \text{for } \lambda\text{-almost every } x \in \mathbf{R}. \tag{1}$$

$$If \le Ig: \quad If(x) \le Ig(x) \quad \text{for } x \in \mathbf{R}. \tag{2}$$

$$I^*f \le I^*g: \quad I^*f(x) \le I^*g(x) \quad \text{for } x \in \mathbf{R}. \tag{3}$$

The same three notions of partial order are expressed in terms of the functionals associated with f and g as follows. It is an exercise in measure theory to prove that the two definitions are equivalent for each partial order.

$$f \le g: \quad \int fh\,d\lambda \le \int gh\,d\lambda \quad \text{for all } h \ge 0. \tag{1}$$

$$If \le Ig: \quad \int fh\,d\lambda \le \int gh\,d\lambda \quad \text{for all decreasing } h \ge 0. \tag{2}$$

$$I^*f \le I^*g: \quad \int fh\,d\lambda \le \int gh\,d\lambda \quad \text{for all increasing } h \ge 0. \tag{3}$$

Decreasing Envelopes.

Among all decreasing functions that lie above a given non-negative function f, there is a unique least one. This is our first example of a monotone envelope, the least decreasing majorant of f. Note that we need a notion of the order of functions in order to interpret the words "least" and "majorant". Here we are using the first partial order defined above: For $f \ge 0$, let $f^{\downarrow}$ be the least$^{(1)}$ decreasing majorant$^{(1)}$ of f. That is, let $f^{\downarrow}$ be the unique non-negative function satisfying $f \le f^{\downarrow}$, $f^{\downarrow}$ decreasing, and whenever $f \le g$ with g decreasing then $f^{\downarrow} \le g$.

The second kind of monotone envelope is the greatest decreasing minorant, where again, "greatest" and "minorant" are understood to refer to the usual partial order on functions: For $f \ge 0$, let $f_{\downarrow}$ be the greatest$^{(1)}$ decreasing minorant$^{(1)}$ of f. That is, let $f_{\downarrow}$ be the unique non-negative function satisfying $f \ge f_{\downarrow}$, $f_{\downarrow}$ decreasing, and whenever $f \ge g$ with g decreasing then $f_{\downarrow} \ge g$.

By changing the partial order, we change the meanings of the words "least" and "majorant" and arrive at a new least decreasing majorant, called the *level function* of f. For $f \ge 0$, let f^o be the least$^{(2)}$ decreasing majorant$^{(2)}$ of f. That is, let f^o be the unique non-negative function satisfying $If \le If^o$, f^o decreasing, and whenever $If \le Ig$ with g decreasing then $If^o \le Ig$.

These envelopes arise naturally in connection with norm inequalities for various positive operators. Applications of monotone envelopes to rearrangement invariant spaces and to Lorentz spaces in particular are important. They are also a feature of work involving concave and quasiconcave functions and therefore of the theory of interpolation spaces and interpolation of operators. Pointwise formulas are available for each envelope and have been the main approach to working with them in the past. However, as we will see, formulas given in terms of functionals provide a useful alternative approach and are well worth the extra effort required to prove them.

Pointwise versus Functional Formulas.

Establishing the existence and the following pointwise formulas for $f^{\downarrow}$ and $f_{\downarrow}$ is another exercise in measure theory. We have,

$$f^{\downarrow}(x) = \operatorname*{ess\,sup_{\lambda}}_{y \geq x} f(y) \quad \text{and} \quad f_{\downarrow}(x) = \operatorname*{ess\,inf_{\lambda}}_{y \leq x} f(y).$$

A bit more work is required to construct a pointwise formula for the level function. If the measure λ is Lebesgue measure on the half line then f^o is the derivative of the least concave majorant of If. For a general measure λ, one introduces the notion of a λ-concave majorant and appeals to the Radon-Nikodym derivative to get the pointwise description of f^o. Specifically,

$$f^o = \left(\frac{d\mu}{d\lambda}\right),$$

where $\mu(-\infty, x]$ is the least λ-concave majorant of $If(x)$. See [27] for details.

Functional formulas for the first two decreasing envelopes are simple in form and, interestingly, involve the second partial order. For $g \geq 0$,

$$\int f^{\downarrow} g \, d\lambda = \sup_{Ih \leq Ig} \int fh \, d\lambda$$

and

$$\int f_{\downarrow} g \, d\lambda = \inf_{Ih \geq Ig} \int fh \, d\lambda.$$

Although the pointwise description of the level function can be rather complicated to work with, the functional formula is simple. It also demonstrates the similarity between the monotone envelopes with respect to the different partial orders. For decreasing $g \geq 0$,

$$\int f^o g \, d\lambda = \sup_{\substack{Ih \leq Ig \\ h \text{ decr.}}} \int fh \, d\lambda.$$

Proofs of these three functional formulas may be found in [31] but the idea behind the proofs may be simply illustrated by looking at the case of a well-behaved function f. For fixed $g \geq 0$, and any $h \geq 0$ satisfying $Ih \leq Ig$, properties of the envelope $f^{\downarrow}$ and of the second partial order yield

$$\int fh \, d\lambda \leq \int f^{\downarrow} h \, d\lambda \leq \int f^{\downarrow} g \, d\lambda.$$

Taking the supremum over all such functions h yields

$$\int f^{\downarrow} g \, d\lambda \geq \sup_{Ih \leq Ig} \int fh \, d\lambda.$$

A technique called *pushing mass* enables us to reverse this inequality by constructing a non-negative function h from the fixed function g such that $Ih \leq Ig$ and such that both inequalities above reduce to equality. The idea of this construction is to take h to be equal to g off the intervals where $f \neq f^{\downarrow}$ and construct h from g on each of these intervals by "pushing" the mass of g onto the right endpoint of the interval. (Notice that this construction makes h a measure rather than function. Some approximation is necessary to ensure that h remains a function.)

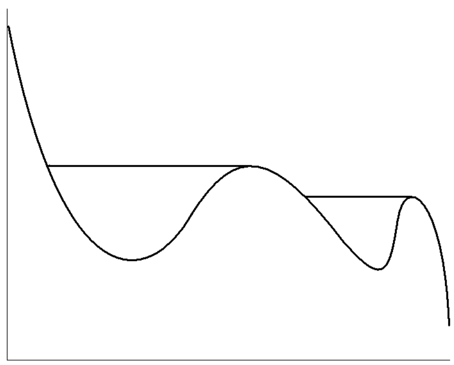

Figure 1: An example of a "well-behaved" function f and its least decreasing majorant $f^{\downarrow}$.

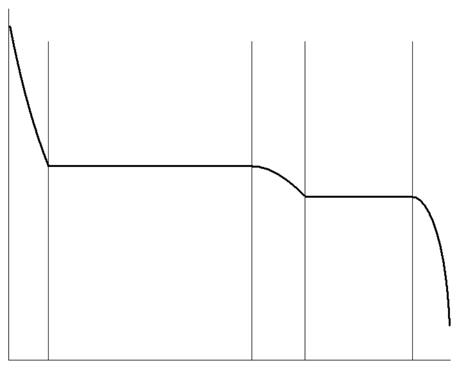

Figure 2: f differs from $f^{\downarrow}$ only on a collection of intervals and $f^{\downarrow}$ is constant on these intervals.

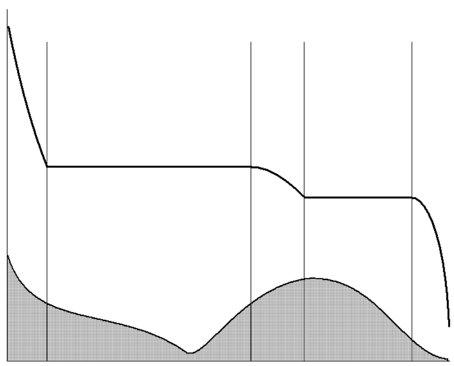

Figure 3: $f^{\downarrow}$ and an "arbitrary" function g. Here g is shown as a shaded mass distribution.

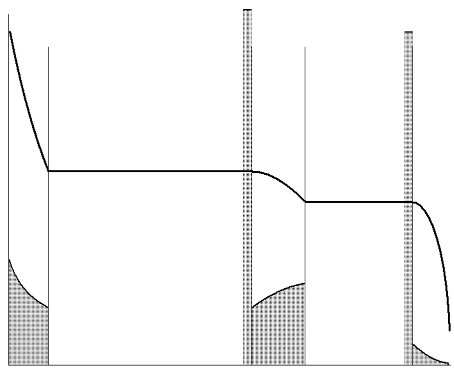

Figure 4: $f^{\downarrow}$ and h. The function h is formed by pushing the mass of g to the right on each interval.

The total mass of g has not changed in forming h and, moreover, the mass of g has been pushed only to the right to form h. It follows that $Ih \leq Ig$. Since $f^{\downarrow}$ is constant on the intervals where f and $f^{\downarrow}$ differ, and g and h have the same mass on each such interval, we see that

$$\int f^{\downarrow} h \, d\lambda = \int f^{\downarrow} g \, d\lambda.$$

The final requirement is that

$$\int f h \, d\lambda = \int f^{\downarrow} h \, d\lambda.$$

But h has been constructed to be zero on each interval where f and $f^{\downarrow}$ differ so this is also satisfied. We conclude that for this h,

$$\int f h \, d\lambda = \int f^{\downarrow} h \, d\lambda = \int f^{\downarrow} g \, d\lambda,$$

which completes our sketch proof of the functional formula for $f^{\downarrow}$,

$$\int f^{\downarrow} g \, d\lambda = \sup_{Ih \leq Ig} \int f h \, d\lambda.$$

A similar argument illustrates the corresponding formula for $f_{\downarrow}$, where this time, mass is pushed to the left in order to construct h from g.

Some different techniques are employed to prove the functional formula for f^{o}. One inequality follow readily from the functional definition of the second partial order. If g and h are decreasing and $Ih \leq Ig$ then

$$\int f h \, d\lambda \leq \int f^{o} h \, d\lambda \leq \int f^{o} g \, d\lambda.$$

Taking the supremum over all such functions h yields

$$\int f^{o} g \, d\lambda \geq \sup_{\substack{Ih \leq Ig \\ h \text{ decr.}}} \int f h \, d\lambda.$$

Pushing mass fails to prove the reverse inequality in this case because the construction of h from g as above does not preserve monotonicity and so does not produce a decreasing h even though g is decreasing. Instead, a family of averaging operators is employed to complete the argument.

