ANGULAR EQUIVALENCE OF NORMED SPACES

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Abstract. Angular equivalence is introduced and shown to be an equivalence relation among the norms on a fixed real vector space. It is a finer notion than the usual (topological) notion of norm equivalence. Angularly equivalent norms share certain geometric properties: A norm that is angularly equivalent to a uniformly convex norm is itself uniformly convex. The same is true for strict convexity. Extreme points of the unit balls of angularly equivalent norms occur on the same rays, and if one unit ball is a polyhedron so is the other.

Among norms arising from inner products, two norms are angularly equivalent if and only if they are topological equivalent. But, unlike topological equivalence, angular equivalence is able to distinguish between different norms on a finite-dimensional space. In particular, no two ℓ^p norms on $\mathbb{R}^n$ are angularly equivalent.

1. Introduction and Definition

Two norms on a real vector space are equivalent if both give rise to the same notion of convergence. A wide variety of functional analysis results concern only the topology a norm generates, not the specific values taken by a given norm. In such a setting, choosing the most convenient one among several, or many, topologically equivalent norms can clarify arguments and simplify proofs.

In other situations, specific properties of individual norms are central to the theory. For example, uniform convexity is an important property of a norm that may not be shared by a topologically equivalent norm.

It is our object here to introduce a finer equivalence of norms, one that preserves certain properties of the norm that simple topological equivalence does not. The idea is straightforward; two norms are *angularly equivalent* if, over all pairs of non-zero vectors, the angle between the pair, determined by one norm, is comparable to the angle between the same pair, determined by the other norm. This will be made precise shortly.

Although the theory of angles in normed spaces cannot match the elegance of its counterpart for inner product spaces, it serves here to give us a means of defining an equivalence of norms that compares vectors two by two rather than one at at time, as topological equivalence does. Thus, angular equivalence emerges as a kind of "second order" equivalence compared to the "first order" topological equivalence.

Definition 1. *Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a real vector space X are topologically equivalent provided there exist positive constants m, M such that for all $x, y \in X$,*

$$m\|x\|_1 \le \|x\|_2 \le M\|x\|_1. \tag{1.1}$$

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Very little is needed to give the definition of angular equivalence besides an accessible concept of angle in normed spaces. We define an angle based on the g-functional, introduced in [5] and studied in [2, Chapter 4] and references there. It it closely connected to smoothness and convexity properties of the unit ball and lends itself to straightforward calculations.

Fix vectors x and y in a real vector space X, with norm $\|\cdot\|$. A few applications of the triangle inequality are enough to show that $\frac{1}{t}(\|x+ty\|-\|x\|)$ is a non-decreasing function of t taking $(-\infty,0)\cup(0,\infty)$ into $[-\|y\|,\|y\|]$. It follows that both

$$g^+(x,y)=\|x\|\lim_{t\to 0+}\tfrac{1}{t}(\|x+ty\|-\|x\|) \quad\text{and}\quad g^-(x,y)=\|x\|\lim_{t\to 0-}\tfrac{1}{t}(\|x+ty\|-\|x\|)$$

exist, and satisfy

(1.2)

$$-\|x\|\|y\|\le\|x\|(\|x\|-\|x-y\|)\le g^-(x,y)\le g^+(x,y)\le\|x\|(\|x+y\|-\|x\|)\le\|x\|\|y\|.$$

Definition 2. *Suppose $\|\cdot\|$ is a norm on a real vector space X. The g-functional relative to $\|\cdot\|$ is the map $g: X\times X\to[0,\infty)$ given by, $g=\frac{1}{2}(g^-+g^+)$. If x and y are non-zero vectors in X, the norm angle*[1] *from x to y is $\theta(x,y)$, defined by $0\le\theta\le\pi$ and*

$$\cos\theta(x,y)=\frac{g(x,y)}{\|x\|\|y\|}.$$

We will refrain from referring to the norm angle "between" x and y, since the norm angle from x to y may not coincide with the norm angle from y to x. If the norm in X arises from an inner product, it is easy to see that norm angles agree with angles defined by the inner product. To see that $\theta(x,y)$ does not depend on the lengths of x and y, make the substitution $s=bt/a$ in the defining limits to get $g(ax,by)=abg(x,y)$ whenever $a,b>0$. A little extra care shows that this equation holds for any $a,b\in\mathbb{R}$ so, in particular, $g(x,-y)=-g(x,y)$.

Definition 3. *Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a real vector space X are angularly equivalent provided there exists a constant C such that for all non-zero $x,y\in X$,*

$$\tan(\theta_2(x,y)/2)\le C\tan(\theta_1(x,y)/2). \tag{1.3}$$

Here $\theta_1(x,y)$ and $\theta_2(x,y)$ are the norm angles from x to y relative to $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Also, $\tan(\pi/2)$ is taken to be $+\infty$.

It is clear from the definition that angular equivalence is a reflexive and transitive relation. To see that it is also symmetric, replace y by $-y$ in (1.3). Then, for $j=1,2$, $g_j(x,-y)=-g_j(x,y)$, $\cos\theta_j(x,-y)=-\cos\theta_j(x,y)$ and

$$\begin{aligned}\tan(\theta_j(x,y)/2)&=\left(\frac{1-\cos\theta_j(x,y)}{1+\cos\theta_j(x,y)}\right)^{1/2}\\&=\left(\frac{1+\cos\theta_j(x,-y)}{1-\cos\theta_j(x,-y)}\right)^{1/2}=\frac{1}{\tan(\theta_j(x,-y)/2)}.\end{aligned}$$

So if (1.3) holds for all non-zero x,y then so does

(1.4)

$$\tan(\theta_1(x,y)/2)=\frac{1}{\tan(\theta_1(x,-y)/2)}\le\frac{C}{\tan(\theta_2(x,-y)/2)}=C\tan(\theta_2(x,y)/2).$$

[1] The term "g-angle" is in use, coined by Pavle Miličić for an angle based on a symmetrized g-functional.

Thus angular equivalence is an equivalence relation.

Since (1.3) implies (1.4) with the same constant C, angular equivalence has a strong form of symmetry. It follows that the constant C is necessarily at least 1.

Remark 1. *We will see later that for "most" x and y, g^- and g^+ coincide. When they differ, the choice of g as the average of the two is unimportant for our purposes; the notion of angular equivalence does not depend on the value of g at these exceptional pairs. Moreover, at pairs for which g^- and g^+ coincide, any semi-inner product necessarily agrees with their common value. So angular equivalence is not dependent on the specific functional used to define it. See* [2] *for a thorough discussion of semi-inner products. This remark is supported by Theorem 3.3 and Corollaries 3.4 and 3.5 below.*

The use of the trigonometric ratio $\tan(\theta/2)$ in the definition of angular equivalence may seem arbitrary, since there are many possible ways to express the condition that two angles be comparable. Our choice is motivated by the situation for inner product spaces, where the ratio $\tan(\theta/2)$ appears in the following sharp inequality relating the best constants for topological and angular equivalence.

Proposition 1.1 ([4])**.** *Let X be a real vector space equipped with inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$, giving rise to norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in the usual way. Let $m, M \in [0,\infty]$ be the best constants in the inequality*

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1, \quad x \in X.$$

Then, for any non-zero $x, y \in X$,

$$\tan(\theta_2(x,y)/2) \leq (M/m)\tan(\theta_1(x,y)/2).$$

The constant M/m is best possible in this inequality. Here θ_1 and θ_2 are the angles between x and y relative to $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. (The usual angles coincide with the norm angles in inner product spaces.)

This result is a reformulation of the generalized Wielandt inequality. The original, based on [8] and appearing in [1], involves the action of an invertible matrix on $\mathbb{C}^n$, relating the angle between a pair of vectors and the angle between the transformed vectors. Since $M/m < \infty$ if and only if the two norms are topologically equivalent, the reformulated inequality shows that if two norms arise from inner products on the same vector space, then the norms are angularly equivalent if and only if they are topologically equivalent.

This is an indication that angular equivalence is not too much finer than topological equivalence; the two notions coincide on an important class of spaces. As an indication that angular equivalence is nonetheless considerably finer than topological equivalence see Corollaries 2.3 and 2.4 below, where we show that, unlike topological equivalence, norms on a finite-dimensional spaces are not all angularly equivalent. (Of course, we have not yet established that angular equivalence is in fact finer than topological equivalence. But we will; see Theorem 4.2.)

2. A First Look at Angular Equivalence

Many properties shared by angularly equivalent norms may be deduced from the definition with minimal calculation, while others emerge only after careful analysis. In this section we present a selection of the former.

Recall that if E is a subset of a real vector space X we say $x \in E$ is an *extreme point* of E provided x is not in any open line segment contained in E. When X is a normed space and E is the closed unit ball of X this is of particular interest in functional analysis. Our first result shows that the unit balls of angularly equivalent norms share the same extreme "directions".

