THE A_p CONDITION AND A POSITIVE, LINEAR OPERATOR

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Abstract. A positive, linear operator is exibited which is bounded on the mixed weighted Lebesgue space $L^p_w[\mathbf{R}](L^1[\mathbf{R}])$ if and only if the weight w satisfies the A_p condition of Muckenhoupt.

1. Introduction

The A_p condition on non-negative weight functions was introduced by Muckenhoupt [3] where he showed that the Hardy-Littlewood Maximal function is bounded on L^p_w if and only if $w \in A_p$. It was later shown that the A_p condition also characterized those weighted Lebesgue spaces on which the Hilbert transform and other singular integrals are bounded. Since that time many difficult problems have seen A_p weights arise naturally in their solution. The weight class is defined as follows: Let $1 < p < \infty$ and $p' = p/(p-1)$. $w \in A_p[\mathbf{R}^n]$ provided

$$\sup \left(\frac{1}{|Q|} \int_Q w \right)^{1/p} \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{1/p'} < \infty,$$

where the supremum is taken over all co-ordinate cubes $Q \subset \mathbf{R}^n$ and $|Q|$ denotes the volume of the cube Q. If $n = 1$ the cubes become intervals and we write A_p for $A_p[\mathbf{R}]$. The definition suggests that the condition is bound up with the geometry of the Euclidean spaces and indeed the Calderon-Zygmund decomposition used in the proof of Muckenhoupt's weighted Maximal Theorem bears this out. It is surprising, therefore, that there should exist a simple averaging operator whose boundedness characterizes the A_p weights. We have the following theorem.

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Theorem 1. *The operator T defined by*

$$Tf(y,h) = \frac{1}{h}\int_{y-h}^{y+h} f(t,h)\,dt$$

is bounded on $L^p_w[\mathbf{R}](L^1[\mathbf{R}])$ if and only if $w \in A_p$.

Here and throughout, the space $L^p_w[\mathbf{R}^n](L^q[\mathbf{R}^m])$, where $1 \le p < \infty$ and $1 \le q < \infty$, is defined to be the collection of those measurable functions $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ satisfying

$$\|f\|_{p,w\,;q} \equiv \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m} |f(x,s)|^q\,ds\right)^{p/q} w(x)\,dx\right)^{1/p} < \infty.$$

The space $L^p_w[\mathbf{R}^n](L^\infty[\mathbf{R}^m])$ consists of those f which satisfy

$$\|f\|_{p,w\,;\infty} \equiv \left(\int_{\mathbf{R}^n}\left(\operatorname*{ess\,sup}_{s\in\mathbf{R}^m} |f(x,s)|\right)^{p} w(x)\,dx\right)^{1/p} < \infty.$$

We conclude this section with some definitions and notation. Theorem 1 is proved in Section 2, and Section 3 contains two n-dimensional analogues.

For each $p \in (0,\infty)$ we set $p' = p/(p-1)$. Also $(1)' = \infty$ and $(\infty)' = 1$. The Hardy-Littlewood Maximal Function M_n is defined by

$$M_n g(x) = \sup \frac{1}{|Q|}\int_Q |g|$$

where the supremum is taken over all co-ordinate cubes $Q \subset \mathbf{R}^n$ centred at x. $L^p_w[\mathbf{R}^n]$ denotes the usual weighted Lebesgue space consisting of those functions f for which the norm $\|f\|_{p,w} \equiv \left(\int |f|^p w\right)^{1/p}$ is finite. Products of the form $0\cdot\infty$ are taken to be zero and we remark that with this convention the weight $w \equiv 0$ is in A_p.

2. Proof of Theorem 1

We begin this section with a useful duality result before turning to the proof of Theorem 1.

Lemma 1 [1, p303]**.** *Suppose that $1 < p < \infty$ and $F, G : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$. Then*

$$\|G\|_{p',w\,;\infty} = \sup\left\{\left|\int_{\mathbf{R}^n}\int_{\mathbf{R}^m} F(x,s)G(x,s)\,ds\,w(x)\,dx\right| : \|F\|_{p,w\,;1} \le 1\right\}.$$

Proof of Theorem 1. Given non-negative functions $f : \mathbf{R}^2 \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ we define

$$I(f,g) = \int_{\mathbf{R}}\int_{\mathbf{R}} f(x,h)\frac{1}{h}\int_{x-h}^{x+h} g(\theta)\,d\theta\,dh\,dx. \tag{1}$$

A straightforward calculation shows that this may also be written as

$$\int_{\mathbf{R}}\int_{\mathbf{R}}\frac{1}{h}\int_{y-h}^{y+h} f(t,h)\,dt\,dh\,g(y)\,dy.$$