If I_k are disjoint bounded intervals define the operator A by

$$Af(x) = \begin{cases} \frac{1}{\lambda(I_k)} \int_{I_k} f \, d\lambda, & x \in I_k \\ f(x), & x \notin \cup_k I_k \end{cases}$$

Note that each different collection $\{I_k\}$ of disjoint bounded intervals defines a new averaging operator A. The collection of all such operators is denoted $\mathcal{A}$. It is easy to check that each $A \in \mathcal{A}$ is formally self-adjoint, that is,

$$\int (Af)g\,d\lambda = \int f(Ag)\,d\lambda.$$

The averaging operator A that we use to prove the functional formula for f^o comes from the function f by defining the intervals I_k to be the bounded components of the open set

$$\{x \in \mathbf{R} : If(x) < If^o(x) \text{ and } If(x-) < If^o(x-)\}$$

with one or both endpoints as appropriate. (Special care has to be taken if the set has an unbounded component.) These intervals are called the *level intervals* of f. The level function, f^o, is constant on each level interval and $Af = f^o$. The effect of averaging f on these particular intervals is that If is increased to its least concave majorant, If^o.

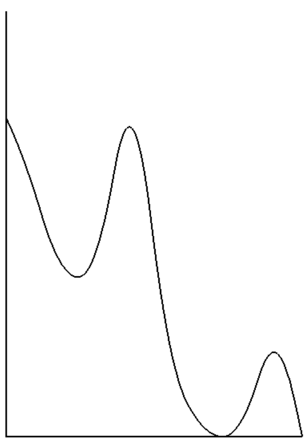

Figure 5: A function f.

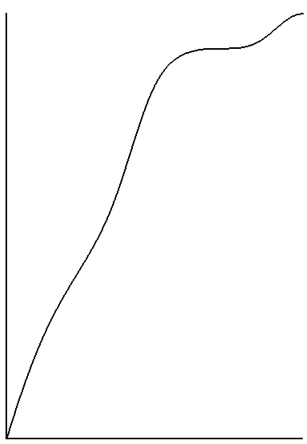

Figure 6: The integral If of f.

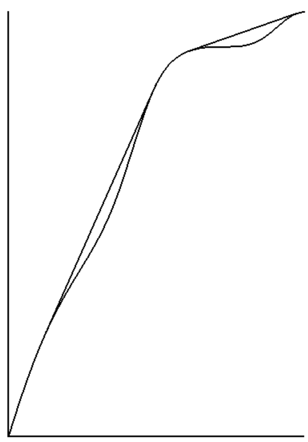

Figure 7: If and If^o, its least concave majorant.

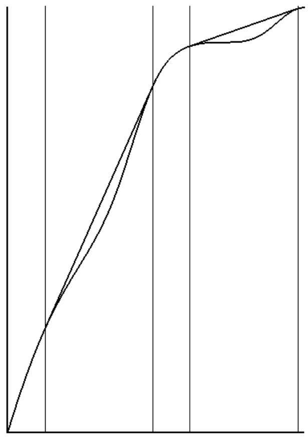

Figure 8: If^o, If and the level intervals of f.

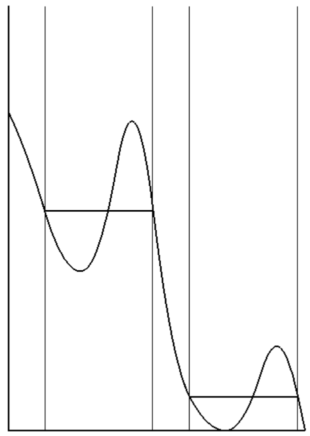

Figure 9: f^o and f. f^o is the result of averaging f on each level interval.

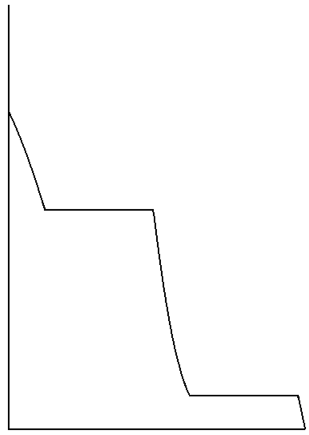

Figure 10: The decreasing function f^o.

Now define $h = Ag$. Since g is decreasing, h is decreasing and $Ig \geq IAg = Ih$. With this h the formal self-adjointness of A provides

$$\int f^o g\, d\lambda = \int (Af)g\, d\lambda = \int f(Ag)\, d\lambda = \int fh\, d\lambda$$

and completes the proof of

$$\int f^o g\, d\lambda = \sup_{\substack{Ih \leq Ig \\ h \text{ decr.}}} \int fh\, d\lambda.$$

We actually get a little more. For decreasing $g \geq 0$,

$$\int f^o g\, d\lambda = \sup_{A \in \mathcal{A}} \int f(Ag)\, d\lambda = \sup_{\substack{Ih \leq Ig \\ h \text{ decr.}}} \int fh\, d\lambda.$$

The Missing Envelopes.

Once these technical arguments have been made we are free to use the functional descriptions of the three monotone envelopes we have defined. Before we continue with applications of monotone envelopes let us pause to consider the missing envelopes. Since we look at both the least decreasing majorant and greatest decreasing minorant and we began with three partial orders, we expect six decreasing envelopes but so far have only considered three. (The increasing envelopes are completely analogous and need not be considered separately.)

The greatest$^{(2)}$ decreasing minorant$^{(2)}$ of f may not exist. This is a consequence of the observation that the lattice of decreasing functions with partial order $If \leq Ig$

is closed under meets but not joins. An example in [31] exhibits two decreasing minorants$^{(2)}$ of a function f such that no decreasing minorant$^{(2)}$ of f is greater$^{(2)}$ than both.

The least$^{(3)}$ decreasing majorant$^{(3)}$ of f may not exist. The lattice of decreasing functions with partial order $I^*f \le I^*g$ has joins but not meets.

The last decreasing envelope of f is the greatest$^{(3)}$ decreasing minorant$^{(3)}$ of f. Suprisingly, this is just f^o, the level function again! The same function f^o that we defined to be the least decreasing majorant of f with respect to the partial order $If \le Ig$ is also the greatest decreasing minorant of f with respect to the partial order $I^*f \le I^*g$. Specifically, $I^*f^o \le I^*f$, f^o is decreasing, and if $I^*g \le I^*f$ and g is decreasing then $I^*g \le I^*f^o$.

Remarks (See [24, 27, 31])

- For general f (possibly taking negative values) we define $f^\downarrow = |f|^\downarrow$ and $f^o = |f|^o$.
- The map $f \mapsto f^\downarrow$ is not linear, but it is sublinear: $(f+g)^\downarrow \le f^\downarrow + g^\downarrow$.
- Clearly $f \le g$ implies $f^\downarrow \le g^\downarrow$ and $f_n \uparrow f$ implies $f_n^\downarrow \uparrow f^\downarrow$.
- The map $f \mapsto f^o$ is not linear, it is not even sublinear.
- It's obvious that if $If \le Ig$ then $If^o \le Ig^o$.
- It's true, but far from obvious, that if $f \le g$ then $f^o \le g^o$.
- It follows that if $f_n \uparrow f$ then $f_n^o \uparrow f^o$.
- The level function can be extended from well-behaved functions to general measurable functions using order instead of continuity.

Application: Transferring Monotonicity.

Let $k(x,t) \ge 0$ be decreasing in t for each x and let K be the integral operator

$$Kf(x) = \int k(x,t) f(t)\, d\lambda(t).$$

The functional definition of the second partial order shows that $Ih \le If$ implies $Kh \le Kf$. Suppose we have a norm (or more generally a functional) that satisfies $\|f\| \le \|g\|$ whenever $f \le g$. Then we can transfer monotonicity from the kernel k to the weight u in certain weighted norm inequalities. The functional descriptions of $u^\downarrow$, $u_\downarrow$ and u^o make the proofs of the following three equivalences very simple and quite similar to each other. We prove only the first. See [31] for details.

The two inequalities

$$\int fu\, d\lambda \le C\|Kf\| \quad \text{and} \quad \int fu^\downarrow\, d\lambda \le C\|Kf\|$$

are equivalent in the sense that if one holds for all $f \ge 0$ then so does the other. Since $u \le u^\downarrow$ it is clear that the second inequality implies the first. Suppose now that the first inequality holds. We have

$$\int fu^\downarrow\, d\lambda = \sup_{Ih \le If} \int hu\, d\lambda \le C \sup_{Ih \le If} \|Kh\| = C\|Kf\|.$$

A similar argument shows that the two inequalities

$$\|Kf\| \le C \int fu\,d\lambda \quad \text{and} \quad \|Kf\| \le C \int fu_\downarrow\,d\lambda$$

are equivalent in the same sense. For the third pair of inequalities we restrict our attention to the non-negative, decreasing functions. The two inequalities

$$\int fu\,d\lambda \le C\|Kf\| \quad \text{and} \quad \int fu^o\,d\lambda \le C\|Kf\|$$

are equivalent in the sense that if one holds for all decreasing $f \ge 0$ then so does the other.

As an example to illustrate the above technique we offer a result involving the weighted Hardy inequality with $p = 1$. Suppose $0 < q < 1$. Let u and w be non-negative functions. The inequality

$$\left(\int_0^\infty \left(\int_0^x f\right)^q w(x)\,dx\right)^{1/q} \le C \int_0^\infty fu$$

holds for all $f \ge 0$ if and only if

$$\left(\int_0^\infty \left(\int_0^x f\right)^q w(x)\,dx\right)^{1/q} \le C \int_0^\infty fu_\downarrow$$

does. The monotonicity of $u_\downarrow$ is the key to showing that the latter inequality holds if and only if

$$\left(\int_0^\infty u_\downarrow(x)^{q/(q-1)} \left(\int_x^\infty w\right)^{q/(1-q)} w(x)\,dx\right)^{(1-q)/q} < \infty.$$

It is important to observe that $u^\downarrow$ arises naturally in this problem and remains essential in the solution. Finiteness of the above integral with $u^\downarrow$ replaced by u is no longer equivalent to the above Hardy inequality. For a proof of this result and an example to show that $u^\downarrow$ is essential see [34].

Banach Function Spaces.

For a proper introduction to the theory of Banach function spaces see [5, 35]. We make some definitions here for easy reference.