Theorem 2.1. *Let X be a real vector space having two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are angularly equivalent and x is a non-zero vector in X. Then $x/\|x\|_1$ is an extreme point of the $\|\cdot\|_1$-unit ball if and only if $x/\|x\|_2$ is an extreme point of the $\|\cdot\|_2$-unit ball.*

Proof. We argue the contrapositive. Suppose $x/\|x\|_2$ is not an extreme point of the $\|\cdot\|_2$-unit ball. Then there are points y and z in X such that $(y+z)/2 = x/\|x\|_2$ and the closed line segment from y to z is contained in the $\|\cdot\|_2$-unit ball. If $s \in [0,1]$ then $(1-s)y+sz$ and $sy+(1-s)z$ are on the line segment and hence in the $\|\cdot\|_2$-unit ball. Thus,

$$2 = \|y+z\|_2 = \|(1-s)y+sz+sy+(1-s)z\|_2 \le \|(1-s)y+sz\|_2+\|sy+(1-s)z\|_2 \le 2.$$

It follows that $\|(1-s)y+sz\|_2 = 1$. In particular, observe that $\|y\|_2 = \|z\|_2 = 1$. Now,

$$\begin{aligned} g_2^{\pm}(y,z) &= \lim_{t\to 0\pm} \tfrac{1}{t}(\|y+tz\|_2 - 1) \\ &= \lim_{s\to 0\pm} \tfrac{1-s}{s}(\|y+\tfrac{s}{1-s}z\|_2 - 1) \\ &= \lim_{s\to 0\pm} \tfrac{1}{s}(\|(1-s)y+sz\|_2 - 1 + s) = 1. \end{aligned}$$

This shows that $g_2(y,z) = 1$, $\cos(\theta_2(y,z)) = 1$, and $\tan(\theta_2(y,z)/2) = 0$. By angular equivalence, $\tan(\theta_1(y,z)/2) = 0$ as well. This implies $\cos(\theta_1(y,z)) = 1$ and hence $g_1(y,z) = \|y\|_1\|z\|_1$. The last statement, which may be written as

$$g_1^-(y,z) + g_1^+(y,z) = 2\|y\|_1\|z\|_1,$$

combined with

$$g_1^-(y,z) \le g_1^+(y,z) \le \|y\|_1(\|y+z\|_1 - \|y\|_1) \le \|y\|_1\|z\|_1,$$

from (1.2), gives

$$\|y\|_1(\|y+z\|_1 - \|y\|_1) = \|y\|_1\|z\|_1.$$

Since $\|y+z\|_1 = \|y\|_1 + \|z\|_1$ and $x/\|x\|_2 = (y+z)/2$, we have

$$\frac{x}{\|x\|_1} = \frac{y+z}{\|y+z\|_1} = \frac{\|y\|_1}{\|y\|_1+\|z\|_1}\frac{y}{\|y\|_1} + \frac{\|z\|_1}{\|y\|_1+\|z\|_1}\frac{z}{\|z\|_1},$$

which is a convex combination of the points $y/\|y\|_1$ and $z/\|z\|_1$. Thus, $x/\|x\|_1$ is an interior point of the line segment from $y/\|y\|_1$ to $z/\|z\|_1$. Since the endpoints of this segment lie in the $\|\cdot\|_1$-unit ball, convexity shows that the entire line segment does. Thus, $x/\|x\|_1$ is not an extreme point of the $\|\cdot\|_1$-unit ball.

Reversing the roles of the two norms gives the other implication. □

A normed space is *strictly convex* if every boundary point of the unit ball is an extreme point. It is immediate that strict convexity is preserved by angular equivalence. The corresponding result for uniform convexity also holds, see Corollary 2.7.

Corollary 2.2. *Suppose X be a real vector space having two angularly equivalent norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Then X is strictly convex when equipped with $\|\cdot\|_1$ if and only if X is strictly convex when equipped with $\|\cdot\|_2$.*

The unit ball of finite-dimensional normed space is *polygonal* if and only if it has only finitely many extreme points, namely, the vertices of the polygon.

Corollary 2.3. *Let X be a finite-dimensional real vector space having two angularly equivalent norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Then the $\|\cdot\|_1$-unit ball is polygonal if and only if the $\|\cdot\|_2$-unit ball is polygonal. In this case, the vertices of the two polygons lie on the same rays.*

Remark 2. *If two norms are angularly equivalent on X then the restrictions of those two norms to any subspace Y of X are also angularly equivalent. This follows easily from the definition. However, the subspace Y may have extreme points that the original space did not have. For every subspace Y, the rays containing extreme points relative to Y are the same for both norms.*

One of the most telling features of angular equivalence is its ability to distinguish between norms on a finite-dimensional space. Here we see that no two ℓ^p norms are angularly equivalent. (This only applies in finite dimensions. The angular equivalence of the usual ℓ^p sequence space norms is a question that does not arise naturally, since they are not norms on the same underlying vector space.)

Corollary 2.4. *Suppose $p, q \in [1, \infty]$. For any integer $n \geq 2$, the ℓ^p and ℓ^q norms on $\mathbb{R}^n$ are angularly equivalent only if $p = q$.*

Proof. If the two norms are angularly equivalent then their restrictions to the subspace $\mathbb{R}^2$ of $\mathbb{R}^n$ (embedded in the usual way) are also angularly equivalent. So we may assume $n = 2$. The unit ball in the ℓ^1 norm is a square with vertices at $(0, \pm 1)$ and $(\pm 1, 0)$. The unit ball in the ℓ^∞ norm is a square with vertices at $(\pm 1, \pm 1)$. When $1 < p < \infty$, the unit ball in the ℓ^p norm is not a polygon. By Corollary 2.3, the ℓ^1 norm and the ℓ^∞ norm are not angularly equivalent to each other or to any other ℓ^p norm.

It remains to consider $p, q \in (1, \infty)$. Fix $s > 0$ and consider the vectors $(1, 0)$ and $(1, s)$. The value of the ℓ^p-norm g-functional for this pair is,

$$\|(1,0)\|_{\ell^p} \lim_{t\to 0} \tfrac{1}{t}(\|(1,0)+t(1,s)\|_{\ell^p} - \|(1,0)\|_{\ell^p}) = \lim_{t\to 0} \tfrac{1}{t}(((1+t)^p + |t|^p s^p)^{1/p} - 1) = 1.$$

Thus $\theta_{\ell^p} \equiv \theta_{\ell^p}((1,0),(1,s))$ satisfies,

$$\cos\theta_{\ell^p} = \frac{1}{(1+s^p)^{1/p}} \quad \text{and} \quad \tan^2\theta_{\ell^p} = \frac{(1+s^p)^{1/p} - 1}{(1+s^p)^{1/p} + 1}.$$

By angular equivalence of the ℓ^p and ℓ^q norms, there exists a $C > 0$, independent of s such that

$$\frac{1}{C} \leq \left(\frac{(1+s^p)^{1/p} - 1}{(1+s^p)^{1/p} + 1}\right)\left(\frac{(1+s^q)^{1/q} + 1}{(1+s^q)^{1/q} - 1}\right) \leq C$$

Taking the limit as $s \to 0+$ it is easy to see that this expression remains bounded above and below only if $p = q$. □

The next theorem may be viewed as a stability result for angular equivalence.

Theorem 2.5. *Suppose* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are angularly equivalent norms on a real vector space* X*. Then the norm* $\|\cdot\|_3 = \|\cdot\|_1 + \|\cdot\|_2$ *is angularly equivalent to* $\|\cdot\|_1$ *and* $\|\cdot\|_2$*.*

Proof. Fix non-zero $x, y \in X$. Let $g_j(x,y)$ and $\theta_j = \theta_j(x,y)$ be the g-functional and norm angle from x to y with respect to the norm $\|\cdot\|_j$, for $j = 1,2,3$. The definition of the g-functional shows that

$$\frac{g_3(x,y)}{\|x\|_3} = \frac{g_1(x,y)}{\|x\|_1} + \frac{g_2(x,y)}{\|x\|_2},$$

or equivalently,

$$\|y\|_3 \cos\theta_3 = \|y\|_1 \cos\theta_1 + \|y\|_2 \cos\theta_2.$$

Therefore,

$$\begin{aligned}\tan^2(\theta_3/2) = \frac{1-\cos\theta_3}{1+\cos\theta_3} &= \frac{\frac{\|y\|_1}{\|y\|_3}(1-\cos\theta_1) + \frac{\|y\|_2}{\|y\|_3}(1-\cos\theta_2)}{\frac{\|y\|_1}{\|y\|_3}(1+\cos\theta_1) + \frac{\|y\|_2}{\|y\|_3}(1+\cos\theta_2)} \\ &= \frac{1 + \frac{\|y\|_2}{\|y\|_1}\frac{1-\cos\theta_2}{1-\cos\theta_1}}{1 + \frac{\|y\|_2}{\|y\|_1}\frac{1+\cos\theta_2}{1+\cos\theta_1}} \tan^2(\theta_1/2).\end{aligned}$$

But for any $A, B > 0$, $\frac{1+A}{1+B} \le 1 + \frac{A}{B}$, so

$$\tan^2(\theta_3/2) \le \left(1 + \frac{\tan^2(\theta_2/2)}{\tan^2(\theta_1/2)}\right)\tan^2(\theta_1/2) \le (1+C^2)\tan^2(\theta_1/2),$$

where C is the constant of angular equivalence of $\|\cdot\|_1$ and $\|\cdot\|_2$ from (1.3). Taking square roots shows that $\|\cdot\|_3$ is angularly equivalent to $\|\cdot\|_1$, and hence to $\|\cdot\|_2$ as well. □

With stability under sums of norms we can easily give an example to show that an angular equivalency class containing inner product norms may contain other norms as well.

Example 1. *A norm that is angularly equivalent to a norm arising from an inner product need not arise from an inner product: Consider the two norms on* $\mathbb{R}^2$ *defined by* $\|(\xi,\eta)\|_1 = (3\xi^2+\eta^2)^{1/2}$ *and* $\|(\xi,\eta)\|_2 = (\xi^2+3\eta^2)^{1/2}$*. These both arise from inner products and are topologically equivalent so by Theorem 1.1 they are angularly equivalent. Theorem 2.5 shows that their sum,* $\|(\xi,\eta)\|_3 = \|(\xi,\eta)\|_1 + \|(\xi,\eta)\|_2$ *is also angularly equivalent. However, it is easy to check that* $\|\cdot\|_3$ *does not arise from an inner product. (For example, the parallelogram law fails for the vectors* $(1,0)$ *and* $(0,1)$*.)*

The stability of angular equivalence does not extend to the maximum of two norms.

