Bootstrapping Weighted Fourier Inequalities

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Abstract

The theory of positive integral operators is applied to convolution operators, giving a method of producing new weighted Fourier inequalities from known ones. The new inequalities produced depend on six parameters; two real indices, two complex-valued measures, and two positive functions. The method may be iterated using the last inequality generated as input to the next stage.

The Fourier transform is defined by

$$\hat{f}(x) = \int_{\mathbb{R}} e^{-ixt} f(t)\, dt$$

for $f \in L^1$, that is, for f satisfying $\int_{\mathbb{R}} |f| < \infty$. It and its extensions are the most studied and most applied operators in all of mathematics. One way that the Fourier transform can be extended to functions not in L^1 is to establish an inequality of the form

$$\left(\int_{\mathbb{R}} |\hat{f}|^q u \right)^{1/q} \le C \left(\int_{\mathbb{R}} |f|^p v \right)^{1/p} \tag{1}$$

for all f in $L^1 \cap L^p(v)$. The Fourier transform then extends by continuity to the closure of $L^1 \cap L^p(v)$ in $L^p(v)$. Since L^1 contains the simple functions, this closure is the whole of $L^p(v)$. Here $1 < p < \infty$, $v \ge 0$, and $L^p(v)$ is the Banach space of functions for which the norm

$$\|f\|_{L^p(v)} \equiv \left(\int_{\mathbb{R}} |f|^p v \right)^{1/p}$$

is finite.

The purpose of this paper is to provide a framework for proving inequalities of the form (1). The idea is to exploit the close relationship of the Fourier transform to the operation of convolution and then to apply techniques from

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the theory of positive integral operators. Although the convolution operators that arise are not necessarily positive, they are trivially majorized by positive convolution operators and this will suffice for our purpose. The main result of the paper is that from a given "input" inequality of the form (1) a parametrized collection of "output" inequalities, again of the form (1), can be deduced.

A single application of the theorem will produce new weighted Fourier inequalities from known ones. However, since the output inequalities are of the same form as the input inequality, it becomes possible to "bootstrap" the production of new inequalities by using the output at one stage as the input at the next. The implications of this sort of iteration are not examined here but will be a subject for further study.

Throughout the paper we adhere to conventions that are more common in the study of positive integral operators than in harmonic analysis generally. When integrals of non-negative functions are involved we will not concern ourselves with convergence; if the integral happens to take the value $+\infty$ then its appearance in formulas is to be interpreted according to arithmetic on $[0,\infty]$. In particular, expressions of the form $0(\infty)$, ∞/∞, $0/0$, 0^0 are all taken to be 0, while $\infty^0 = 1$.

The collection of non-negative, Lebesgue measurable functions on $\mathbb{R}$ will be denoted by L^+ and a *positive operator* is a map from L^+ to $[0,\infty]$. If T and T^* are such operators then T^* is said to be a *formal adjoint* of T provided

$$\int_{\mathbb{R}} (Tf)g = \int_{\mathbb{R}} f(T^*g)$$

for all $f, g \in L^+$.

The following proposition is a special case ($n = 1$ and $r_1 = 1$) of Theorem 2.1 in [4]. Note especially that $C = 1$ when $p = q$, even if $\int_{\mathbb{R}} h^p v = \infty$.

Proposition 1 *Let T be a positive operator on L^+ having a formal adjoint T^*. Suppose $1 < q \le p < \infty$ and $u, h \in L^+$ with $0 < h < \infty$. Set*

$$v = h^{1-p} T^*(u(Th)^{q-1}) \quad \textit{and} \quad C = \left(\int_{\mathbb{R}} h^p v \right)^{1/q-1/p}.$$

Then,

$$\left(\int_{\mathbb{R}} (Tf)^q u \right)^{1/q} \le C \left(\int_{\mathbb{R}} f^p v \right)^{1/p}$$

for all $f \in L^+$.

The next result may be deduced from the last by a duality argument. It is also a special case of Theorem 3.1 of [3]. Again note that $C = 1$ when $p = q$.