A *Banach function space* X is a Banach space of λ-measurable functions satisfying

$$g \in X \text{ and } |f| \le |g| \implies f \in X \text{ and } \|f\|_X \le \|g\|_X.$$

The (Köthe) dual space X' is defined by

$$\|g\|_{X'} = \sup_{f\in X} \frac{\int |fg|\,d\lambda}{\|f\|_X} \quad \text{and} \quad X' = \{g : \|g\|_{X'} < \infty\}.$$

To avoid technicalities we assume X has the *Fatou property*:

$$f_n \uparrow f \text{ and } \|f_n\|_X \text{ bounded} \implies f \in X \text{ and } \|f_n\|_X \uparrow \|f\|_X.$$

It is known that the Fatou property is equivalent to $X = (X')'$.

Down Spaces.

The definition of the dual norm leads immediately to a general Hölder inequality for X and X',

$$\int |fg|\,d\lambda \le \|f\|_X \|g\|_{X'}.$$

This inequality cannot be improved without restrictions on f or g. That is, for fixed g there is an f that makes the ratio of the two sides as close to 1 as desired. Also, assuming the Fatou property, for fixed f there is an g that makes the ratio of the two sides as close to 1 as desired.

However, if f is fixed and g is known to be decreasing, then some improvement can be expected. This is because the functions g that make the ratio of the two sides close to 1 may not happen to be among the decreasing functions. Define the *down space* of X, denoted $D(X)$, by

$$D(X) = \{f : \|f\|_{D(X)} < \infty\} \quad \text{where} \quad \|f\|_{D(X)} = \sup_{0\le g \text{ decr.}} \frac{\int |f| g\,d\lambda}{\|g\|_{X'}}$$

to get, for all (non-negative) decreasing g,

$$\int |f| g\,d\lambda \le \|f\|_{D(X)} \|g\|_{X'}.$$

Since the norm in $D(X)$ is less than or equal to the norm in X, this improves the Hölder inequality above. Before it can be of use, however, it is necessary to understand the norm in $D(X)$.

As we will see, the norms in the down space, $D(X)$ and its dual, $D(X)'$ are related to the norms in X and X' via decreasing envelopes. To make this connection we first note that the definition of the down norm ensures $\|f\|_{D(X)} \le \|f\|_X$ for all $f \in X$. Also, the functional description of the second partial order shows that $\|h\|_{D(X)} \le \|f\|_{D(X)}$ whenever $Ih \le If$.

We begin by looking at the simpler and more general relation between the dual space $D(X)'$ and the monotone envelope $g^{\downarrow}$. See [17] for a statement of this result in greater generality.

Theorem. *For any Banach function space X, a function g is in $D(X)'$ if and only if $g^{\downarrow}$ is in X'. In fact,*

$$\|g\|_{D(X)'} = \|g^{\downarrow}\|_{X'}.$$

Proof. Fix a λ-measurable function g. For any non-negative f,

$$\int f|g|\,d\lambda \le \int f g^{\downarrow}\,d\lambda \le \|f\|_{D(X)} \|g^{\downarrow}\|_{X'}.$$

Taking the supremum over all f yields $\|g\|_{D(X)'} \le \|g^{\downarrow}\|_{X'}$.

On the other hand, using the functional description of $g^\downarrow$, we have

$$\begin{aligned}\int fg^\downarrow\,d\lambda &= \sup_{Ih\le If}\int hg\,d\lambda\\ &\le \sup_{Ih\le If}\|h\|_{D(X)}\|g\|_{D(X)'}\\ &\le \|f\|_{D(X)}\|g\|_{D(X)'}\\ &\le \|f\|_X\|g\|_{D(X)'}.\end{aligned}$$

Taking the supremum over all f yields $\|g\|_{D(X)'} \ge \|g^\downarrow\|_{X'}$ to complete the proof.

The relationship between the down space and the monotone envelope f^o is similar but holds in less generality.

Theorem. *For any Banach function space X, $f \in D(X)$ whenever $f^o \in X$ and*

$$\|f\|_{D(X)} \le \|f^o\|_X.$$

If the averaging operators in $\mathcal{A}$ are contractions on X then $f^o \in X$ whenever $f \in D(X)$ and

$$\|f\|_{D(X)} = \|f^o\|_X.$$

Proof. Since $If \le If^o$ we have

$$\|f\|_{D(X)} \le \|f^o\|_{D(X)} \le \|f^o\|_X.$$

On the other hand, if the operators in $\mathcal{A}$ are contractions on X then it follows from the formal self-adjointness of the operators in $\mathcal{A}$ that they are also contractions on X'. Thus

$$\begin{aligned}\int f^o g\,d\lambda \le \int f^o g^o\,d\lambda &= \sup_{A\in\mathcal{A}}\int fAg^o\,d\lambda\\ &\le \sup_{A\in\mathcal{A}}\|f\|_{D(X)}\|Ag^o\|_{X'} = \sup_{A\in\mathcal{A}}\|f\|_{D(X)}\|AA_g g\|_{X'} \le \|f\|_{D(X)}\|g\|_{X'}.\end{aligned}$$

Here A_g is the averaging operator in $\mathcal{A}$ based on the level intervals of g, so that $A_g g = g^o$. Taking the supremum over all $g \in X'$ yields $\|f\|_{D(X)} \ge \|f^o\|_X$ and completes the proof.

The down spaces give a simple perspective on the D-type Hölder inequalities introduced by Halperin and Lorentz in [12] and [19]. If $f \ge 0$ and g is decreasing then

$$\int fg\,d\lambda \le \|f^o\|_X\|g\|_{X'}.$$

The inequality is sharp if the operators in $\mathcal{A}$ are contractions on X. In fact, we have established the more general fact that, for any $f, g \ge 0$,

$$\int fg\,d\lambda \le \|f\|_{D(X)}\|g\|_{D(X)'} = \|f^o\|_X\|g^\downarrow\|_{X'}.$$

The inequality is sharp if the operators in $\mathcal{A}$ are contractions on X.

Rearrangement Invariant Spaces.

The last theorem leads us to investigate spaces X on which the operators in $\mathcal{A}$ are contractions. A large, well-studied class of spaces with this property is the class of rearrangement invariant spaces. See, for example, [5].

Functions f and g are *equimeasurable* provided

$$\lambda\{x : |f(x)| > \alpha\} = \lambda\{x : |g(x)| > \alpha\}$$

for all $\alpha > 0$. A Banach function space X is *rearrangement invariant* (r.i.) if equimeasurable functions have the same norm in X.

The (generalized) inverse of the decreasing function $\alpha \mapsto \lambda\{x : |f(x)| > \alpha\}$ is called the decreasing rearrangement of f and denoted f^*. For any f and g,

$$\lambda\{x : |f(x)| > \alpha\} = |\{t > 0 : |f^*(t)| > \alpha\}|$$

and

$$\int fg\, d\lambda \leq \int_0^\infty f^* g^*.$$

A Banach function space X is called *universally rearrangement invariant* (u.r.i) if

$$\int_0^t f^* \leq \int_0^t g^* \text{ for all } t > 0 \implies \|f\|_X \leq \|g\|_X.$$

Since equimeasurable functions have the same rearrangement, it follows that a u.r.i. space is always r.i. The converse holds if the underlying measure λ is *resonant,* that is, if for any λ-measurable f and g

$$\sup_{h^* \leq g^*} \int fh\, d\lambda = \int_0^\infty f^* g^*.$$

It is well-known that a σ-finite measure is resonant if and only if it is non-atomic or else purely atomic with all atoms having equal weight.

Here we consider u.r.i. spaces over general measures. This automatically includes all r.i. spaces over resonant measures. We avoid r.i. spaces over measures that are not assumed to be resonant because this setting has some unpleasant complications. For example, the dual of a u.r.i. space is u.r.i. no matter what the measure but the dual of an r.i. space need not be r.i. if the underlying measure is not resonant.

Exercise. *Construct an r.i. space whose dual is not r.i.*

Roughly speaking, if the norm of f in X can be expressed in terms of f^* then X is r.i. For example,

$$\|f\|_{L^1} \equiv \int |f|\, d\lambda = \int_0^\infty f^* \quad \text{and} \quad \|f\|_{L^\infty} \equiv \operatorname*{ess\,sup_\lambda}_{x \in \mathbf{R}} |f(x)| = \sup_{t>0} f^*(t)$$

so L^1 and L^∞ are r.i. spaces. We can do better than this, however. If $\int_0^t f^* \le \int_0^t g^*$ for all $t > 0$ then

$$\|f\|_{L^1} = \lim_{t\to\infty} \int_0^t f^* \le \lim_{t\to\infty} \int_0^t g^* = \|g\|_{L^1}$$

and

$$\|f\|_{L^\infty} = \lim_{t\to 0} \frac{1}{t} \int_0^t f^* \le \lim_{t\to 0} \frac{1}{t} \int_0^t g^* = \|g\|_{L^\infty}$$

so L^1 and L^∞ are u.r.i. The spaces L^1 and L^∞ are much more than just examples of u.r.i. spaces. They are the starting points for a beautiful description of all u.r.i. spaces coming from the theory of interpolation. One consequence of this description is that any operator that is bounded on L^1 and L^∞ is bounded on all u.r.i. spaces.

We can apply this important fact to the averaging operators introduced above. It is a simple matter to verify that every $A \in \mathcal{A}$ is a contraction on both L^1 and L^∞ and thus each $A \in \mathcal{A}$ is a contraction on every u.r.i. space.

Corollary. *If X is a u.r.i space with the Fatou property then $\|f\|_{D(X)} = \|f^o\|_X$ and $\|f\|_{D(X)'} = \|f^\downarrow\|_{X'}$.*

Explicit expressions for the down norms of L^1 and L^∞ are easy to find and will eventually lead us to a complete description of the down spaces for all u.r.i. spaces.

$$\|f\|_{D(L^1)} = \|f^o\|_{L^1} = \int f^o \, d\lambda = \int A_{|f|}|f| \, d\lambda = \int |f| \, d\lambda = \|f\|_{L^1}$$

Thus $D(L^1) = L^1$ with identical norms.