Example 2. *The maximum of two angularly equivalent norms is a norm that need not be angularly equivalent to the original two. Let* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *be the norms in the previous example, set* $\|(\xi,\eta)\|_4 = \max(\|(\xi,\eta)\|_1, \|(\xi,\eta)\|_2)$*, and let* $g_j(x,y)$ *and* $\theta_j = \theta_j(x,y)$ *be the g-functional and norm angle from* x *to* y *with respect to the norm* $\|\cdot\|_j$*, for* $j = 1,4$*. Now let* $s > 0$ *and take* $x = (1,1)$ *and* $y = (1-s, 1+s)$*. Calculations show that*

$$\|x+ty\|_1 = 2\sqrt{(1+t)^2 - (1+t)ts + t^2s^2} \quad \textit{and} \quad g_1(x,y) = 4-2s;$$

and also that

$\|x+ty\|_4 = 2\sqrt{(1+t)^2+(1+t)|t|s+t^2s^2}, \quad g_4^{\pm}(x,y) = 4\pm 2s, \quad \text{and } g_4(x,y) = 4.$ *But* $\|x\|_1 = \|x\|_4 = 2$, $\|y\|_1 = 2\sqrt{1-s+s^2}$ *and* $\|y\|_4 = 2\sqrt{1+s+s^2}$*. So, using the definition of* $\cos\theta_1(x,y)$ *and* $\cos\theta_4(x,y)$*, we get,*

$$\tan^2(\theta_1(x,y)/2) = \frac{3s^2}{4(1-s/2+\sqrt{1-s+s^2})^2}$$

and

$$\tan^2(\theta_4(x,y)/2) = \frac{s+s^2}{(1+\sqrt{1+s+s^2})^2}.$$

Clearly, there is no constant C *for which* $\tan(\theta_4(x,y)/2) \le C\tan(\theta_1(x,y)/2)$ *as* $s \to 0+$*. Thus* $\|\cdot\|_4$ *is not angularly equivalent to* $\|\cdot\|_1$*.*

We close this section with the promised proof that uniform convexity is preserved by angular equivalence. A real vector space X, with norm $\|\cdot\|$, is *uniformly convex* provided that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x-y\| \ge \varepsilon$ then $\|(x+y)/2\| \le 1-\delta$. First we show that unform convexity can be characterized in terms of the norm angle. The idea appears in [7], where closely related results are proved.

Theorem 2.6. *Let* X *be a real vector space with norm* $\|\cdot\|$ *and let* $\theta(x,y)$ *denote the norm angle from* x *to* y*. Then* X *is uniformly convex if and only if for all* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* $\|x\| = \|y\| = 1$ *and* $\|x-y\| \ge \varepsilon$ *then* $\tan(\theta(x,y)/2) \ge \delta$*.*

Proof. Suppose X is uniformly convex. Fix $\varepsilon > 0$ and choose an $\eta > 0$ so that $\|(x+y)/2\| \le 1-\eta$ whenever $\|x\| = \|y\| = 1$ and $\|x-y\| \ge \varepsilon$. Set $\delta = \sqrt{\eta}$. If $\|x\| = \|y\| = 1$ and $\|x-y\| \ge \varepsilon$ then (1.2) shows that $-1 \le g(x,y) \le \|x+y\|-1 \le 1$ so

$$\tan(\theta(x,y)/2) = \sqrt{\frac{1-g(x,y)}{1+g(x,y)}} \ge \sqrt{\frac{1-g(x,y)}{2}} \ge \sqrt{1-\left\|\frac{x+y}{2}\right\|} \ge \sqrt{\eta} = \delta.$$

For the converse, fix $\varepsilon > 0$ and choose an $\eta > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x-y\| \ge \varepsilon/4$ then $\tan(\theta(x,y)/2) \ge \eta$. Set $\delta = \min(\eta^2/(1+\eta^2), \varepsilon/4)$. Now suppose $\|x\| = \|y\| = 1$ and $\|x-y\| \ge \varepsilon$, and set $z = x+y$. If $z = 0$, then the desired conclusion, $\|z/2\| \le 1-\delta$, is trivial. Otherwise,

$$\|(2-\|z\|)x - \|z\|((z/\|z\|)-x)\| = \|x-y\| \ge \varepsilon,$$

so either $\|(2-\|z\|)x\| \ge 2\delta$ or $\|\|z\|((z/\|z\|)-x)\| \ge \varepsilon - 2\delta$. The first of the two implies that $\|z/2\| \le 1-\delta$ and completes the proof. The choice of δ ensures that $\varepsilon - 2\delta \ge \varepsilon/2$ so the second implies $\|(z/\|z\|)-x\| \ge \varepsilon/(2\|z\|) \ge \varepsilon/4$, and we have

$$\eta \le \tan(\theta(z/\|z\|,x)/2) = \sqrt{\frac{1-g(z/\|z\|,x)}{1+g(z/\|z\|,x)}}.$$

But this is equivalent to $g(z/\|z\|,x) \le (1-\eta^2)/(1+\eta^2)$. Using (1.2) again, this time with x and y replaced by z and x, respectively, gives $\|z\|-1 \le g(z/\|z\|,x)$ so we have,

$$\|z/2\| \le \tfrac{1}{2}(1+g(z/\|z\|,x)) \le \frac{1}{2}\Big(1+\frac{1-\eta^2}{1+\eta^2}\Big) = 1 - \frac{\eta^2}{1+\eta^2} \le 1-\delta.$$

□

The next result uses the previous theorem so it is included here despite its dependence on Theorem 4.2 below.

Corollary 2.7. *Suppose X be a real vector space having two angularly equivalent norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Then X is uniformly convex when equipped with $\|\cdot\|_1$ if and only if X is uniformly convex when equipped with $\|\cdot\|_2$.*

Proof. Let C be the constant in the definition of angular equivalence so that, using symmetry and (1.3), for all non-zero x and y in X we have $\tan(\theta_1(x,y)/2) \le C\tan(\theta_2(x,y)/2)$.

Theorem 4.2 shows that $\|\cdot\|_1$ and $\|\cdot\|_2$ are also topologically equivalent so there exist constants m and M such that (1.1) holds.

Suppose X is uniformly convex when equipped with $\|\cdot\|_1$ and fix $\varepsilon > 0$. Choose $\eta > 0$ so that, if $\|x\|_1 = \|y\|_1 = 1$ and $\|x-y\|_1 \ge m\varepsilon/(2M)$, then $\tan(\theta_1(x,y)/2) \ge \eta$. Set $\delta = \eta/C$.

Let $\|x\|_2 = \|y\|_2 = 1$ and $\|x-y\|_2 \ge \varepsilon$, and set $\hat{x} = x/\|x\|_1$ and $\hat{y} = y/\|y\|_1$. Clearly, $\|\hat{x}\|_1 = \|\hat{y}\|_1 = 1$. Also, (as in the proof of the Dunkl-Williams inequality,)

$$\begin{aligned}\varepsilon \le \|x-y\|_2 &= \left\| \frac{\hat{x}}{\|\hat{x}\|_2} - \frac{\hat{y}}{\|\hat{x}\|_2} + \frac{\hat{y}}{\|\hat{x}\|_2} - \frac{\hat{y}}{\|\hat{y}\|_2} \right\|_2 \\ &\le \frac{\|\hat{x}-\hat{y}\|_2}{\|\hat{x}\|_2} + \frac{|\|\hat{x}\|_2 - \|\hat{y}\|_2|}{\|\hat{x}\|_2} \le \frac{2\|\hat{x}-\hat{y}\|_2}{\|\hat{x}\|_2}.\end{aligned}$$

But $\|\hat{x}\|_2 = \|x\|_2/\|x\|_1 \ge m$ so $\|\hat{x}-\hat{y}\|_1 \ge (1/M)\|\hat{x}-\hat{y}\|_2 \ge m\varepsilon/(2M)$. This ensures that $\tan(\theta_1(\hat{x},\hat{y})/2) \ge \eta$. But $\theta_1(\hat{x},\hat{y}) = \theta_1(x,y)$ so angular equivalence implies $\tan(\theta_2(x,y)/2) \ge \eta/C = \delta$. This shows that X is uniformly convex when equipped with $\|\cdot\|_2$. For the other implication, reverse the roles of $\|\cdot\|_1$ and $\|\cdot\|_2$. □

3. Norms in the Plane

Since angular equivalence is defined in terms of pairs of vectors, it is natural to study it first in two-dimensional spaces. Our analysis in the plane is more than the investigation of a special case, however. It enables us to establish results for real vector spaces of any dimension: It is evident from the definition that, if a pair of norms on a real vector space X are angularly equivalent then the restrictions of those norms to any subspace of X are also angularly equivalent. We will make use of the equally evident converse: If the restrictions of two norms on X to any two-dimensional subspace of X are angularly equivalent *with constant C*, then the original norms are also angularly equivalent with the same constant C. Uniform control of the constant is essential here.

In the plane we can study norms by viewing the boundary of their unit balls as polar functions. The following setup and notation will be used throughout this section and the next. Here and throughout, the symbol "$\angle$" denotes the usual angle in the plane, not the norm angle.

Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^2$ and define $r > 0$ by requiring $\|r(t)(\cos t, \sin t)\| = 1$ for all $t \in \mathbb{R}$. Then r is a strictly positive, π-periodic function and the unit ball, $\{\rho(\cos t, \sin t) : 0 \le \rho \le r(t), t \in \mathbb{R}\}$, is a closed, convex set with $(0,0)$ in its interior. The convexity of the unit ball readily implies that r is continuous. Since r is continuous and periodic it attains its minimum and maximum values; both are strictly positive. Call them r_m and r_M, respectively. Let $O = (0,0)$ and parametrize the boundary of the ball by setting $P_t = r(t)(\cos t, \sin t)$ for all $t \in \mathbb{R}$. If $0 < |\alpha - \beta| < \pi$, set

$$\psi(\alpha,\beta) = \begin{cases} \angle OP_\alpha P_\beta, & \alpha > \beta, \\ \pi - \angle OP_\alpha P_\beta, & \alpha < \beta. \end{cases}$$

Also define $\varphi^-(\alpha) = \lim_{\beta\to\alpha-} \psi(\alpha,\beta)$ and $\varphi^+(\alpha) = \lim_{\beta\to\alpha+} \psi(\alpha,\beta)$. These exist by property (ii) below.