Proposition 2 *Let T be a positive operator on L^+ having a formal adjoint T^*. Suppose $1 < q \le p < \infty$ and $v, h \in L^+$ with $0 < h < \infty$. Set*

$$u = h(T(v^{1-p'}(T^*h)^{p'-1}))^{1-q} \quad \textit{and} \quad C = \left(\int_{\mathbb{R}} h^{q'} u^{1-q'} \right)^{1/q-1/p}.$$

Then,

$$\left(\int_{\mathbb{R}} (Tf)^q u\right)^{1/q} \le C \left(\int_{\mathbb{R}} f^p v\right)^{1/p}$$

for all $f \in L^+$.

Observe that if $p > q$, then the formulas for the constants C given in these two propositions may take useful alternative forms. In the first,

$$\int_{\mathbb{R}} h^p v = \int_{\mathbb{R}} hT^*(u(Th)^{q-1}) = \int_{\mathbb{R}} (Th)u(Th)^{q-1} = \int_{\mathbb{R}} (Th)^q u$$

so $C = \left(\int_{\mathbb{R}} (Th)^q u\right)^{1/q-1/p}$. In the second,

$$\int_{\mathbb{R}} h^{q'} u^{1-q'} = \int_{\mathbb{R}} hT(v^{1-p'}(T^*h)^{p'-1}) = \int_{\mathbb{R}} (T^*h)v^{1-p'}(T^*h)^{p'-1}$$

so $C = \left(\int_{\mathbb{R}} (T^*h)^{p'} v^{1-p'}\right)^{1/q-1/p}$.

Let a and b be finite, complex-valued Borel measures on $\mathbb{R}$ and let $|a|$ and $|b|$ denote their absolute values. Then

$$\int_{\mathbb{R}} d|a| < \infty \quad \text{and} \quad \int_{\mathbb{R}} d|b| < \infty.$$

It is routine to check that

$$\hat{a}(x) = \int_{\mathbb{R}} e^{-ixt}\, da(t) \quad \text{and} \quad \check{b}(x) = \int_{\mathbb{R}} e^{ixt}\, db(t)$$

are well-defined functions in L^∞. It is also routine to verify that

$$f * a(x) = \int_{\mathbb{R}} f(x-t)\, da(t) \quad \text{and} \quad f * b(x) = \int_{\mathbb{R}} f(x-t)\, db(t)$$

are well-defined functions in L^1 when $f \in L^1$ and in L^∞ when $f \in L^\infty$. A little more work is required to verify that for $f \in L^1$,

$$(f * a)\hat{} = \hat{f}\hat{a} \quad \text{and} \quad (f\check{b})\hat{} = \hat{f} * b. \tag{2}$$

To establish (2) observe that the bounded function $\check{b}$ has a Fourier transform in the distributional sense and that Theorem 7.7 of [2] shows that $\hat{\check{b}} = 2\pi b$. (Note that the notation $\check{b}$ has a different meaning in Rudin's book than it does here.) With this in hand, if we view a and b as tempered distributions, then Theorem 7.19 of [2] shows that both statements in (2) hold for all f in the space of rapidly decreasing functions. However, the space of rapidly decreasing functions is a dense subset of L^1 and it is easy to verify that $f \mapsto (f*a)\hat{}$, $f \mapsto \hat{f}\hat{a}$, $f \mapsto (f\check{b})\hat{}$ and $f \mapsto \hat{f} * b$ are all continuous maps from L^1 to L^∞. Thus, the identities in (2) extend to be valid for all $f \in L^1$.

Define the positive convolution operators K_a and K_b by

$$K_a f = f * |a| \quad \text{and} \quad K_b f = f * |b|,$$

for all $f \in L^+$. If $f \in L^1 \cup L^\infty$ then $f*a$ and $f*b$ are well-defined, $|f*a| \le K_a|f|$, and $|f*b| \le K_b|f|$. It is an essential feature of the argument to come that the convolution operators are majorized by positive convolution operators.