Recall that $If(x) = \int_{(-\infty,x]} f \, d\lambda$ and set $\Lambda(x) = \int_{(-\infty,x]} d\lambda$. Then

$$\|f\|_{D(L^\infty)} = \|f^o\|_{L^\infty} = \lim_{x\to-\infty} f^o(x) = \lim_{x\to-\infty} If^o(x)/\Lambda(x) = \operatorname*{ess\,sup_\lambda}_{\mathbf{R}} If^o/\Lambda.$$

But If^o is the least λ-concave majorant of If and it is easy to check that the function

$$(\operatorname*{ess\,sup_\lambda}_{\mathbf{R}} I|f|/\Lambda)\Lambda$$

is a particular λ-concave majorant of $I|f|$. It follows that

$$\operatorname*{ess\,sup_\lambda}_{\mathbf{R}} If^o/\Lambda \le \operatorname*{ess\,sup_\lambda}_{\mathbf{R}} I|f|/\Lambda.$$

Since $I|f| \le If^o$ this inequality is actually equality and we have

$$\|f\|_{D(L^\infty)} = \operatorname*{ess\,sup_\lambda}_{\mathbf{R}} I|f|/\Lambda = \|I|f|/\Lambda\|_{L^\infty}.$$

It is important to point out that the down space of a u.r.i. space need not be u.r.i. For example, $D(L^\infty)$ is not u.r.i. Let λ be Lebesgue measure on $(0,\infty)$. For each $y > 1$ set $f_y = \chi_{(y-1,y)}$. Then $f_y^* = \chi_{(0,1)}$ for all y so the f_y are all equimeasurable. However, $f_y^o = (1/y)\chi_{(0,y)}$ so

$$\|f_y\|_{D(L^\infty)} = \|f^o\|_{L^\infty} = 1/y.$$

Weighted Lebesgue and Lorentz Spaces.

There are L^p spaces of functions defined on any fixed measure space. A feature of weighted Lebesgue spaces is that they may be viewed as Banach function spaces with respect to various measures. The choice of measure does not change the underlying Banach space or its Banach dual but it does have implications when considering rearrangement invariance and the Köthe dual.

Let $1 \le p < \infty$ and let $w : \mathbf{R} \to [0, \infty]$ be λ-measurable. The weighted L^p space with norm

$$\|f\|_{L^p_\lambda(w)} = \left(\int |f|^p w \, d\lambda \right)^{1/p} = \|f\|_{L^p_{w\lambda}}$$

may be viewed as a Banach function space of λ-measurable functions or as a Banach function space of $(w\lambda)$-measurable functions. With the right choice of measure, a weighted L^p space is u.r.i.

In the first case the dual norm of g is

$$\sup_f \frac{\int fg \, d\lambda}{\|f\|_{L^p_\lambda(w)}} = \left(\int |g|^{p'} w^{1-p'} \, d\lambda \right)^{1/p'} = \|g\|_{L^{p'}_\lambda(w^{1-p'})}$$

and in the second case the dual norm of g is

$$\sup_f \frac{\int fgw \, d\lambda}{\|f\|_{L^p_{w\lambda}}} = \left(\int |g|^{p'} w \, d\lambda \right)^{1/p'} = \|g\|_{L^{p'}_{w\lambda}}$$

These two norms lead to different, but isometrically isomorphic, spaces of functions.

With this in hand we consider the dual of Lorentz space. Let λ be a resonant measure, let $1 \le p \le \infty$ and suppose w is a decreasing function on $(0, \infty)$. Then $\Lambda_p(w)$ is the space of all λ-measurable functions f for which

$$\|f\|_{\Lambda_p(w)} \equiv \|f^*\|_{L^p_w} < \infty.$$

The dual of $\Lambda_p(w)$ has norm

$$\|g\|_{\Lambda_p(w)'} = \sup_f \frac{\int fg \, d\lambda}{\|f\|_{\Lambda_p(w)}} = \sup_{f^*} \frac{\int_0^\infty f^*(g^*/w) \, w}{\|f^*\|_{L^p_w}} = \|g^*/w\|_{D(L^{p'}_w)}.$$

Since $L^{p'}_w$ is r.i. with respect to the measure $w(x)\,dx$, we have

$$\|g\|_{\Lambda_p(w)'} = \|(g^*/w)^o\|_{L^{p'}_w}.$$

The above calculation is valid even when w is not decreasing, although in that case the "norm" in $\Lambda_p(w)$ does not satisfy the triangle inequality.

Since pointwise formulas for the level function are not easy to work with, this formula can be unwieldy. However, it does show that equivalent norms for $D(L^{p'}_w)$ give equivalent norms for the dual of $\Lambda_p(w)$.

Equivalent Norms for the Down Spaces.
Let $\Lambda(x) = \lambda(-\infty, x]$ and define the dual operators P and Q by

$$Pg(x) = \frac{1}{\Lambda(x)} \int_{(-\infty,x]} g\, d\lambda \quad \text{and} \quad Qf(x) = \int_{[x,\infty)} f \frac{d\lambda}{\Lambda}$$

Note that if $g \geq 0$ is decreasing then Pg is decreasing and $g \leq Pg$. Also, if $f \geq 0$ then Qf is decreasing.

Theorem. *If X is u.r.i. and Q is bounded on X then $\|f\|_{D(X)} \approx \|Qf\|_X$ for all non-negative f.*

Proof. If g is decreasing then

$$\int fg\, d\lambda \leq \int f(Pg)\, d\lambda = \int (Qf)g\, d\lambda \leq \|Qf\|_X \|g\|_{X'}.$$

Taking the supremum over all decreasing g gives $\|f\|_{D(X)} \leq \|Qf\|_X$.

On the other hand, since X is u.r.i. and Q is bounded on X, X' is u.r.i and P is bounded on X'. Thus, $\|Pg^o\|_{X'} \leq C\|g^o\|_{X'} \leq C\|g\|_{X'}$ for any $g \geq 0$ and so

$$\int (Qf)g\, d\lambda \leq \int (Qf)g^o\, d\lambda = \int f(Pg^o)\, d\lambda \leq C\|f\|_{D(X)}\|g\|_{X'}.$$

It follows that $\|Qf\|_X \lessapprox \|f\|_{D(X)}$ to complete the proof.

There is also an equivalent norm for $D(X)$ involving the P operator. Let L map f to the constant function

$$Lf(x) = Pf(\infty) = \frac{\int f\, d\lambda}{\int\, d\lambda}.$$

Then $Lf = Lf^o$. Note that if λ is an infinite measure then $Lf(x) = 0$ for all f. The following technical lemma appears in [28].

Lemma. *If λ is resonant and X is r.i. then $\|(P+L)f^o\|_X \leq 3\|(P+L)f\|_X$.*

Theorem. *If λ is resonant, X is r.i., and the operator $P+L$ is bounded on X then $\|f\|_{D(X)} \approx \|(P+L)f\|_X$.*

Proof. $\|f\|_{D(X)} = \|f^o\|_X$ and

$$\tfrac{1}{3}\|f^o\|_X \leq \tfrac{1}{3}\|Pf^o\|_X \leq \|(P+L)f\|_X \leq \|(P+L)f^o\|_X \leq C\|f^o\|_X.$$

These two equivalent norms extend earlier results involving the norm of $D(X)$, see [12, 19, 1, 23, 14]. Suppose $1 < p < \infty$ and $W(x) = \int_0^x w$ and consider the

D-type Hölder inequality. If $f \geq 0$ and g is decreasing then

$$\int_0^\infty fg \leq \left(\int_0^\infty \left(\int_x^\infty \frac{f}{W}\right)^{p'} w(x)\,dx\right)^{1/p'} \left(\int_0^\infty g^p w\right)^{1/p}$$

$$\int_0^\infty fg \leq 3\left(\int_0^\infty \left(\frac{\int_0^x f}{\int_0^x w} + \frac{\int_0^\infty f}{\int_0^\infty w}\right)^{p'} w(x)\,dx\right)^{1/p'} \left(\int_0^\infty g^p w\right)^{1/p}$$

We also easily deduce a known result for the dual of Lorentz space: For any g,

$$\|g\|_{\Lambda_p(w)'} \approx \left(\int_0^\infty \left(\int_x^\infty \frac{g^*}{W}\right)^{p'} w(x)\,dx\right)^{1/p'}$$

$$\|g\|_{\Lambda_p(w)'} \approx \left(\int_0^\infty \left(\frac{\int_0^x g^*}{\int_0^x w} + \frac{\int_0^\infty g^*}{\int_0^\infty w}\right)^{p'} w(x)\,dx\right)^{1/p'}$$

Application: Factoring Hardy's Inequality.

Hardy's inequality is often viewed as an inequality in ℓ^p sequence spaces and even more often as an inequality in L^p for Lebesgue measurable functions on the half line $(0,\infty)$. Its applications and generalizations have received a great deal of attention from [13] to [22, 18] and the references therein. For many years the two views developed more or less separately, each with their particular techniques. We argued in [32] that the natural setting for Hardy's inequality is as an L^p_λ space inequality for functions with respect to a general measure λ. A great many results in both the sequence case and the case of Lebesgue measurable functions can be achieved much more simply from that point of view.

Consider the following question: For which measures λ, μ does $P : L^p_\lambda \to L^q_\mu$ boundedly? Because there are two indices and two measures in this formulation one does not expect a simple answer but surprisingly, a simple answer is available. The techniques used to provide this answer vary greatly and can be quite technical. We show that all the major results can be deduced from a simple factorization where the Hardy inequality is used on a single space (one index and one measure) and the condition which ensures the boundedness of P arises naturally from the requirement that the monotone functions in one L^p space are embedded in another.