Lemma 3.1. *Suppose $\alpha, \beta \in \mathbb{R}$ with $0 < |\alpha - \beta| < \pi$. Then:*

(i) $\psi(\alpha,\beta) + \alpha = \psi(\beta,\alpha) + \beta$;
(ii) $\psi(\alpha,t)$ *is non-decreasing for t in* $(\alpha - \pi, \alpha) \cup (\alpha, \alpha + \pi)$;
(iii) $0 < \psi(\alpha,\beta) < \pi$ *and, if* $0 < |\alpha - \beta| \le \pi/2$, *then*
$$0 < \tan^{-1}(r_m/r_M) \le \psi(\alpha,\beta) \le \pi - \tan^{-1}(r_m/r_M) < \pi;$$
(iv) $r(\alpha)/r(\beta) = \cos(\alpha-\beta) + \sin(\alpha-\beta)\cot\psi(\alpha,\beta)$ *and ψ is continuous;*
(v) *the left derivative of r at α exists and equals $r(\alpha)\cot\varphi^-(\alpha)$, the right derivative of r at α exists and equals $r(\alpha)\cot\varphi^+(\alpha)$, and r is differentiable wherever $\varphi^- = \varphi^+$;*
(vi) $\varphi^-(\alpha) \le \varphi^+(\alpha)$ *and if $\alpha < \beta$ then* $\varphi^+(\alpha) + \alpha \le \varphi^-(\beta) + \beta$;
(vii) φ^- *is left continuous and* φ^+ *is right continuous;*
(viii) φ^- *and* φ^+ *are continuous at all but countably many points, and φ^- is continuous at α if and only if φ^+ is continuous at α if and only if* $\varphi^-(\alpha) = \varphi^+(\alpha)$.

Proof. The angles of triangle $P_\alpha O P_\beta$ add to π. If $\alpha > \beta$ they are $\alpha - \beta$, $\psi(\alpha,\beta)$, and $\pi - \psi(\beta,\alpha)$ and if $\alpha < \beta$ they are $\beta - \alpha$, $\psi(\beta,\alpha)$ and $\pi - \psi(\alpha,\beta)$. This proves (i).

To prove (ii), suppose $s < t$ and both lie in $(\alpha - \pi, \alpha) \cup (\alpha, \alpha + \pi)$. If $s < t < \alpha$ then, by convexity, the segment $P_s P_\alpha$ intersects the segment OP_t. It follows that,

$$\psi(\alpha,s) = \angle 0P_\alpha P_s \le \angle 0P_\alpha P_t = \psi(\alpha,t).$$

If $s < \alpha < t$ then the segment $P_s P_t$ intersects the segment OP_α. Thus

$$\psi(\alpha,s) + \pi - \psi(\alpha,t) = \angle OP_\alpha P_s + \angle OP_\alpha P_t = \angle P_s P_\alpha P_t \le \pi.$$

If $\alpha < s < t$ then the segment $P_\alpha P_t$ intersects the segment OP_s. It follows that,

$$\pi - \psi(\alpha,s) = \angle 0P_\alpha P_s \ge \angle 0P_\alpha P_t = \pi - \psi(\alpha,t).$$

In each of the three cases we have $\psi(\alpha,s) \le \psi(\alpha,t)$, as required.

Since $r(\alpha) > 0$, $r(\beta) > 0$, and $0 < |\alpha - \beta| < \pi$, $\triangle P_\alpha O P_\beta$ is non-degenerate. In particular, $0 < \psi(\alpha,\beta) < \pi$, the first statement of (iii). For the other, suppose $0 < \alpha - \beta < \pi/2$. Then, using (ii),

$$\psi(\alpha,\beta) \ge \psi(\alpha,\alpha - \pi/2) = \tan^{-1}(r(\alpha-\pi/2)/r(\alpha)) \ge \tan^{-1}(r_m/r_M) > 0$$

and

$$\psi(\alpha,\beta) \le \psi(\alpha,\alpha + \pi/2) = \pi - \tan^{-1}(r(\alpha+\pi/2)/r(\alpha)) \le \pi - \tan^{-1}(r_m/r_M) < \pi.$$

The sine law in $\triangle OP_\alpha P_\beta$ gives

$$\frac{\sin\psi(\alpha,\beta)}{r(\beta)} = \frac{\sin(\angle OP_\alpha P_\beta)}{r(\beta)} = \frac{\sin(\angle OP_\beta P_\alpha)}{r(\alpha)} = \frac{\sin(\pi - \psi(\alpha,\beta) - (\alpha-\beta))}{r(\alpha)}$$

when $\alpha > \beta$ and

$$\frac{\sin(\pi - \psi(\alpha,\beta))}{r(\beta)} = \frac{\sin(\angle OP_\alpha P_\beta)}{r(\beta)} = \frac{\sin(\angle OP_\beta P_\alpha)}{r(\alpha)} = \frac{\sin(\psi(\alpha,\beta) - (\beta-\alpha))}{r(\alpha)}$$

when $\alpha < \beta$. These both reduce to the equation in (iv). Continuity of r now implies continuity of ψ.

To prove (v) we use (iv) to get

$$\frac{r(\alpha) - r(s)}{\alpha - s} = r(s)\left(\frac{\cos(\alpha - s) - 1}{\alpha - s} + \frac{\sin(\alpha - s)}{\alpha - s}\cot\psi(\alpha, s)\right).$$

The definitions of φ^+ and φ^- (and the continuity of r) show that the left and right derivatives of r exist and are equal to,

$$\lim_{s\to\alpha-} \frac{r(\alpha) - r(s)}{\alpha - s} = r(\alpha)\cot\varphi^-(\alpha) \quad \text{and} \quad \lim_{s\to\alpha+} \frac{r(\alpha) - r(s)}{\alpha - s} = r(\alpha)\cot\varphi^+(\alpha),$$

respectively. It is immediate that r is differentiable wherever $\varphi^+ = \varphi^-$.

By (ii),

$$\varphi^-(\alpha) = \lim_{s\to\alpha-} \psi(\alpha, s) \le \lim_{t\to\alpha+} \psi(\alpha, t) = \varphi^+(\alpha).$$

Also, if $\alpha < \beta$ then, by (i) and (ii),

$$\begin{aligned}\varphi^+(\alpha) + \alpha &= \lim_{t\to\alpha+} \psi(\alpha, t) + \alpha \le \psi(\alpha,\beta) + \alpha \\ &= \psi(\beta,\alpha) + \beta \le \lim_{s\to\beta-} \psi(\beta, s) + \beta = \varphi^-(\beta) + \beta.\end{aligned}$$

These prove (vi).

For (vii), observe that if $\gamma > \alpha$, applying (vi), (ii), and then (iv) gives

$$\begin{aligned}\varphi^+(\alpha) &\le \lim_{s\to\alpha+} \varphi^-(s) + s - \alpha \le \lim_{s\to\alpha+} \varphi^+(s) \\ &= \lim_{s\to\alpha+}\lim_{t\to s+} \psi(s,t) \le \lim_{s\to\alpha+} \psi(s,\gamma) = \psi(\alpha,\gamma).\end{aligned}$$

Letting $\gamma \to \alpha+$ shows $\varphi^+(\alpha) \le \lim_{s\to\alpha+} \varphi^+(s) \le \varphi^+(\alpha)$. Thus, φ^+ is right-continuous at α. A similar argument shows that φ^- is left-continuous.

By (vi), the functions $s \mapsto \varphi^-(s) + s$ and $t \mapsto \varphi^+(t) + t$ are both non-decreasing and hence continuous except at countably many points. It follows that φ^- and φ^+ are both continuous except at countably many points, the first statement of (viii). The second may be deduced from the following consequence of (vi) and (vii):

$$\begin{aligned}\varphi^-(\alpha) &= \lim_{s\to\alpha-} \varphi^-(s) \le \lim_{s\to\alpha-} \varphi^+(s) \le \varphi^-(\alpha) \\ &\le \varphi^+(\alpha) \le \lim_{t\to\alpha+} \varphi^-(t) \le \lim_{t\to\alpha+} \varphi^+(t) = \varphi^+(\alpha).\end{aligned}$$

□

The g-functional and norm angle for a norm on $\mathbb{R}^2$ can be expressed in terms of the functions r, ψ and ϕ.

Theorem 3.2. *Suppose* $\|\cdot\|$ *is a norm on* $\mathbb{R}^2$*, define* r*,* ψ *and* $\varphi^\pm$ *as above, and let* $Q_t = (\cos t, \sin t)$ *for* $t \in \mathbb{R}$*, If* $a, b > 0$ *and* $0 < |\alpha - \beta| < \pi$*, then*

$$g(aQ_\alpha, bQ_\beta) = \frac{ab}{r(\alpha)^2}\left(\cos(\alpha-\beta) + \tfrac{1}{2}(\cot\varphi^-(\alpha) + \cot\varphi^+(\alpha))\sin(\alpha-\beta)\right),$$

$$\cos\theta(aQ_\alpha, bQ_\beta) = \frac{\cot(\alpha-\beta) + \frac{1}{2}(\cot\varphi^-(\alpha) + \cot\varphi^+(\alpha))}{\cot(\alpha-\beta) + \cot\psi(\alpha,\beta)}, \quad \text{and}$$

$$\tan^2(\theta(aQ_\alpha, bQ_\beta)/2) = \frac{\cot\psi(\alpha,\beta) - \frac{1}{2}(\cot\varphi^-(\alpha) + \cot\varphi^+(\alpha))}{2\cot(\alpha-\beta) + \cot\psi(\alpha,\beta) + \frac{1}{2}(\cot\varphi^-(\alpha) + \cot\varphi^+(\alpha))}.$$

If $a, b > 0$ *and* $\alpha = \beta$ *then*

$$g(aQ_\alpha, bQ_\beta) = \frac{ab}{r(\alpha)^2}, \quad \cos\theta(aQ_\alpha, bQ_\beta) = 1, \quad \text{and} \quad \tan^2(\theta(aQ_\alpha, bQ_\beta)/2) = 0.$$