The two propositions above provide weighted Lebesgue norm inequalities for the positive operators K_a and K_b that can be used to give inequalities for convolution by a and b. To apply them we need to show that K_a and K_b have formal adjoints. In the following calculation we write $\tilde{a}$ for the complex-valued measure defined by $\tilde{a}(x) = a(-x)$.

If $f, g \in L^+$ then interchanging the order of integration and making the substitution $y = x - t$ yields

$$\begin{aligned}
\int_{\mathbb{R}} K_a f(x) g(x)\, dx &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t)\, d|a|(t)\, g(x)\, dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(x)\, dx\, d|a|(t) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(y+t)\, dy\, d|a|(t) \\
&= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(y+t)\, d|a|(t)\, dy \\
&= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(y-t)\, d|\tilde{a}|(t)\, dy \\
&= \int_{\mathbb{R}} f(y) K_{\tilde{a}} g(x)\, dx
\end{aligned}$$

This shows that $K_a^* = K_{\tilde{a}}$. Obviously, $K_b^* = K_{\tilde{b}}$ as well.

Before introducing any technical details we give a sketch of the argument behind Theorem 1 below. We suppose that the Fourier inequality (1) is known to be valid for some fixed p_0, q_0, u_0, and v_0. For each appropriate function g, define $f = (g*a)/\check{b}$ and verify that $\hat{g} = (\hat{f}*b)/\hat{a}$.

For $p_1 \ge p_0$, $q_1 \le q_0$ and arbitrary positive functions h_a and h_b we apply the above two propositions to give formulas for u_1 and v_1 so that

$$\begin{array}{ccccccc}
L^{p_1}(v_1) & \longrightarrow & L^{p_0}(v_0) & \longrightarrow & L^{q_0}(u_0) & \longrightarrow & L^{q_1}(u_1) \\
g & \longmapsto & f & \longmapsto & \hat{f} & \longmapsto & \hat{g}
\end{array}$$

The arrow in the middle corresponds to the known "input" Fourier inequality and the other two arrows correspond to convolution inequalities for the operators

$$g \mapsto (g*a)/\check{b} \quad \text{and} \quad \hat{f} \mapsto (\hat{f}*b)/\hat{a}.$$

The inequality relating $\hat{g}$ and g that results from this composition is just (1) with new indices p_1 and q_1 and new weights u_1 and v_1. This is our "output" inequality.

Theorem 1 *Suppose C_0 is a positive constant, p_0 and q_0 are indices in $(1, \infty)$, and u_0 and v_0 are non-negative weight functions such that the Fourier inequality*

$$\left(\int_{\mathbb{R}} |\hat{f}|^{q_0} u_0\right)^{1/q_0} \leq C_0 \left(\int_{\mathbb{R}} |f|^{p_0} v_0\right)^{1/p_0} \tag{3}$$

holds for all $f \in L^1$. Let a and b be finite complex-valued Borel measures and h_a and h_b be positive functions on $\mathbb{R}$. For p_1 and q_1 satisfying $1 < p_0 \leq p_1 < \infty$ and $1 < q_1 \leq q_0 < \infty$ set

$$w_a = |\check{b}|^{-p_0} v_0 \quad \text{and} \quad v_1 = h_a^{1-p_1} K_a^*(w_a (K_a h_a)^{p_0 - 1}),$$
$$w_b = h_b (K_b (u_0^{1-q_0'} (K_b^* h_b)^{q_0' - 1}))^{1-q_1} \quad \text{and} \quad u_1 = |\hat{a}|^{q_1} w_b.$$

Also set

$$C_a = \left(\int_{\mathbb{R}} h_a^{p_1} v_1\right)^{1/p_0 - 1/p_1} \qquad \text{and} \quad C_b = \left(\int_{\mathbb{R}} h_b^{q_1'} w_b^{1-q_1'}\right)^{1/q_1 - 1/q_0}.$$

If $\check{b}$ is bounded away from zero, then the Fourier inequality

$$\left(\int_{\mathbb{R}} |\hat{g}|^{q_1} u_1\right)^{1/q_1} \leq C_1 \left(\int_{\mathbb{R}} |g|^{p_1} v_1\right)^{1/p_1} \tag{4}$$

holds for all $g \in L^1$. Here $C_1 = C_b C_0 C_a$.