The "One Hardy Inequality" needed to carry out this factorization follows readily from the classical Hardy inequality, together with standard results from the theory of rearrangements. If $p > 1$ then $P : L^p_\lambda \to L^p_\lambda$, specifically,

$$\left(\int \left|\frac{1}{\Lambda(x)}\int_{(-\infty,x]} f\,d\lambda\right|^p d\lambda\right)^{1/p} \leq p'\left(\int |f|^p\,d\lambda\right)^{1/p}.$$

The existence of the level function provides a crucial reduction of the above question. See [24, 25]. The Hardy averaging operator is bounded if and only if its restriction to decreasing functions is bounded. Specifically, $P : L^p_\lambda \to L^q_\mu$ is bounded if and only if $P : L^p_\lambda \cap \{\text{decr.}\} \to L^q_\mu$ is bounded. Moreover the bound is the same. In the proof of this reduction, one direction is obvious and the other follows from a simple estimate using the level function.

$$\|Pf\|_{L^q_\mu} \le \|Pf^o\|_{L^q_\mu} \le C\|f^o\|_{L^p_\lambda} \le C\|f\|_{L^p_\lambda}$$

A decreasing function f is less than or equal to its average, Pf. This observation, together with the One Hardy Inequality above, shows that $\|f\|_{L^p_\lambda}$ and $\|Pf\|_{L^p_\lambda}$ are equivalent for decreasing functions. Therefore $P : L^p_\lambda \to L^q_\mu$ if and only if $L^p_\lambda \cap \{\text{decr.}\} \overset{\text{id}}{\hookrightarrow} L^q_\mu$. This can also be viewed as a factorization where the Hardy operator is applied only to the decreasing functions in a single space,

$$L^p_\lambda \cap \{\text{decr.}\} \xrightarrow{P} L^p_\lambda \cap \{\text{decr.}\} \overset{\text{id}}{\hookrightarrow} L^q_\mu$$

For this factorization to give a simple answer to our question we need a simple characterization of the embedding step. We need to know for which measures λ, μ is

$$L^p_\lambda \cap \{\text{decr.}\} \overset{\text{id}}{\hookrightarrow} L^q_\mu ?$$

This splits naturally into two cases. If $p \le q$ then an application of Minkowski's integral inequality yields the characterizing condition,

$$\sup_x \mu(-\infty, x]^{1/q} \Lambda(x)^{-1/p} < \infty.$$

To handle the case $q < p$ we make the obvious substitution to reduce to the case $q = 1$ and then use the first equivalent norm for $D(X) = D(L^{p/q}_\lambda)$. The embedding holds if and only if $Q\mu \in L^{p/(p-q)}_\lambda$ where $Q\mu(x) = \int_{[x,\infty)} \frac{d\mu}{\Lambda}$.

Related Inequalities.

As an illustration of the flexibility of this method we show how easily a class of related inequalities can be characterized along with the Hardy inequality itself. Suppose $\varphi : (0,\infty) \to \mathbf{R}$ is either concave and increasing or convex and decreasing. Define the operator T on $f \ge 0$ by

$$Tf(x) = (\varphi^{-1} \circ P(\varphi \circ f))(x) = \varphi^{-1}\left(\frac{1}{\Lambda(x)} \int_{(-\infty,x]} \varphi(f(t))\, d\lambda(t)\right).$$

Then $T : L^p_\lambda \to L^q_\mu$ if and only if $P : L^p_\lambda \to L^q_\mu$. To see this, observe first that Jensen's inequality implies $Tf \le Pf$ so the boundedness of P implies the boundedness of T. On the other hand, for a decreasing function f we have $f \le Tf$ so the boundedness of T implies the boundedness of the embedding of decreasing functions. As we have just seen, this implies that P is bounded.

In particular, this gives a characterization for the boundedness of the Geometric Mean Operator, just take $\varphi(t) = \log(t)$. It also gives a characterization of Hardy's inequality for negative indices by taking $\varphi(t) = 1/t$.

3. Quasiconcave Functions and Fourier Inequalities

We require the following facts from the previous section. If I_k are disjoint bounded intervals define the operator $A \in \mathcal{A}$ by

$$Af(x) = \begin{cases} \frac{1}{\lambda(I_k)} \int_{I_k} f \, d\lambda, & x \in I_k \\ f(x), & x \notin \cup_k I_k. \end{cases}$$

Let $f \geq 0$. For all decreasing $g \geq 0$ the level function f^o of f satisfies

$$\int f^o g \, d\lambda = \sup_{A \in \mathcal{A}} \int f(Ag) \, d\lambda = \sup_{\substack{Ih \leq Ig \\ h \text{ decr.}}} \int fh \, d\lambda.$$

Suppose $0 < q < 1$. The Hardy inequality with $p = 1$,

$$\left(\int_0^\infty \left(\int_0^x f \right)^q w(x) \, dx \right)^{1/q} \leq C \int_0^\infty fu,$$

holds for all $f \geq 0$ if and only if

$$\left(\int_0^\infty u_\downarrow(x)^{q/(q-1)} \left(\int_x^\infty w \right)^{q/(1-q)} w(x) \, dx \right)^{(1-q)/q} < \infty.$$

Embedding Quasiconcave Functions.

A Lebesgue measurable function f on $(0, \infty)$ is *quasiconcave* if $f(x)$ is increasing and $f(x)/x$ is decreasing. (The term quasiconcave is sometimes used to denote any function equivalent to a quasiconcave function but we do not make that definition here.) It is convenient to introduce *generalized quasiconcavity* as well. For $\alpha + \beta > 0$, let $\Omega_{\alpha,\beta}$ be the collection of functions f such that $x^\alpha f(x)$ is increasing and $x^{-\beta} f(x)$ is decreasing. Clearly, $\Omega_{0,1}$ is the collection of quasiconcave functions.

In view of our experience with embedding the decreasing functions it seems reasonable to ask the following question. For which α, β, u and w is

$$L^p(w) \cap \Omega_{\alpha,\beta} \overset{\text{id}}{\hookrightarrow} L^q(u)$$

bounded? See [15, 20, 21] for some results on this and related questions. We will see later that an understanding of these embeddings will lead to applications involving the dual of Lorentz spaces and to Fourier inequalities between Lorentz spaces.

Following the method introduced in [29] we work on the above question in this form: We wish to find all α, β, u and w for which

$$\sup_{f \in \Omega_{\alpha,\beta}} \frac{\|f\|_{L^q(u)}}{\|f\|_{L^p(w)}} < \infty.$$

The first step is to replace f by $f^{1/p}$ to reduce to the case $p = 1$. Note that $f \in \Omega_{\alpha,\beta}$ if and only if $f^p \in \Omega_{p\alpha,p\beta}$

Next, for $f \in \Omega_{\alpha,\beta}$ let $g(x) = x^{\alpha/(\alpha+\beta)} f(x^{1/(\alpha+\beta)})$ to reduce to the case $\alpha = 0$ and $\beta = 1$. It is easy to check that $g \in \Omega_{0,1}$, $\|f\|_{L^q(u)} = \|g\|_{L^q(U)}$ and $\|f\|_{L^1(w)} = \|g\|_{L^1(W)}$, for appropriate weights U and W, given in terms of u and w.

These two observations reduce the question to finding all u and w for which

$$\sup_{g\in\Omega_{0,1}} \frac{\|g\|_{L^q(U)}}{\int gW} < \infty.$$

Operators that Map (almost) Onto $\Omega_{\alpha,\beta}$.

Introduce the operators H_α, H^β and H_α^β by

$$\begin{aligned} H_\alpha h(x) =& x^{-\alpha} \int_0^x t^\alpha h(t)\, dt, \quad H^\beta h(x) = x^\beta \int_x^\infty t^{-\beta} h(t)\, dt \quad \text{and} \\ H_\alpha^\beta h(x) =& H_\alpha h(x) + H^\beta h(x) = \int_0^\infty \min((t/x)^\alpha, (x/t)^\beta) h(t)\, dt. \end{aligned}$$

It is easy to check that $\int_0^\infty (H_\alpha^\beta f) g = \int_0^\infty f(H_\beta^\alpha g)$. Also, and most importantly for us, if $h \ge 0$ then $H_\alpha^\beta h \in \Omega_{\alpha,\beta}$.

It is a well-known fact that every quasiconcave function is equivalent to its least concave majorant. Since every concave function is differentiable almost everywhere it is easy to see that every concave function is the limit of an increasing sequence of functions in the range of H_0^1. The conclusion, stated more precisely, is that if $g \in \Omega_{0,1}$ then there exists $h_n \ge 0$ such that $H_0^1 h_n \uparrow \tilde{g}$ and $\frac{1}{2}\tilde{g} \le g \le \tilde{g}$. This enables us to replace the supremum over all $g \in \Omega_{0,1}$ by a supremum over all non-negative functions. Now we wish to find those u and w for which

$$\sup_{h\ge 0} \frac{\|H_0^1 h\|_{L^q(U)}}{\int (H_0^1 h) W} < \infty.$$

Using $H_0^1 = H_0 + H^1$ in the numerator and moving the operator to W in the denominator reduces the supremum to two weighted Hardy inequalities. Specifically,

$$\sup_{h\ge 0} \frac{\|H_0 h\|_{L^q(U)}}{\int h(H_1^0 W)} < \infty \quad \text{and} \quad \sup_{h\ge 0} \frac{\|H^1 h\|_{L^q(U)}}{\int h(H_1^0 W)} < \infty$$

Our work on the Hardy inequality with $p = 1$ provides a characterization for the finiteness of each of these suprema.

Necessary and Sufficient Conditions.

Combining the two conditions we get above into one condition and carefully making our way back from the case $a = 0$, $b = 1$, and $p = 1$ to the general case yields a simple answer to our original question. As usual with such embeddings, it splits into two cases. If $0 < p \le q < \infty$ then

$$\sup_{f\in\Omega_{\alpha,\beta}} \frac{\|f\|_{L^q(u)}}{\|f\|_{L^p(v)}} \approx \sup_{t>0}[H^{p\alpha}_{p\beta}v(t)]^{-1/p}[H^{q\alpha}_{q\beta}u(t)]^{1/q}.$$

If $0 < q < p < \infty$ and $1/r = 1/q - 1/p$ then

$$\sup_{f\in\Omega_{\alpha,\beta}} \frac{\|f\|_{L^q(u)}}{\|f\|_{L^p(v)}} \approx \left(\int_0^\infty (H^{p\alpha}_{p\beta}v)^{-r/p}(H^{q\alpha}_{q\beta}u)^{r/p}u\right)^{1/r}.$$

An alternate form of this will be useful. After integration by parts we see that the last integral is equivalent to

$$\left(\int_0^\infty H^{q\alpha}u(t)^{r/q}H^{p\alpha}_{p\beta}v(t)^{-r/q}H_{p\beta}v(t)\,\frac{dt}{t}\right)^{1/r} + \frac{\|x^{-\alpha}\|_{L^q(u)}}{\|x^{-\alpha}\|_{L^p(v)}}$$
$$+\left(\int_0^\infty H_{q\beta}u(t)^{r/q}H^{p\alpha}_{p\beta}v(t)^{-r/q}H^{p\alpha}v(t)\,\frac{dt}{t}\right)^{1/r} + \frac{\|x^{\beta}\|_{L^q(u)}}{\|x^{\beta}\|_{L^p(v)}}$$

Application: The Dual of the Lorentz Gamma Space.