If $a, b > 0$ *and* $|\alpha - \beta| = \pi$ *then*

$$g(aQ_\alpha, bQ_\beta) = -\frac{ab}{r(\alpha)^2}, \quad \cos\theta(aQ_\alpha, bQ_\beta) = -1, \quad \text{and} \quad \tan^2(\theta(aQ_\alpha, bQ_\beta)/2) = \infty.$$

Proof. Since a and b are non-zero, for t sufficiently close to zero we may define c and $\gamma \in (\alpha - \pi/2, \alpha + \pi/2)$ as functions of t, by requiring that $cQ_\gamma = aQ_\alpha + tbQ_\beta$. Evidently, both c and γ are differentiable. Equating components, we have the equations,

$$c\cos\gamma = a\cos\alpha + tb\cos\beta \quad \text{and} \quad c\sin\gamma = a\sin\alpha + tb\sin\beta.$$

Differentiating these with respect to t gives

$$c'\cos\gamma - c\gamma'\sin\gamma = b\cos\beta \quad \text{and} \quad c'\sin\gamma + c\gamma'\cos\gamma = b\sin\beta.$$

This system is readily solved to yield $c' = b\cos(\beta - \gamma)$ and $c\gamma' = b\sin(\beta - \gamma)$. The definition of r ensures that $\|r(t)Q_t\| = 1$ for all t, so

$$\tfrac{1}{t}(\|cQ_\gamma\| - \|aQ_\alpha\|) = \frac{1}{t}\left(\frac{c}{r(\gamma)} - \frac{a}{r(\alpha)}\right) = \frac{\frac{c-a}{t}r(\alpha) - \frac{r(\gamma)-r(\alpha)}{\gamma-\alpha}a\frac{\gamma-\alpha}{t}}{r(\alpha)r(\gamma)}.$$

As $t \to 0$ we have

$$c \to a, \quad \gamma \to \alpha, \quad \tfrac{c-a}{t} \to c'(0) = b\cos(\beta-\alpha) \quad \text{and} \quad a\tfrac{\gamma-\alpha}{t} \to a\gamma'(0) = b\sin(\beta-\alpha).$$

Notice that γ is a strictly monotone function of t in a neighbourhood of 0. Lemma 3.1(v) shows that

$$\lim_{t\to 0\pm}\frac{r(\gamma)-r(\alpha)}{\gamma-\alpha} = r(\alpha)\cot\varphi^\pm(\alpha) \quad \text{or} \quad \lim_{t\to 0\pm}\frac{r(\gamma)-r(\alpha)}{\gamma-\alpha} = r(\alpha)\cot\varphi^\mp(\alpha),$$

depending on whether γ is increasing or decreasing. In either case,

$$\begin{aligned} g(aQ_\alpha, bQ_\beta) &= \tfrac{1}{2}\|aQ_\alpha\|\left(\lim_{t\to 0+}\tfrac{1}{t}(\|cQ_\gamma\| - \|aQ_\alpha\|) + \lim_{t\to 0-}\tfrac{1}{t}(\|cQ_\gamma\| - \|aQ_\alpha\|)\right) \\ &= \frac{a}{r(\alpha)}\frac{b\cos(\beta-\alpha)r(\alpha) - r(\alpha)\frac{1}{2}(\cot\varphi^+(\alpha) + \cot\varphi^-(\alpha))b\sin(\beta-\alpha)}{r(\alpha)^2} \\ &= \frac{ab}{r(\alpha)^2}\left(\cos(\alpha-\beta) + \tfrac{1}{2}(\cot\varphi^-(\alpha) + \cot\varphi^+(\alpha))\sin(\alpha-\beta)\right). \end{aligned}$$

Using this formula and Lemma 3.1(iv), we get

$$\cos\theta(aQ_\alpha, bQ_\beta) = \frac{g(aQ_\alpha, bQ_\beta)}{\|aQ_\alpha\|\|bQ_\beta\|} = \frac{g(aQ_\alpha, bQ_\beta)}{ab/(r(\alpha)r(\beta))} = \frac{\cot(\alpha-\beta) + \frac{1}{2}(\cot\varphi^-(\alpha) + \cot\varphi^+(\alpha))}{\cot(\alpha-\beta) + \cot\psi(\alpha,\beta)}.$$

But $\tan^2(\theta/2) = (1-\cos\theta)/(1+\cos\theta)$, so,

$$\tan^2(\theta(aQ_\alpha, bQ_\beta)/2) = \frac{\cot\psi(\alpha,\beta) - \frac{1}{2}(\cot\varphi^-(\alpha) + \cot\varphi^+(\alpha))}{2\cot(\alpha-\beta) + \cot\psi(\alpha,\beta) + \frac{1}{2}(\cot\varphi^-(\alpha) + \cot\varphi^+(\alpha))}.$$

The formulas in the case $\alpha = \beta$ and $|\alpha - \beta| = \pi$ are easily established directly from the definitions of the g-functional and norm angle. We omit the details. □

If we have two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on $\mathbb{R}^2$ we define r_j, ψ_j and $\varphi_j^\pm$ as above based on $\|\cdot\|_j$, for $j = 1, 2$. The previous theorem shows that the angular equivalence of these two norms can be expressed in the form,

$$\frac{\cot\psi_2(\alpha,\beta) - A_2}{2\cot(\alpha-\beta) + \cot\psi_2(\alpha,\beta) + A_2} \le C^2 \frac{\cot\psi_1(\alpha,\beta) - A_1}{2\cot(\alpha-\beta) + \cot\psi_1(\alpha,\beta) + A_1} \tag{3.1}$$

for all $\alpha, \beta \in \mathbb{R}$ with $0 < |\alpha - \beta| < \pi$. Here $A_j = \frac{1}{2}(\cot\varphi_j^-(\alpha) + \cot\varphi_j^+(\alpha))$ for $j = 1, 2$. The cases $\beta = \alpha$ and $\beta = \alpha \pm \pi$ need not be included, as $\tan(\theta_2(aQ_\alpha, bQ_\beta)/2) \le C\tan(\theta_1(aQ_\alpha, bQ_\beta)/2)$ holds trivially in those cases.

However, one can avoid discontinuities of the functions $\varphi_1^\pm$ and $\varphi_2^\pm$. Next we see that it is enough to consider only values of α for which A_1 and A_2 simplify.

Theorem 3.3. *The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^2$ are angularly equivalent with constant C if and only if (3.1) holds for all α, β satisfying*

$$\varphi_1^+(\alpha) = \varphi_1^-(\alpha), \quad \varphi_2^+(\alpha) = \varphi_2^-(\alpha), \quad \varphi_1^+(\beta) = \varphi_1^-(\beta), \quad \varphi_2^+(\beta) = \varphi_2^-(\beta), \tag{3.2}$$

and $0 < |\alpha - \beta| < \pi$.

Proof. If the two norms are angularly equivalent, then by Theorem 3.2, (3.1) holds for all α, β such that $0 < |\alpha - \beta| < \pi$. Conversely, suppose (3.1) holds for all α, β satisfying (3.2) and $0 < |\alpha - \beta| < \pi$. We will show that (3.1) holds for all α, β satisfying $0 < |\alpha - \beta| < \pi$.

By Lemma 3.1(viii) the set $E = \{t \in \mathbb{R} : \varphi_1^+(t) = \varphi_1^-(t), \varphi_2^+(t) = \varphi_2^-(t)\}$ has a countable complement and is therefore dense in $\mathbb{R}$. By hypothesis, for each $\alpha \in E$, inequality (3.1) holds for all $\beta \in E \cap ((\alpha-\pi, \alpha) \cup (\alpha, \alpha+\pi))$. The continuity of ψ_1 and ψ_2, from Lemma 3.1(iv), shows that it remains valid for all $\beta \in (\alpha-\pi, \alpha) \cup (\alpha, \alpha+\pi)$. Fix a $\beta \in \mathbb{R}$. Then (3.1) holds for all $\alpha \in E \cap ((\beta-\pi, \beta) \cup (\beta, \beta+\pi))$. Taking one-sided limits of (3.1) with respect to α, but only through points of E, gives, for each $\alpha \in (\beta-\pi, \beta) \cup (\beta, \beta+\pi)$, the two inequalities, (one with "+" and one with "−")

$$\frac{\cot\psi_2(\alpha,\beta) - \cot\varphi_2^\pm(\alpha)}{2\cot(\alpha-\beta) + \cot\psi_2(\alpha,\beta) + \cot\varphi_2^\pm(\alpha)} \le \frac{C^2(\cot\psi_1(\alpha,\beta) - \cot\varphi_1^\pm(\alpha))}{2\cot(\alpha-\beta) + \cot\psi_1(\alpha,\beta) + \cot\varphi_1^\pm(\alpha)}. \tag{3.3}$$

Fix an $\alpha \in (\beta-\pi, \beta) \cup (\beta, \beta+\pi)$. A convexity argument will show that for the fixed β and α, (3.3) implies (3.1).

Let $A_j(z) = (1-z)\cot\varphi_j^-(\alpha) + z\cot\varphi_j^+(t)$ for $j = 1, 2$. The inequality

$$\frac{\cot\psi_2(\alpha,\beta) - A_2(z)}{2\cot(\alpha-\beta) + \cot\psi_2(\alpha,\beta) + A_2(z)} \leq C^2 \frac{\cot\psi_1(\alpha,\beta) - A_1(z)}{2\cot(\alpha-\beta) + \cot\psi_1(\alpha,\beta) + A_1(z)} \tag{3.4}$$

holds for $z = 0, 1$ because of (3.3). If we show that it holds for $z = 1/2$ then we have (3.1).

It is important to point out that in each of fractions in (3.4) the numerator and denominator cannot both be zero, lest $\cot(\alpha-\beta) + \cot\psi_j(\alpha,\beta) = 0$. If this were true, then either $\psi_j(\alpha,\beta) = \beta - \alpha$ or $\psi_j(\alpha,\beta) = \beta - \alpha + \pi$. But by Lemma 3.1(i) this is $\psi_j(\beta,\alpha) = 0$ or $\psi_j(\beta,\alpha) = \pi$, contrary to Lemma 3.1(iii).