Proof. Let $g \in L^1$ and set $f = (g * a)/\check{b}$. Since $\check{b}$ is bounded away from zero, $1/\check{b} \in L^\infty$ so $f \in L^1$. Taking the Fourier transform of both sides of the equation $g * a = f\check{b}$ and using (2) yields $\hat{g}\hat{a} = \hat{f} * b$.

Proposition 2 shows that

$$\left(\int_{\mathbb{R}} (K_b |\hat{f}|)^{q_1} w_b\right)^{1/q_1} \leq C_b \left(\int_{\mathbb{R}} |\hat{f}|^{q_0} u_0\right)^{1/q_0}$$

and the trivial estimate $|\hat{g}\hat{a}| = |\hat{f} * b| \leq K_b |\hat{f}|$ gives

$$\left(\int_{\mathbb{R}} |\hat{g}|^{q_1} u_1\right)^{1/q_1} \leq C_b \left(\int_{\mathbb{R}} |\hat{f}|^{q_0} u_0\right)^{1/q_0}. \tag{5}$$

Proposition 1 shows that

$$\left(\int_{\mathbb{R}} (K_a |g|)^{p_0} w_a\right)^{1/p_0} \leq C_a \left(\int_{\mathbb{R}} |g|^{p_1} v_1\right)^{1/p_1}$$

and the trivial estimate $|f\check{b}| = |g * a| \leq K_a |g|$ gives

$$\left(\int_{\mathbb{R}} |f|^{p_0} v_0\right)^{1/p_0} \leq C_a \left(\int_{\mathbb{R}} |g|^{p_1} v_1\right)^{1/p_1}. \tag{6}$$

The three inequalities (5), (3), and (6) combine to yield (4) as required. This completes the proof.

When all indices are taken equal to 2 the theorem simplifies substantially.

Corollary 1 *Suppose C is a positive constant and u_0 and v_0 are non-negative weight functions such that the Fourier inequality*

$$\int_{\mathbb{R}} |\hat{f}|^2 u_0 \le C \int_{\mathbb{R}} |f|^2 v_0 \tag{7}$$

holds for all $f \in L^1$. Let a and b be finite complex-valued Borel measures and h_a and h_b be positive functions on $\mathbb{R}$. Set

$$v_1 = \frac{1}{h_a} K_a^* \left(\frac{v_0(K_a h_a)}{|\check{b}|^2} \right) \quad \text{and} \quad u_1 = \frac{|\hat{a}|^2 h_b}{K_b \left(\frac{K_b^* h_b}{u_0} \right)}.$$

If $\check{b}$ is bounded away from zero, then the Fourier inequality

$$\int_{\mathbb{R}} |\hat{g}|^2 u_1 \le C \int_{\mathbb{R}} |g|^2 v_1 \tag{8}$$

holds for all $g \in L_1$.

A further simplification yields an attractive concrete collection of weighted Fourier inequalities that may be compared with results from [1].

Corollary 2 *If a is a finite positive Borel measure and h is a positive function on $\mathbb{R}$ then*

$$\int_{\mathbb{R}} |\hat{g}|^2 |\hat{a}|^2 \le 2\pi \int_{\mathbb{R}} |g|^2 \frac{\tilde{a} * (a * h)}{h} \tag{9}$$

for all $g \in L_1$. Here $\tilde{a}$ is defined by $\tilde{a}(x) = a(-x)$.

Proof. In Corollary 1, take $u_0 = v_0 \equiv 1$ and observe that (7) holds with $C = 2\pi$. Take $h_b \equiv 1$, $h_a = h$ and take b to be the Dirac measure at zero. Then $\check{b} \equiv 1$ and both K_b and K_b^* reduce to identity operators. Since a is assumed to be positive, K_a and K_a^* are just convolution by a and $\tilde{a}$, respectively.

References

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