The Lorentz space $\Gamma_p(v)$ is defined to be the collection of λ-measurable functions such that

$$\|f\|_{\Gamma_p(v)} \equiv \|f^{**}\|_{L^p(v)} < \infty.$$

Here $f^{**}(x) = \frac{1}{x}\int_0^x f^*$ and f^* is the non-increasing rearrangement of f with respect to the measure λ.

The dual norm for this space has been shown to be another Lorentz Γ-space in [11, 9, 10]. The work above on embeddings of quasiconcave functions provides another expression for the dual norm.

Suppose $1 < p < \infty$ and λ is a resonant measure. Then

$$\|g\|_{\Gamma_{p,\lambda}(v)'} \approx \|g^*\|_{L^{p'}(v_0)} + \|g^{**} - g^*\|_{L^{p'}(v_\infty)} + V_0\|g^*\|_{L^\infty} + V_\infty\|g^*\|_{L^1}$$

where

$$v_0(t) = \frac{1}{t}\left(\frac{1}{t^p}\int_0^t v(x)\,dx + \int_t^\infty v(x)\,\frac{dx}{x^p}\right)^{-p'}\frac{1}{t^p}\int_0^t v(x)\,dx,$$
$$v_\infty(t) = \frac{1}{t}\left(\frac{1}{t^p}\int_0^t v(x)\,dx + \int_t^\infty v(x)\,\frac{dx}{x^p}\right)^{-p'}\int_t^\infty v(x)\,\frac{dx}{x^p},$$
$$V_0 = \left(\int_0^\infty x^{-p}v(x)\,dx\right)^{-1/p} \text{ and } V_\infty = \left(\int_0^\infty v(x)\,dx\right)^{-1/p}.$$

For details of the proof see [29]. An unusual feature of this expression for the dual norm is the appearance of the term $g^{**} - g^*$. Spaces defined using this expression are called Lorentz S-spaces, see [8], and defined (modulo constant functions) by

$$\|g\|_{S_p(w)} = \|g^{**} - g^*\|_{L^p(w)}.$$

Why Should $f^{**} - f^*$ Appear?.

Given g we can solve the equation $f^{**}(t) = \frac{1}{t} g^{**}(\frac{1}{t})$ and check that $g^{**}(t) = \frac{1}{t} f^{**}(\frac{1}{t})$ as well.

Moreover, if $w(t) = v(\frac{1}{t}) t^{p-2}$ then $v(t) = w(\frac{1}{t}) t^{p-2}$ and

$$\|f\|_{\Gamma_p(v)} = \|f^{**}\|_{L^p(v)} = \|g^{**}\|_{L^p(w)} = \|g\|_{\Gamma_p(w)}$$

so the correspondence between f and g above gives an isometry of Lorentz Γ-spaces. However,

$$\|f\|_{\Lambda_p(v)} = \|f^*\|_{L^p(v)} = \|g^{**} - g^*\|_{L^p(w)} = \|g\|_{S_p(w)}$$

Evidently this isometry between Γ-spaces does not extend to the larger Λ-spaces. In fact, its extension interchanges the factors in the intersection $\Gamma_p(v) = \Lambda_p(v) \cap S_p(v)$, taking $\Lambda_p(v)$ to $S_p(w)$ and $S_p(v)$ to $\Lambda_p(w)$ while preserving the Γ-spaces.

We have seen that the dual of a Λ space is a Γ-space. Via this isometry we see that the dual of an S-space is also a Γ-space. It is natural then, that the dual of a Γ-space should have aspects of both Λ- and S-spaces. Our expression for the dual of the Γ-space makes this clear.

The Fourier Transform on Lorentz Spaces.

For an application of both monotone envelopes and the embedding of quasiconcave functions we turn to the Fourier transform on $\mathbf{R}^n$ defined by

$$\mathcal{F}f(x) = \int_{\mathbf{R}^n} e^{-ix\cdot y} f(y)\,dy$$

For related work on the boundedness of the Fourier transform see [2, 4, 3]. Since $\mathcal{F} : L^1 \to L^\infty$ and $\mathcal{F} : L^2 \to L^2$, there is a $D > 0$ such that

$$\int_0^z (\mathcal{F}f)^*(t)^2\,dt \le D \int_0^z \left(\int_0^{1/t} f^* \right)^2 dt, \quad z > 0.$$

This is a result from [16] and applies to any operator that maps $L^1 \to L^\infty$ and $L^2 \to L^2$, that is, to every operator of type $(1, \infty)$ and $(2, 2)$.

Fix an f and let $h(t) = (1/D)(\mathcal{F}f)^*(t)^2$ and $g(t) = \left(\int_0^{1/t} f^*\right)^2$. Observe that $Ih \le Ig$, h is decreasing, and $g \in \Omega_{2,0}$.

Let $u, v : [0, \infty) \to [0, \infty]$ and let C be the best constant in

$$\left(\int_0^\infty (\mathcal{F}f)^*(t)^q u(t)\,dt \right)^{1/q} \le C \left(\int_0^\infty \left(\int_0^{1/t} f^* \right)^p v(t)\,dt \right)^{1/p}$$

A simple change of variable shows that this inequality expresses the boundedness of the Fourier transform as a map from $\Gamma^p(w) \to \Lambda^q(u)$. Here $w(t) = t^{p-2} v(1/t)$.

We ask for which weights u and v is this C finite?

First we look at a sufficient condition for the finiteness of C. With g and h as above we see that if $q \geq 2$ then

$$C^2/D \leq \sup_{g\in\Omega_{2,0}} \sup_{\substack{Ih\leq Ig \\ h \text{ decr.}}} \frac{\|h\|_{L^{q/2}(u)}}{\|g\|_{L^{p/2}(v)}} = \sup_{g\in\Omega_{2,0}} \sup_{A\in\mathcal{A}} \frac{\|Ag\|_{L^{q/2}(u)}}{\|g\|_{L^{p/2}(v)}}$$

The corresponding necessary condition depends on the features of the Fourier transform and requires a certain amount of careful construction, see [30]. The result is this, if $q \geq 0$ then

$$C^2 \geq (const) \sup_{z>0} \sup_{A\in\mathcal{A}} \frac{\|A\omega_z\|_{L^{q/2}(u)}}{\|\omega_z\|_{L^{p/2}(v)}}$$

Here $\omega_z(t) = \min(z^{-2}, t^{-2})$.

Restricting our attention to the case where necessity and sufficiency overlap we see that if $0 < p \leq 2 \leq q < \infty$ then

$$\left(\int_0^\infty (\mathcal{F}f)^*(t)^q u(t)\,dt\right)^{1/q} \leq C\left(\int_0^\infty \left(\int_0^{1/t} f^*\right)^p v(t)\,dt\right)^{1/p}$$

holds if and only if

$$\sup_{z>0} \sup_{A\in\mathcal{A}} \frac{\|A\omega_z\|_{L^{q/2}(u)}}{\|\omega_z\|_{L^{p/2}(v)}} < \infty.$$

Since the sufficient condition applies to more operators than the necessary condition we can draw the following conclusion from the existence of a necessary and sufficient condition, even without appealing to the form of the condition: If $0 < p \leq 2 \leq q < \infty$ and the Fourier transform is bounded from $\Gamma_p(w)$ to $\Gamma_q(u)$ then so is every operator of type $(1,\infty)$ and $(2,2)$.

Before we can reasonably say that we have characterized the Fourier inequality with the necessary and sufficient condition just given we have to ask an important question. Is the weight condition easier to handle than the original inequality? It is not clear that this is the case in general, but in the important special case $q = 2$ the condition becomes very simple indeed.

If $0 < p \leq 2$ then

$$\begin{aligned}\sup_{z>0} \sup_{A\in\mathcal{A}} \frac{\|A\omega_z\|_{L^1(u)}}{\|\omega_z\|_{L^{p/2}(v)}} &= \sup_{z>0} \frac{\|\omega_z\|_{L^1(u^o)}}{\|\omega_z\|_{L^{p/2}(v)}} \\ &\approx \sup_{x>0} \left(\frac{1}{x^2}\int_0^x u^o\right)^{1/2} \left(\frac{1}{x^p}\int_0^x v(t)\,dt + \int_x^\infty v(t)\,\frac{dt}{t^p}\right)^{-1/p}\end{aligned}$$

The finiteness of the above expression is necessary and sufficient for any one of the following equivalent statements:

$$\mathcal{F} : \Gamma_p(w) \to \Gamma_2(u)$$
$$\mathcal{F} : \Gamma_p(w) \to \Lambda_2(u)$$
$$\mathcal{F} : \Gamma_p(w) \to \Gamma_2(u^o)$$
$$\mathcal{F} : \Gamma_p(w) \to \Lambda_2(u^o)$$

The Fourier Weight Condition and an Optimal R.I. Space.

Let $p \geq 1$. A concrete expression for

$$\sup_{A \in \mathcal{A}} \left(\int (Ag)^p u \right)^{1/p} = \sup_{\substack{Ih \leq Ig \\ h \text{ decr.}}} \left(\int h^p u \right)^{1/p}$$

for all decreasing g (or just for $g \in \Omega_{2,0}$) would extend the weight characterization for the Fourier transform on Lorentz spaces.

Define the space $\Theta_p(u)$ by its norm,

$$\|g\|_{\Theta_p(u)} = \sup_{\substack{Ih \leq Ig^* \\ h \text{ decr.}}} \left(\int h^p u \right)^{1/p} = \sup_{h^{**} \leq g^{**}} \left(\int (h^*)^p u \right)^{1/p}.$$

It can be shown that $\Theta_p(u)$ is a Banach space for any u and that we have the embeddings

$$\Gamma_p(u) \subset \Theta_p(u) \subset \Lambda_p(u).$$

In fact, $\Theta_p(u)$ is the largest r.i. space contained in $\Lambda_p(u)$ and if $\Lambda_p(u)$ can be renormed to become a Banach space then all three spaces coincide, up to equivalent norms.

The connection with monotone envelopes is that if $p = 1$ then $\Theta_1(u) = \Lambda_1(u^o)$.