Define $f : [0,1] \to [0,\infty)$ by

$$\begin{aligned} f(z) = C^2&(\cot\psi_1(\alpha,\beta) - A_1(z))(2\cot(\alpha-\beta) + \cot\psi_2(\alpha,\beta) + A_2(z)) \\ &- (\cot\psi_2(\alpha,\beta) - A_2(z))(2\cot(\alpha-\beta) + \cot\psi_1(\alpha,\beta) + A_1(z)). \end{aligned}$$

First consider the case $\beta < \alpha < \beta + \pi$. Lemma 3.1(ii) shows that for $j = 0, 1$ and $0 \leq z \leq 1$,

$$0 < \psi_j(\alpha,\beta) \leq \varphi_j^-(\alpha) \leq \varphi_j^+(\alpha) \leq \psi_j(\alpha,\beta+\pi) < \pi$$

and we get,

$$\cot\psi_j(\alpha,\beta) \geq \cot\varphi_j^-(\alpha) \geq A_j(z) \geq \cot\varphi_j^+(\alpha) \geq \cot\psi_j(\alpha,\beta+\pi).$$

Since $r(\beta) = r(\beta+\pi)$, Lemma 3.1(iv) implies

$$\cos(\alpha-\beta) + \sin(\alpha-\beta)\cot\psi_j(\alpha,\beta) = \cos(\alpha-\beta-\pi) + \sin(\alpha-\beta-\pi)\cot\psi_j(\alpha,\beta+\pi)$$

so $\cot\psi_j(\alpha,\beta+\pi) = -2\cot(\alpha-\beta) - \cot\psi_j(\alpha,\beta)$. We conclude that the numerators and denominators of both sides of inequality (3.4) are all non-negative. It follows that (3.4) holds if and only if $f(z) \geq 0$.

The second case is similar. If $\beta - \pi < \alpha < \beta$ then Lemma 3.1(ii) shows that for $j = 0, 1$ and $0 \leq z \leq 1$,

$$0 < \psi_j(\alpha,\beta-\pi) \leq \varphi_j^-(\alpha) \leq \varphi_j^+(\alpha) \leq \psi_j(\alpha,\beta) < \pi$$

and we get,

$$\cot\psi_j(\alpha,\beta-\pi) \geq \cot\varphi_j^-(\alpha) \geq A_j(z) \geq \cot\varphi_j^+(\alpha) \geq \cot\psi_j(\alpha,\beta).$$

Since $r(\beta) = r(\beta-\pi)$, Lemma 3.1(iv) implies

$$\cos(\alpha-\beta+\pi) + \sin(\alpha-\beta+\pi)\cot\psi_j(\alpha,\beta-\pi) = \cos(\alpha-\beta) + \sin(\alpha-\beta)\cot\psi_j(\alpha,\beta)$$

so $\cot\psi_j(\alpha,\beta-\pi) = -2\cot(\alpha-\beta) - \cot\psi_j(\alpha,\beta)$. This time the numerators and denominators of both sides of inequality (3.4) are all non-positive but again (3.4) holds if and only if $f(z) \geq 0$.

It remains to show that $f(1/2) \geq 0$.

The function $f(z)$ is a quadratic polynomial and the coefficient of z^2 is

$$-(C^2-1)(\cot\varphi_1^-(t) - \cot\varphi_1^+(t))(\cot\varphi_2^-(t) - \cot\varphi_2^+(t)) \leq 0$$

so it is a concave function. Since (3.4) holds for $z = 0$ and $z = 1$, $f(0) \geq 0$ and $f(1) \geq 0$. It follows that $f(z) \geq 0$ for all $z \in [0,1]$ and in particular $f(1/2) \geq 0$. □

This theorem may be used to simplify verification of angular equivalence of norms on $\mathbb{R}^2$ but it also shows that the notion of angular equivalence does not depend on the value of the two g-functionals at the exceptional pairs for which either g-functional fails to satisfy $g^-(x,y) = g^+(x,y)$.

Corollary 3.4. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the real vectors space X. To show that the norms are angularly equivalent with constant C it is enough to verify (1.3) for pairs x, y satisfying $g_1^+(x,y) = g_1^-(x,y)$ and $g_2^+(x,y) = g_2^-(x,y)$*

Proof. Suppose that (1.3) holds for pairs x, y satisfying $g_j^+(x,y) = g_j^-(x,y)$ for $j = 1,2$. A calculation shows that this case includes all linearly dependent pairs. So fix a pair of independent vectors x and y in X. We identify their two-dimensional span with $\mathbb{R}^2$ by the map $(\xi,\eta) \mapsto \xi x + \eta y$ so that $\|(\xi,\eta)\|_j = \|\xi x + \eta y\|_j$ is a norm on $\mathbb{R}^2$ for $j = 1,2$. This identification is isometric in both norms so it does not affect g-functional calculations or norm angles.

Using these norms on $\mathbb{R}^2$ we define φ_j^+ and φ_j^- for $j = 1,2$. Suppose the conditions (3.2) hold for some α, β satisfying $0 < |\alpha - \beta| < \pi$. The proof of Theorem 3.2 shows that $g_j^+(aQ_\alpha, bQ_\beta) = g_j^-(aQ_\alpha, bQ_\beta)$ for $j = 1,2$ and for any $a, b > 0$. But the points $aQ_\alpha, bQ_\beta \in \mathbb{R}^2$ correspond to vectors in the span of x and y. Our hypothesis, shows that $\tan(\theta_2(aQ_\alpha, bQ_\beta)/2) \le C \tan(\theta_1(aQ_\alpha, bQ_\beta)/2)$. By Theorem 3.2 inequality (3.1) also holds. Now Theorem 3.3 shows that the two norms on $\mathbb{R}^2$ are angularly equivalent with constant C. The same is true for the original two norms on the span of x and y. In particular, (1.3) holds for the vectors x, y. This completes the proof.

□

Let X be a real vector space. A *semi-inner product in the sense of Lumer-Giles*, is a map $[\cdot,\cdot] : X \times X \to [0,\infty)$ that is linear in the first variable, homogeneous in the second, positive definite, and satisfies $[x,y]^2 \le [x,x][y,y]$. It follows that $x \mapsto [x,x]^{1/2}$ is a norm on X. See [2, Chapter 2] and the references there. It is natural to define the angle $\theta_{[\cdot,\cdot]}$ from x to y, associated with semi-inner product $[\cdot,\cdot]$, by requiring that $0 \le \theta \le \pi$ and

$$\cos\theta_{[\cdot,\cdot]} = \frac{[y,x]}{[y,y]^{1/2}[x,x]^{1/2}}.$$

Corollary 3.6 in [2] shows that any semi-inner product in the sense of Lumer-Giles satisfies $g^-(x,y) \le [y,x] \le g^+(x,y)$. In particular, $[y,x] = g(x,y)$ whenever $g^-(x,y) = g^+(x,y)$ for all $x, y \in X$. This proves the following.

Corollary 3.5. *Let X be a real vector space and let $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ be semi-inner products in the sense of Lumer-Giles, giving rise to norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. If (1.3) holds with $\theta_{[\cdot,\cdot]_1}$ and $\theta_{[\cdot,\cdot]_2}$ replacing the norm angles θ_1 and θ_2 then $\|\cdot\|_1$ and $\|\cdot\|_2$ are angularly equivalent.*

It is worth pointing out that if X is a normed space, setting $[y,x] = g(x,y)$ defines a semi-inner product in the sense of Lumer-Giles only under additional conditions on the norm. See [2, Definition 4.2.9 and Proposition 4.2.10].

4. Angular Equivalence Implies Topological Equivalence

The techniques developed in the last section are used to prove that angular equivalence is finer than topological equivalence. The definitions of r_j, ψ_j, and $\varphi_j^\pm$, relative to the norm $\|\cdot\|_j$, for $j = 1, 2$, that were introduced at the beginning of Section 3 will be used throughout. We begin by showing that the angular equivalence of two norms gives a relationship between their ψ and φ-functions.

Lemma 4.1. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be angularly equivalent norms on $\mathbb{R}^2$. If $0 < \alpha - \beta < \pi/2$, then*

$$(\alpha + \varphi_2^-(\alpha)) - (\beta + \varphi_2^+(\beta)) \le MC^2((\alpha + \varphi_1^-(\alpha)) - (\beta + \varphi_1^+(\beta))),$$

where $M = \sup\{\csc^2 \psi_1(t,s) : s, t \in (\beta, \alpha), s \ne t\}$.

Proof. Let E be the set of points in (β, α) at which both $\varphi_1^+ = \varphi_1^-$ and $\varphi_2^+ = \varphi_2^-$. By Lemma 3.1(viii), E contains all but countably many points of (β, α) and φ_1^+, φ_1^-, φ_2^+, and φ_2^- are continuous there. Fix $s, t \in E$ and let $\bar t \in (\beta, \alpha)$. The cosecant function decreases on $(0, \pi/2)$ and increases on $(\pi/2, \pi)$. So if y lies between $\psi_1(t, s)$ and $\psi_1(t, \bar t)$ we have

$$\csc^2 y \le \max(\csc^2 \psi_1(t, \bar t), \csc^2 \psi_1(t, s)) \le M.$$

The mean value theorem implies that,

$$|\cot \psi_1(t, s) - \cot \psi_1(t, \bar t)| \le M|\psi_1(t, \bar t) - \psi_1(t, s)|.$$

Letting $\bar t \to t$, we have

$$|\cot \psi_1(t, s) - \cot \varphi_1^+(t)| \le M(|\varphi_1^+(t) - \psi_1(t, s)|. \tag{4.1}$$

On the other hand, the function $\csc^2$ is bounded below by 1 so

$$|\cot \psi_2(t, s) - \cot \varphi_2^+(t)| \ge |\varphi_2^+(t) - \psi_2(t, s)|. \tag{4.2}$$

By Lemma 3.1(iii) the functions ψ_1 and ψ_2 each take values in a compact subset of $(0, \pi)$. So do their limits, φ_1^+ and φ_2^+. But the cotangent function is unbounded near 0. Therefore, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |t - s| < \delta$ then

$$\frac{|2\cot(t-s) + \cot \psi_2(t, s) + \cot \varphi_2^+(t)|}{|2\cot(t-s) + \cot \psi_1(t, s) + \cot \varphi_1^+(t)|} \le 1 + \varepsilon.$$

Since $t \in E$, the angular equivalence of $\|\cdot\|_1$ and $\|\cdot\|_2$, in the form (3.1), becomes

$$\frac{\cot \psi_2(t, s) - \cot \varphi_2^+(t)}{2\cot(t-s) + \cot \psi_2(\alpha, \beta) + \cot \varphi_2^+(t)} \le C^2 \frac{\cot \psi_1(t, s) - \cot \varphi_1^+(t)}{2\cot(t-s) + \cot \psi_1(t, s) + \cot \varphi_1^+(t)}.$$

Note that both sides of the inequality are non-negative. Combining it with the previous estimate gives

$$|\cot \psi_2(t, s) - \cot \varphi_2^+(t)| \le (1+\varepsilon)C^2 |\cot \psi_1(t, s) - \cot \varphi_1^+(t)| \tag{4.3}$$

whenever $0 < |t - s| < \delta$.