Recognising the operator $Tf(y,h) = \frac{1}{h}\int_{y-h}^{y+h} f(t,h)\,dt$ in the line above yields the following alternate expression for $I(f,g)$.

$$I(f,g) = \int_{\mathbf{R}}\int_{\mathbf{R}} Tf(y,h)\,dh\,g(y)\,dy. \tag{2}$$

We now rewrite lines (1) and (2) introducing the weight w. The case $w \equiv 0$ is trivial so we restrict our attention to weights w which are not identically zero. Note that in this case we may deduce that w is non-zero almost everywhere whether we begin with the assumption $w \in A_p$ or, in the other direction, with the assumption that T is bounded on $L^p_w[\mathbf{R}](L^1[\mathbf{R}])$.

$$I(f,g) = \int_{\mathbf{R}}\int_{\mathbf{R}} f(x,h)\left[\frac{1}{h}\int_{x-h}^{x+h} g(\theta)\,d\theta\,w(x)^{-1}\right]dh\,w(x)\,dx. \tag{1$'$}$$

$$I(f,g) = \int_{\mathbf{R}}\left[\int_{\mathbf{R}} Tf(y,h)\,dh\,w(y)^{1/p}\right]\left[g(y)w(y)^{(1-p')/p'}\right]dy. \tag{2$'$}$$

Using the above two expressions for $I(f,g)$, the duality argument is now standard. In view of (2$'$), $T : L^p_w[\mathbf{R}](L^1[\mathbf{R}]) \to L^p_w[\mathbf{R}](L^1[\mathbf{R}])$ if and only if there exists a $C > 0$ such that

$$\sup\left\{I(f,g) : \|g\|_{p',w^{1-p'}} \le 1\right\} \le C\|f\|_{p,w\,;1} \tag{3}$$

for all $f \ge 0$. Since $I(f,g)$ is positive homogeneous in both f and g, (3) is equivalent to

$$\sup\{I(f,g) : \|f\|_{p,w\,;1} \le 1\} \le C\|g\|_{p',w^{1-p'}} \tag{4}$$

for all $g \ge 0$. If we apply Lemma 1 to the expression (1$'$), (4) becomes

$$\left(\int_{\mathbf{R}}\left(\sup_{h\in\mathbf{R}}\frac{1}{h}\int_{x-h}^{x+h} g(\theta)\,d\theta\right)^{p'} w(x)^{1-p'}\,dx\right)^{1/p'} \le C\|g\|_{p',w^{1-p'}} \tag{5}$$

for all $g \ge 0$. This last equivalent statement is just the boundedness of the Maximal Function M_1 on the space $L^{p'}_{w^{1-p'}}$. By [3] this is equivalent to $w^{1-p'} \in A_{p'}$. It is immediate from the definition of A_p that this is equivalent to $w \in A_p$. This completes the proof.

3. Higher Dimensional Results

Theorems 2 and 3 of this section are proved in much the same way as Theorem 1. Therefore they are given here without proof.

Theorem2. *Suppose $1 < p < \infty$. Define the operator T_n by*

$$T_n f(y,t) = \frac{1}{|h|^n} \int_{\substack{|y_i - t_i| < |h| \\ \text{for all } i}} f(t,h)\, dt, \qquad t = (t_1, \ldots, t_n) \in \mathbf{R}^n,$$

where $h \in \mathbf{R}$ and $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$. T is bounded on $L^p_w[\mathbf{R}^n](L^1[\mathbf{R}])$ if and only if $w \in A_p[\mathbf{R}^n]$.

We define the weight class A^*_p as follows: $w \in A^*_p$ provided

$$\sup \left(\frac{1}{|R|} \int_R w \right)^{1/p} \left(\frac{1}{|R|} \int_R w^{1-p'} \right)^{1/p'} < \infty,$$

where the supremum is taken over all co-ordinate rectangles $R \subset \mathbf{R}^n$ and $|R|$ denotes the volume of the rectangle R. The condition A^*_p characterizes those weights for which the strong maximal function is bounded on $L^p_w[\mathbf{R}^n]$. See [2, p451ff] for that result and for the definition of the strong maximal function.

Theorem3. *Suppose $1 < p < \infty$. Define the operator S_n by*

$$S_n f(y,t) = \frac{1}{|h_1 \cdots h_n|} \int_{\substack{|y_i - t_i| < |h_i| \\ \text{for all } i}} f(t,h)\, dt, \qquad t = (t_1, \ldots, t_n) \in \mathbf{R}^n,$$

where $h = (h_1, \ldots, h_n) \in \mathbf{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$. T is bounded on $L^p_w[\mathbf{R}^n](L^1[\mathbf{R}^n])$ if and only if $w \in A^*_p[\mathbf{R}^n]$.

References

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