A Calderón Couple of Down Spaces

Our construction of down spaces and their duals demonstrates a surprising connection between the two monotone envelopes, the least decreasing majorant and the level function. One envelope gives an expression for the norm of the down space and the other an expression for its dual. Specifically, if X is a u.r.i. function space then

$$\|f\|_{D(X)} = \|f^o\|_X \quad \text{and} \quad \|g\|_{D(x)'} = \|g^{\downarrow}\|_{X'}.$$

A fundamental result of Calderón [7] shows that the u.r.i. spaces are precisely the exact interpolation spaces between L^1_λ and L^∞_λ. In this section we follow [33], investigate the interpolation properties of the corresponding down spaces and prove a very strong result. Specifically, we show that $(D(L^1_\lambda), D(L^\infty_\lambda))$ is a Calderón couple of spaces and from this we deduce that the down spaces of u.r.i. spaces are precisely the exact interpolation spaces between $D(L^1_\lambda)$ and $D(L^\infty_\lambda)$. Although the pair $(L^1_\lambda, L^\infty_\lambda)$ is self-dual, the pair $(D(L^1_\lambda), D(L^\infty_\lambda))$ is not so we also investigate the dual pair $(D(L^\infty_\lambda)', D(L^1_\lambda)')$, achieving almost as strong a result.

Interpolation of Operators.

Let (X_1, X_2) be a couple of Banach spaces contained in $X_1 + X_2$. For a careful definition of compatible couples of Banach spaces and for definitions and proofs of the other ideas introduced briefly below, see [6].

An operator $T : X_1 + X_2 \to X_1 + X_2$ is called *admissible* if

$$\|T\|_{X_1 \to X_1} \leq 1 \quad \text{and} \quad \|T\|_{X_2 \to X_2} \leq 1.$$

A Banach space X which is intermediate between X_1 and X_2, ie.,

$$X_1 \cap X_2 \hookrightarrow X \hookrightarrow X_1 + X_2,$$

is an *exact interpolation space* if $\|T\|_{X \to X} \leq 1$ for every admissible T. Define the Peetre K-functional by

$$K(t, x; X_1, X_2) = \inf_{x = x_1 + x_2} \|x_1\|_{X_1} + t\|x_2\|_{X_2}$$

If Φ is a Banach Function Space of Lebesgue measurable functions, containing the function $t \mapsto \min(1, t)$, then Φ is called a parameter of the $\mathcal{K}$-method. The space $K_\Phi(X_1, X_2)$, of all $x \in X_1 + X_2$ for which

$$\|x\|_{K_\Phi(X_1, X_2)} \equiv \|K(\cdot, x; X_1, X_2)\|_\Phi < \infty,$$

is an exact interpolation space between X_1 and X_2.

If the inequality $K(t, x; X_1, X_2) \leq K(t, y; X_1, X_2)$ for all $t > 0$ implies that there exists an admissible T such that $Ty = x$, then (X_1, X_2) is called a *Calderón couple.*

For a Calderón couple, the K_Φ are the only exact interpolation spaces.

Theorem.. *$(L^1_\lambda, L^\infty_\lambda)$ is a Calderón couple.*

An Informal K-functional Calculation.

Fix a σ-finite measure λ on $\mathbf{R}$ and set $L^1 \equiv L^1_\lambda$, $L^\infty \equiv L^\infty_\lambda$.

For $f \geq 0$, $K(t, f; L^1, L^\infty) = \inf_{f = f_1 + f_2} \|f_1\|_{L^1} + t\|f_2\|_{L^\infty}$.

If $f = f_1 + f_2$ with $\|f_2\|_{L^\infty} \equiv \sup_x |f_2(x)| = \alpha$ then there is a clear best choice for f_2, namely, $f_2 = \min(f, \alpha)$. In this case $f_1 = f - f_2 = (f - \alpha)^+$ and it follows that $f_1^* = (f^* - \alpha)^+$. Choose y so that $f^*(y) = \alpha$. Then

$$\begin{aligned} K(t, f; L^1, L^\infty) &= \inf_{f = f_1 + f_2} \int f_1 \, d\lambda + t \sup_x |f_2(x)| \\ &= \inf_\alpha \int (f - \alpha)^+ \, d\lambda + t\alpha \\ &= \inf_y \int_0^y f^* + (t - y) f^*(y) \end{aligned}$$

Set the derivative to zero: $f^*(y) - f^*(y) + (t - y)\frac{df^*}{dy}(y) = 0$ so $y = t$. Therefore

$$K(t, f; L^1, L^\infty) = \int_0^t f^*$$

The informality of this calculation is evident in our use of phrases like "clear best choice", "it follows that", and our differentiation of a function that may not, in fact, be differentiable. With sufficient care, however, this argument may be made precise. The result itself is well known.

Universally Rearrangement Invariant Spaces.

Once the K-functional is known it is simple and instructive to see how (L^1, L^∞) being a Calderón couple provides a connection between u.r.i. spaces and exact interpolation spaces.

Suppose X satisfies $L^1 \cap L^\infty \hookrightarrow X \hookrightarrow L^1 + L^\infty$.

If X is u.r.i. then let T be an admissible operator and $g \in X$. For $t > 0$,

$$\begin{aligned}\int_0^t (Tg)^* &= \inf_{f_1+f_2=Tg} \|f_1\|_{L^1} + t\|f_2\|_{L^\infty} \\ &\le \inf_{g_1+g_2=g} \|Tg_1\|_{L^1} + t\|Tg_2\|_{L^\infty} \\ &\le \inf_{g_1+g_2=g} \|g_1\|_{L^1} + t\|g_2\|_{L^\infty} = \int_0^t g^*\end{aligned}$$

Since X is u.r.i., $Tg \in X$ and $\|Tg\|_X \le \|g\|_X$. Thus T is a contraction on X and so X is an exact interpolation space between L^1 and L^∞.

Conversely, if X is an exact interpolation space between L^1 and L^∞, then suppose $\int_0^t f^* \le \int_0^t g^*$. Since (L^1, L^∞) is a Calderón couple, there exists an admissible T such that $Tg = f$. This T is a contraction on X, so $f \in X$ and $\|f\|_X \le \|g\|_X$. Thus X is u.r.i.

Two More Informal K-functional Calculations.

Recall that $D(L^1) = L^1$ with identical norms and, writing $D^\infty = D(L^\infty)$, that

$$\|f\|_{D^\infty} = \|f^o\|_{L^\infty} \quad \text{and} \quad \|f\|_{(D^\infty)'} = \|f^\downarrow\|_{L^1}.$$

Our last K-functional calculation relied on the fact that one of the spaces involved was L^∞. We can imitate it closely to find the K- functional of $((D^\infty)', L^\infty)$. See also [26].

For $f \ge 0$, $K(t, f; (D^\infty)', L^\infty) = \inf_{f=f_1+f_2} \|f_1^\downarrow\|_{L^1} + t\|f_2\|_{L^\infty}$.

If $f = f_1 + f_2$ with $\|f_2\|_{L^\infty} \equiv \sup_x |f_2(x)| = \alpha$ then there is a clear best choice for f_2, namely, $f_2 = \min(f, \alpha)$. In this case $f_1 = f - f_2 = (f - \alpha)^+$ and it follows that $f_1^\downarrow = (f^\downarrow - \alpha)^+$. Choose y so that $(f^\downarrow)^*(y) = \alpha$. Then

$$\begin{aligned}K(t, f; (D^\infty)', L^\infty) &= \inf_\alpha \int (f^\downarrow - \alpha)^+ \, d\lambda + t\alpha \\ &= \inf_y \int_0^y (f^\downarrow)^* + (t - y)(f^\downarrow)^*(y)\end{aligned}$$

Set the derivative to zero: $(f^\downarrow)^*(y) - (f^\downarrow)^*(y) + (t - y)\frac{d[(f^\downarrow)^*]}{dy}(y) = 0$ so $y = t$. Therefore

$$K(t, f; (D^\infty)', L^\infty) = \int_0^t (f^\downarrow)^*$$

In the calculation of $K(t,f;L^1,D^\infty)$, neither space is L^∞ so a slightly different approach has to be taken.

Let $\Lambda(x) = \int_{(-\infty,y]} d\lambda$ and recall that in addition to the formula for $\|f\|_{D^\infty}$ involving the level function, we also have

$$\|f\|_{D^\infty} = \sup_y \frac{1}{\Lambda(y)} \int_{(-\infty,y]} f\, d\lambda.$$

For $f \geq 0$, $K(t,f;L^1,D^\infty) = \inf_{f=f_1+f_2} \|f_1\|_{L^1} + t\|f_2^o\|_{L^\infty}$.

If $f = f_1 + f_2$ view $\|f_1\|_{L^1}$ as fixed. Then f_2 has fixed mass and f_2^o is minimized by taking the mass of f_2 as far right as possible. The clear best choice (ignoring atoms) is $f_1 = f\chi_{(-\infty,x]}$ and $f_2 = f\chi_{(x,\infty)}$ for some x. Thus

$$\begin{aligned} K(t,f;L^1,D^\infty) &= \inf_x \int_{(-\infty,x]} f\, d\lambda + t \sup_{y>x} \frac{1}{\Lambda(y)} \int_{(x,y]} f\, d\lambda \\ &= \inf_x \sup_{y>x} \left(1 - \frac{t}{\Lambda(y)}\right) \int_{(-\infty,x]} f\, d\lambda + \frac{t}{\Lambda(y)} \int_{(-\infty,y]} f\, d\lambda \end{aligned}$$

This is the least Λ-concave majorant of $\int_{(-\infty,\Lambda^{-1}(t)]} f\, d\lambda$. Therefore

$$K(t,f;L^1,D^\infty) = \int_{(-\infty,\Lambda^{-1}(t)]} f^o\, d\lambda = \int_0^t (f^o)^*$$

Constructing Admissible Operators.

Now that we have the K-functional for (L^1,D^∞) we can address the problem of showing it is a Calderón couple. First suppose λ is Lebesgue measure on $(0,\infty)$.

To show that (L^1,D^∞) is a Calderón couple, we start with the inequality

$$\int_0^t f^o \leq \int_0^t g^o, \quad t > 0,$$

and produce an operator T such that

$$\|T\|_{L^1\to L^1} \leq 1, \quad \|T\|_{D^\infty \to D^\infty} \leq 1, \quad \text{and} \quad Tg = f.$$

Actually, we produce three admissible maps to get from g to f,

$$g \mapsto g^o \mapsto f^o \mapsto f$$

The map $g \mapsto g^o$ is essentially the averaging operator A_g from $\mathcal{A}$, although the possibility of an unbounded interval makes for some technical complications.