Now suppose that $s,t\in E$ and $s<t$. By Lemma 3.1(i), $t-s=\psi_j(s,t)-\psi_j(t,s)$, for $j=1,2$, so (4.3) combines with (4.2) and (4.1) to give

$$\begin{aligned}&(t+\varphi_2^+(t))-(s+\varphi_2^+(s))\\ &\le|\varphi_2^+(t)-\psi_2(t,s)|+|\psi_2(s,t)-\varphi_2^+(s)|\\ &\le|\cot\psi_2(t,s)-\cot\varphi_2^+(t)|+|\cot\varphi_2^+(s)-\cot\psi_2(s,t)|\\ &\le(1+\varepsilon)C^2(|\cot\psi_1(t,s)-\cot\varphi_1^+(t)|+|\cot\varphi_1^+(s)-\cot\psi_1(s,t)|)\\ &\le(1+\varepsilon)MC^2(|\varphi_1^+(t)-\psi_1(t,s)|+|\psi_1(s,t)-\varphi_1^+(s)|)\\ &=(1+\varepsilon)MC^2(\varphi_1^+(t)-\psi_1(t,s)+\psi_1(s,t)-\varphi_1^+(s))\\ &=(1+\varepsilon)MC^2((t+\varphi_1^+(t))-(s+\varphi_1^+(s))).\end{aligned}$$

Removal of the absolute value signs to get the second-last line is justified by Lemma 3.1(ii). Now suppose $\beta<\bar\beta<\bar\alpha<\alpha$ with $\bar\alpha,\bar\beta\in E$. Since E is dense (β,α) we can choose $t_0=\bar\beta<t_1<\cdots<t_n=\bar\alpha$ with $t_k-t_{k-1}<\delta$ and $t_k\in E$ for each k. Then

$$\begin{aligned}(\bar\alpha+\varphi_2^+(\bar\alpha))-(\bar\beta+\varphi_2^+(\bar\beta))&=\sum_{k=1}^n(t_k+\varphi_2^+(t_k))-(t_{k-1}+\varphi_2^+(t_{k-1}))\\ &\le(1+\varepsilon)MC^2\sum_{k=1}^n(t_k+\varphi_1^+(t_k))-(t_{k-1}+\varphi_1^+(t_{k-1}))\\ &=(1+\varepsilon)MC^2((\bar\alpha+\varphi_1^+(\bar\alpha))-(\bar\beta+\varphi_1^+(\bar\beta))).\end{aligned}$$

Letting $\bar\beta$ decrease to β through E, $\bar\alpha$ increase to α through E, and $\varepsilon\to0$ we have

$$(\alpha+\varphi_2^-(\alpha))-(\beta+\varphi_2^+(\beta))\le MC^2((\alpha+\varphi_2^-(\alpha))-(\beta+\varphi_2^+(\beta)).$$

□

Somewhat surprisingly, the next result, initially expected to be easy, turned out to be the main result of the current article because of its involved proof. It is hoped that greater familiarity with angular equivalence will reveal a simpler one.

Theorem 4.2. *Let X be a real vector space with norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are angularly equivalent then they are also topologically equivalent.*

Proof. Let $C\ge1$ be the constant in the definition of angular equivalence, that is, suppose that for all non-zero $x,y\in X$,

$$\tan(\theta_2(x,y)/2)\le C\tan(\theta_1(x,y)/2).$$

We will show that for all non-zero $\bar x,\bar y\in X$,

$$\frac{\|\bar x\|_1\|\bar y\|_2}{\|\bar x\|_2\|\bar y\|_1}\le 40C^2. \tag{4.4}$$

Topological equivalence follows from this by fixing any non-zero $\bar y\in X$ to get $m\|\bar x\|_1\le\|\bar x\|_2$ with $m=\|\bar y\|_2/(40C^2\|y\|_1)$ and, interchanging $\bar x$ and $\bar y$ in (4.4), $\|\bar x\|_2\le M\|\bar x\|_1$ with $M=40C^2\|\bar y\|_2/\|\bar y\|_1$.

If $\bar x$ and $\bar y$ are multiples of one another then (4.4) holds trivially so we assume henceforth that they are independent. Let Z be the two-dimensional subspace of X spanned by $\bar x$ and $\bar y$. Choose vectors $x,y\in Z$ satisfying three conditions:

(4.5)

$$\|x\|_2=\|y\|_2=1;\quad \|y\|_1\le\frac{\|z\|_1}{\|z\|_2}\le\|x\|_1 \text{ for } 0\ne z\in Z;\quad \|x\|_1\le\|x+ty\|_1 \text{ for } t\ge0.$$

To see that this is possible, take x and y to give the maximum and minimum, respectively, of the continuous function $z \mapsto \|z\|_1$ on the compact set $\{z \in Z : \|z\|_2 = 1\}$. For any non-zero $z \in Z$, $z/\|z\|_2$ is in this set so the first two conditions hold. But they also hold with y replaced by $-y$. The third condition must hold either with y or with $-y$; otherwise, there would exist $s, t > 0$ such that $\|x+ty\|_1 < \|x\|_1$ and $\|x - sy\|_1 < \|x\|_1$, which leads to the contradiction,

$$\|x\|_1 = \|\tfrac{s}{s+t}(x + ty) + \tfrac{t}{s+t}(x - sy)\|_1 \leq \tfrac{s}{s+t}\|x + ty\|_1 + \tfrac{t}{s+t}\|x - sy\|_1 < \|x\|_1.$$

With x and y now fixed, we identify Z with $\mathbb{R}^2$ by the linear map $(\xi, \eta) \mapsto \xi x + \eta y$ so that $\|(\xi, \eta)\|_1 = \|\xi x + \eta y\|_1$ and $\|(\xi, \eta)\|_2 = \|\xi x + \eta y\|_2$ give two norms on $\mathbb{R}^2$. These make the identification isometric in both norms so g-functional calculations are not affected. In particular, the two norms on $\mathbb{R}^2$ are angularly equivalent with constant C. Define r_j, ψ_j and $\varphi_j^{\pm}$ as above for $j = 1, 2$. Note that the polar graph of r_j coincides with the boundary of the $\|\cdot\|_j$-unit ball—we refer to it simply as the *r_j-curve*.

The first condition of (4.5) ensures that $r_2(0) = 1$ and $r_2(\pi/2) = 1$. For convenience we set $a = r_1(0)$ and $b = r_1(\pi/2)$. The second condition implies that,

$$\frac{\|\bar{x}\|_1\|\bar{y}\|_2}{\|\bar{x}\|_2\|\bar{y}\|_1} \leq \frac{\|x\|_1}{\|y\|_1} = \frac{b}{a},$$

so we may complete the proof of (4.4) and the theorem by showing $b/a \leq 40C^2$. Suppose instead that $b/a > 40C^2$. First, we set up to apply Lemma 4.1 with $\alpha = \tan^{-1} 3$ and $\beta = 0$. Estimates will be needed for $\alpha + \varphi_1^-(\alpha) - \varphi_1^+(0)$, $\alpha + \varphi_2^-(\alpha) - \varphi_2^+(0)$, and $M = \sup\{\csc^2(\psi_1(s, t)) : s, t \in (0, \alpha), s \neq t\}$.

For the remainder of the proof we will frequently apply the definitions of ψ and $\varphi^{\pm}$ and the properties in Lemma 3.1(i) and (ii) without explicit reference. Also, as above, "angle" refers to the usual angle in $\mathbb{R}^2$, not the norm angle.

The third condition of (4.5) shows that, in the first quadrant, the r_1-curve lies to the left of the line $x = a$. Therefore $\psi_1(0, t) \geq \pi/2$ for $t > 0$ and, in the limit, $\varphi_1^+(0) \geq \pi/2$. Also, the r_1-curve intersects the line $y = 3x$ on the segment from $(0, 0)$ to $(a, 3a)$. Therefore $\psi_1(\pi/2, \alpha)$ is less than or equal to the measure of the angle from $(0, 0)$ to $(0, b)$ to $(a, 3a)$. That is,

$$\psi_1(\pi/2, \alpha) \leq \cot^{-1}(b/a - 3). \tag{4.6}$$

Note that since $C \geq 1$ our assumption that $b/a > 40C^2$ implies $b/a > 3$. Now,

$$\alpha + \varphi_1^-(\alpha) - \varphi_1^+(0) \leq \alpha + \psi_1(\alpha, \pi/2) - \pi/2 = \psi_1(\pi/2, \alpha) \leq \cot^{-1}(b/a - 3). \tag{4.7}$$

To estimate $\varphi_2^+(0)$ we use the angular equivalence hypothesis applied to the two norm angles from $(1, 0)$ to $(\cos(-t), \sin(-t))$, for $t > 0$. In the form (3.1), it implies

$$\frac{\cot \psi_2(0, -t) - A_2}{2 \cot t + \cot \psi_2(0, -t) + A_2} \leq C^2 \frac{\cot \psi_1(0, -t) - A_1}{2 \cot t + \cot \psi_1(0, -t) + A_1}.$$

where $A_j = \frac{1}{2}(\cot\varphi_j^-(0) + \cot\varphi_j^+(0))$, $j = 1, 2$. By Lemma 3.1(iii), A_1, A_2, ψ_1 and ψ_2 are uniformly bounded away from zero, but $\cot t \to \infty$ as $t \to 0+$. Thus,

$$\begin{aligned}
&\cot\varphi_2^-(0) - \cot\varphi_2^+(0)\\
&= 2\lim_{t\to 0+} \frac{\cot\psi_2(0,-t) - A_2}{2\cot t + \cot\psi_2(0,-t) + A_2}(2\cot t + \cot\psi_2(0,-t) + A_2)\\
&\le 2\lim_{t\to 0+} C^2\frac{\cot\psi_1(0,-t) - A_1}{2\cot t + \cot\psi_1(0,-t) + A_1}(2\cot t + \cot\psi_2(0,-t) + A_2)\\
&= C^2(\cot\varphi_1^-(0) - \cot\varphi_1^+(0)).
\end{aligned}$$

Returning to the second condition of (4.5) we see that for each t,

$$\frac{r_1(t)}{r_2(t)} = \frac{\|(\cos t)x + (\sin t)y\|_2}{\|(\cos t)x + (\sin t)y\|_1} \ge \frac{1}{\|x\|_1} = a.$$

Recalling that $r_2(0) = 1$ and applying Lemma 3.1(v), we get

$$0 \le \lim_{t\to 0-} \frac{1}{t}\left(a - \frac{r_1(t)}{r_2(t)}\right) = \lim_{t\to 0-} a\frac{r_2(t)-1}{tr_2(t)} - \frac{r_1(t)-a}{tr_2(t)} = a\cot\varphi_2^-(0) - a\cot\varphi_1^-(0).$$

Thus, $\cot\varphi_2^-(0) \ge \cot\varphi_1^-(0)$ and we have

$$\begin{aligned}
\cot\varphi_2^+(0) &\ge \cot\varphi_2^-(0) - C^2(\cot\varphi_1^-(0) - \cot\varphi_1^+(0))\\
&\ge (1-C^2)\cot\varphi_1^-(0) + C^2\cot\varphi_1^+(0)\\
&\ge (1-C^2)\cot\psi_1(0,-\pi/2) + C^2\cot\psi_1(0,\pi/2)\\
&= (1-C^2)(a/b) + C^2(-a/b)\\
&\ge -2C^2a/b.
\end{aligned}$$

We conclude that

$$\varphi_2^+(0) \le \cot^{-1}(-2C^2a/b) = \pi - \cot^{-1}(2C^2a/b).$$

The r_2-curve passes through $(-1, 0)$ and $(0, 1)$ so, in the first quadrant, it lies below the line $y = x + 1$. Thus, the r_2-curve intersects the line $y = 3x$ on the segment from $(0, 0)$ to $(1/2, 3/2)$. It follows that $\psi_2(\alpha, 0)$ is greater than or equal to the measure of the apex angle of the isosceles triangle with vertices $(0, 0)$, $(1/2, 3/2)$, and $(1, 0)$. The latter is $\pi - 2\alpha$ so we have

$$\alpha + \varphi_2^-(\alpha) - \varphi_2^+(0) \ge \alpha + \psi_2(\alpha,0) - \varphi_2^+(0) \ge \cot^{-1}(2C^2a/b) - \alpha. \tag{4.8}$$

Finally we estimate M. Since $b/a > 40C^2 > 3 + 1/3$, $\cot^{-1}(b/a - 3) < \cot^{-1}(1/3) = \alpha$ so (4.6) implies

$$\psi_1(\alpha,\pi/2) = \psi_1(\pi/2,\alpha) - \alpha + \pi/2 \le \cot^{-1}(b/a-3) - \alpha + \pi/2 \le \pi/2.$$

Thus, for $s, t \in [0, \alpha)$ with $s \ne t$,

$$\psi_1(t,s) \le \psi_1(t,\alpha) = \psi_1(\alpha,t) + \alpha - t \le \psi_1(\alpha,t) + \alpha \le \psi_1(\alpha,\pi/2) + \alpha \le \pi/2 + \alpha.$$

Combining this with,

$$\psi_1(t,s) \ge \psi_1(t,0) = \psi_1(0,t) - t \ge \varphi_1^+(0) - t \ge \pi/2 - t \ge \pi/2 - \alpha$$

gives $\pi/2 - \alpha \le \psi_1(t, s) \le \pi/2 + \alpha$ and hence

$$\csc^2\psi_1(t,s) \le \csc^2(\pi/2-\alpha) = 10.$$

So $M \leq 10$. Now Lemma 4.1 gives,

$$\alpha + \varphi_2^-(\alpha) - \varphi_2^+(0) \leq 10C^2(\alpha + \varphi_1^-(\alpha) - \varphi_1^+(0)).$$

Using (4.7) and (4.8), we have

$$\cot^{-1}(2C^2a/b) - \alpha \leq 10C^2 \cot^{-1}(b/a - 3).$$

Since $\cot^{-1} u \leq 1/u$ when $u > 2/\pi$, and $C \geq 1$, our assumption $b/a > 40C^2$ yields,

$$10C^2 \cot^{-1}(b/a - 3) \leq 10C^2 \cot^{-1}(40C^2 - 3C^2) \leq 10/37 < 0.271$$

and

$$\cot^{-1}(2C^2a/b) - \alpha \geq \cot^{-1}(2/40) - \tan^{-1} 3 > 0.271.$$

This contradiction completes the proof. □

5. Further Work

There are many natural, fundamental questions about angular equivalence still to be investigated. We list a few in the hope that interested readers will contribute to the theory. Throughout, $\|\cdot\|_1$ and $\|\cdot\|_2$ are angularly equivalent norms on a real vector space X. By Theorem 4.2 the two norms are also topologically equivalent.

Since the completions of $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on a common vector space it makes sense to ask,

Question 1. *Are the completions of angularly equivalent norms again angularly equivalent?*

If Y is a subspace of X then it is closed with respect to $\|\cdot\|_1$ if and only if it is closed with respect to $\|\cdot\|_2$. Each of the two norms give rise to a quotient norm on X/Y defined by, $\|x + Y\|_j = \inf_{y \in Y} \|x + y\|_j$, $j = 1, 2$.

Question 2. *Do angularly equivalent norms induce angularly equivalent norms on quotient spaces?*

The vector space X^*, of continuous linear functionals on X, is the same for both norms. Thus, their respective dual norms $\|\cdot\|_1^*$ and $\|\cdot\|_2^*$ are norms on a common vector space. These dual norms are not, in general, angularly equivalent. See Example 3, below. So it is natural to ask,

Question 3. *Under what conditions on angularly equivalent norms are their dual norms also angularly equivalent?*

Example 3. *The dual norms of angularly equivalent norms need not be angularly equivalent: Consider the two weighted ℓ^1 norms, $\|(\xi, \eta)\|_1 = 2|\xi| + |\eta|$ and $\|(\xi, \eta)\|_2 = |\xi| + 2|\eta|$ on $\mathbb{R}^2$. A calculation shows that, with θ_j denoting the $\|\cdot\|_j$-norm angle from (ξ, η) to (μ, ν),*

$$\begin{aligned}\tan^2(\theta_2/2) &= \frac{|\mu| - \mu \operatorname{sgn} \xi + 2(|\nu| - \nu \operatorname{sgn} \eta)}{|\mu| + \mu \operatorname{sgn} \xi + 2(|\nu| + \nu \operatorname{sgn} \eta)} \\ &\leq 4 \frac{2(|\mu| - \mu \operatorname{sgn} \xi) + |\nu| - \nu \operatorname{sgn} \eta}{2(|\mu| + \mu \operatorname{sgn} \xi) + |\nu| + \nu \operatorname{sgn} \eta} = 4 \tan^2(\theta_1/2).\end{aligned}$$

(The operator sgn *takes the value 1, -1 or 0 when its argument is positive, negative or zero, respectively.) Thus, the two norms are angularly equivalent. However, their dual norms are given by $\|(\xi, \eta)\|_1^* = \max(|\xi|/2, |\eta|)$ and $\|(\xi, \eta)\|_2^* = \max(|\xi|, |\eta|/2)$, respectively. These have polygonal unit balls in which the vertices are not on the*

same rays. One unit ball has vertices at $(\pm 2, \pm 1)$ *and the other has vertices at* $(\pm 1, \pm 2)$*. By Corollary 2.3 the dual norms are not angularly equivalent.*

The Cartesian product and tensor product of normed vector spaces can be given various different, but topologically equivalent, norms. This leads to the question,

Question 4. *How should the norm of a Cartesian or tensor product be defined to ensure that product norms are angularly equivalent whenever the norms on the factors are angularly equivalent?*

An Orlicz space can be equipped with either of two topologically equivalent norms, the Luxemburg norm or the Orlicz norm. Both are needed because each arises naturally from the other when considering the norm on the dual space. For the special case of L^p spaces, these two norms coincide (up to a constant multiple) and hence are angularly equivalent. However, the examples following [3, Theorem 10] show that there are Orlicz spaces which are strictly convex when equipped with the Luxemburg norm but are not strictly convex when equipped with the Orlicz norm. By Corollary 2.2 the two norms are not angularly equivalent for such spaces. So we ask,

Question 5. *For which Orlicz spaces are the Luxemburg norm and the Orlicz norm angularly equivalent?*

A similar question could be posed for many other families. For example, topologically equivalent norms abound on the much-studied families of Banach spaces defined by Hardy, Sobolev, Besov, Triebel, Lizorkin and others.

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