Is A_g admissible? Each operator $A \in \mathcal{A}$ is easily seen to be a contraction on L^1. Since A is self-adjoint, to see that it is a also contraction on D^∞, it is enough to show that it is a contraction on $(D^\infty)'$. But,

$$\|Af\|_{(D^\infty)'} = \int (Af)^\downarrow \leq \int (A(f^\downarrow))^\downarrow = \int A(f^\downarrow) = \int f^\downarrow = \|f\|_{(D^\infty)'}.$$

The map $f^o \mapsto f$ is also a kind of averaging operator, based on the level intevals of f. We skip over the special attention which must be paid to the unbounded level interval if there is one. The operator we need is

$$B_f h(x) = \begin{cases} f\frac{\int_{I_k} h}{\int_{I_k} f}, & x \in I_k \\ h(x), & x \notin \cup_k I_k. \end{cases}$$

The proof of admissibility of this operator B_f is similar (but not identical) to the proof that A_g is admissible. We omit the details.

The map from $g^o \mapsto f^o$ will be the limit of a sequence of averaging operators, each on a single interval. Once again, we skip over the complications that arise when the intervals are unbounded.

We are free to suppose here that f and g are decreasing so $f = f^o$ and $g = g^o$. Let $q_1, q_2, q_3, \ldots$ be a countable dense subset of $(0, \infty)$, perhaps the rational numbers.

With $g = g_0$, suppose intervals $I_1, \ldots, I_{n-1}$ and functions $g_1, \ldots, g_{n-1}$ have been constructed so that $If \le Ig_k$ for each $k < n$. Let ℓ_n be the tangent line to the concave function If at the point q_n and let I_n be the interval where $\ell_n \le Ig_{n-1}$. The concave function $\min(Ig_{n-1}, \ell_n)$ has a derivative almost everywhere so we may define g_n by $Ig_n = \min(Ig_{n-1}, \ell_n)$.

Figure 11: If, Ig, and a tangent line to If at q_0.

Figure 12: If and Ig_1.

Figure 13: If, Ig_1, and a tangent line to If at q_1.

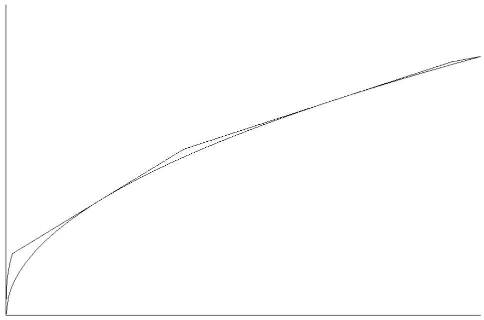

Figure 14: If and Ig_2.

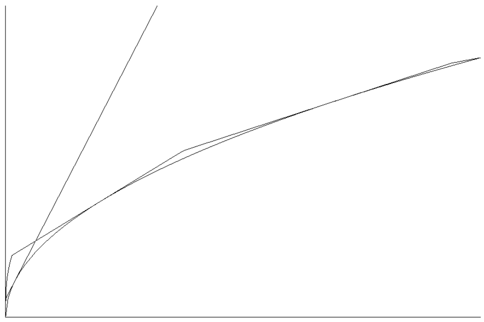

Figure 15: If, Ig_2, and a tangent line to If at q_3.

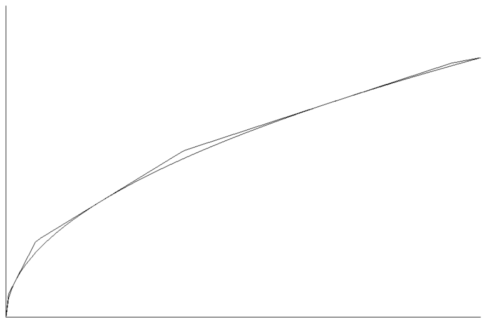

Figure 16: If and Ig_3.

For each n take A_n to be the averaging operator on the single interval I_n and observe that $g_n = A_n g_{n-1}$. Set $Th(x) = \lim_{n\to\infty}(A_n \dots A_1)h(x)$ and verify that the limit exists and defines an admissible operator. This operator T applied to the function g satisfies $ITg = \lim_{n\to\infty} ITg_n = If$ and hence $Tg = f$ as required.

This shows that (L^1, D^∞) is a Calderón couple in the case of Lebesgue measure on the half line.

Aside: The Modulus of Absolute Continuity.

In working with the limit above we actually define $I(Th)$ first, then show it is absolutely continuous and define Th as its derivative. To do this, we require an estimate of the modulus of absolute continuity of a function Ih on an interval. (The h may not be positive.)

There is no such thing.

After asking around and trying to look it up in a few texts, I realized why no one has bothered to mention it. The natural definition would be

$$\omega_{\rm abs}(Ih,[0,x],\delta) = \sup\left\{\sum_{j=1}^{J}|Ih(x_j)-Ih(x_{j-1})| : 0\le x_j\le x, \sum_{j=1}^{J}|x_j-x_{j-1}|<\delta\right\}$$

but it quickly turns into something more familiar,

$$\omega_{\rm abs}(Ih,[0,x],\delta)=\int_0^\delta h^*.$$

General Measures.

Let L^1_λ, D^∞_λ be spaces of λ-measurable functions and L^1, D^∞ be spaces of Lebesgue measurable functions on $(0,\infty)$.

Assume, as usual, that $\Lambda(x)=\lambda(-\infty,x]<\infty$ for each $x\in\mathbf{R}$. Let $\varphi(t)=\inf\{x\in\mathbf{R}:t\le\Lambda(x)\}$ be the generalized inverse of Λ and set $\Phi f=f\circ\varphi$. The non-empty intervals among $(\Lambda(x)-,\Lambda(x))$ are disjoint and the corresponding averaging operator, A_λ, is a projection onto the range of Φ.

$$L^1+D^\infty \underset{A_\lambda}{\overset{\rm id}{\leftrightarrows}} A_\lambda(L^1+D^\infty) \underset{\Phi^{-1}}{\overset{\Phi}{\leftrightarrows}} L^1_\lambda+D^\infty_\lambda$$

One must check that $(f^o)^*=\Phi(f^o)=(\Phi f)^o$.

To see that $(L^1_\lambda,D^\infty_\lambda)$ is a Calderón couple for arbitrary λ, we apply the case of Lebesgue measure. If $\int_0^t(f^o)^*\le\int_0^t(g^o)^*$ then $\int_0^t(\Phi f)^o\le\int_0^t(\Phi g)^o$ so there is an admissible $T:L^1+D^\infty\to L^1+D^\infty$ such that $T\Phi g=\Phi f$. It follows that $\Phi^{-1}A_\lambda T\Phi:L^1_\lambda+D^\infty_\lambda\to L^1_\lambda+D^\infty_\lambda$ is admissible and $(\Phi^{-1}A_\lambda T\Phi)g=f$.

Summary and Comparison. Let $\mathrm{Int}(X_1,X_2)$ denote the set of all exact interpolation spaces between X_1 and X_2. Here L^1 and L^∞ are understood to be spaces of λ-measurable functions on $\mathbf{R}$.

The Couple (L^1,L^∞)

- $K(t,f;L^1,L^\infty)=\int_0^t f^*$
- (L^1,L^∞) is a Calderón couple
- $X\in\mathrm{Int}(L^1,L^\infty)$ if and only if X is u.r.i.

The Couple (L^1,D^∞)

- $L^\infty\subset D^\infty$
- $K(t,f;L^1,D^\infty)=\int_0^t(f^o)^*$
- (L^1,D^∞) is a Calderón couple
- $Y\in\mathrm{Int}(L^1,D^\infty)$ if and only if $\|f\|_Y=\|f^o\|_X$ for some u.r.i. space X
- $Y\in\mathrm{Int}(L^1,D^\infty)$ and has the Fatou property if and only if $Y=D(X)$, with identical norms, for some u.r.i. space X with the Fatou property

The Dual Couple $((D^\infty)', L^\infty)$
• $(D^\infty)' \subset L^1$
• $K(t, f; (D^\infty)', L^\infty) = \int_0^t (f^\downarrow)^*$
• Open question: Is $((D^\infty)', L^\infty)$ a Calderón couple?
• If $Z \in \mathrm{Int}((D^\infty)', L^\infty)$ then $\|f\|_{Z''} = \|f^\downarrow\|_{X'}$ for some u.r.i. space X
• If $\|f\|_Z = \|f^\downarrow\|_{X'}$ for a u.r.i. space X then $Z \in \mathrm{Int}((D^\infty)', L^\infty)$
• $Z \in \mathrm{Int}((D^\infty)', L^\infty)$ and has the Fatou property if and only if $Z = D(X)'$, with identical norms, for some u.r.i. space X with the Fatou property

Answers To All The Exercises.

To find an r.i. space whose dual is not r.i., let δ_x denote the measure consisting of a single atom of mass 1 at x and set $\lambda = \delta_1 + 2\delta_2 + 3\delta_3$. Identify the λ-measurable function f with $(f(1), f(2), f(3)) \in \mathbf{R}^3$ and let X be the weighted L^1 space with norm $\|(a,b,c)\|_X = |a| + 3|b| + 4|c|$. If two elements of this space are equimeasurable, say $(a,b,c)^* = (d,e,f)^*$ (with $a,b,c,d,e,f \geq 0$,) then an easy argument shows that either $(a,b,c) = (d,e,f)$ or else $a = b = f$, and $c = d = e$. In the first case the norms are trivially equal and in the second case,

$$\|(a,b,c)\|_X = a + 3b + 4c = 4a + 4d = d + 3e + 4f = \|(d,e,f)\|_X.$$

This shows that X is a rearrangement invariant space. (Of course, as we may readily verify by looking at $(0,0,1)$ and $(3,0,0)$, X is a not a universally rearrangement invariant space.)

The norm in X' is the norm in a weighted L^∞ space,

$$\|(r,s,t)\|_{X'} = \sup_{a,b,c \geq 0} \frac{|r|a + 2|s|b + 3|t|c}{a + 3b + 4c} = \max(|r|, \tfrac{2}{3}|s|, \tfrac{3}{4}|t|).$$

To see that X' is not r.i. observe that $(1,1,2)^* = (2,2,1)^*$ but

$$\|(1,1,2)\|_{X'} = \frac{3}{2} \neq 2 = \|(2,2,1)\|_{X'}.$$

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