A CONDITION FOR CONVEXITY OF A PRODUCT OF POSITIVE DEFINITE QUADRATIC FORMS

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Abstract. A sufficient condition for the convexity of a finite product of positive definite quadratic forms is given in terms of the condition numbers of the underlying matrices. When only two factors are involved the condition is also necessary. This complements and improves a result recently obtained by Zhao [Convexity Conditions and the Legendre-Fenchel Transform for the Product of Finitely Many Positive Definite Quadratic Forms, Applied Mathematics and Optimization, Volume 62, (2010) Number 3, 411-434]. As a special case, a necessary and sufficient condition is given for the Kantorovich function $(x^TAx)(x^TA^{-1}x)$, where A is positive definite, to be convex.

Key words. Legendre-Fenchel transform, quadratic form, positive definite matrix, condition number.

AMS subject classifications. 90C25, 65K05, 65F15, 15A48

1. Introduction. Given a function $h : \mathbf{R^n} \to \mathbf{R}$, its Legendre-Fenchel conjugate (LF-conjugate for short), which is also widely referred to as the Legendre-Fenchel transform of h [1, 2, 4, 6, 8], is defined as,

$$h^*(x) = \sup_{y\in\mathbf{R^n}} x^Ty - h(y).$$

The LF-conjugate has a significant impact in many areas. It plays an essential role in developing the convex optimization theory and algorithms (e.g., [3, 5, 12]); it is also widely used in matrix analysis and eigenvalue optimization [9, 10, 11].

In this paper, we consider finite products of positive definite quadratic forms. If A is a real symmetric positive definite matrix we let q_A denote the quadratic form

$$q_A(y) = \frac{1}{2}y^TAy.$$

It is easy to verify that q_A is a convex function on $\mathbf{R^n}$, and well known (see, e.g., [12]) that the LF-conjugate of q_A is also a positive definite quadratic form; specifically,

$$q_A^*(y) = \frac{1}{2}y^TA^{-1}y.$$

From a fast computation and practical application point of view, it is interesting and important to know the LF-conjugate of the product of two positive definite quadratic forms. This problem was posed by Hiriart-Urruty as an open question in the field of nonlinear analysis and optimization [7] and recently studied by Y. B. Zhao in [13]. Zhao also considered products of finitely many positive definite quadratic forms in [14]. Before introducing his result, we need to introduce some notation.

We write $A \succ 0$ to mean that A is a real symmetric positive definite matrix, and let $\kappa(A)$ denote the condition number of A; i.e., $\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$, the ratio of its

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largest and smallest eigenvalues. Fix $m \geq 2$, $n \times n$ matrices $A_1, \dots, A_m \succ 0$, and let $f : \mathbf{R^n} \to \mathbf{R}$ be the product $q_{A_1} \dots q_{A_m}$, i.e.,

$$f(y) = \prod_{i=1}^{m} \frac{1}{2} y^T A_i y.$$

For f to be a convex function on $\mathbf{R^n}$ it is necessary and sufficient that the Hessian matrix $\nabla^2 f(y)$ of f be positive semi-definite at each point y. For $y \neq 0$, the gradient and the Hessian matrix of f are given by,

$$\nabla f(y) = 2f(y) \sum_{i=1}^{m} \frac{A_i y}{y^T A_i y},\ \nabla^2 f(y) = 2f(y) \left(\sum_{i=1}^{m} \frac{A_i}{y^T A_i y} + 2 \sum_{i=1}^{m} \sum_{j \neq i} \frac{A_i y y^T A_j}{y^T A_i y y^T A_j y} \right).$$

Since $f(y) > 0$ whenever $y \neq 0$, the convexity of f reduces to showing that

$$\sum_{i=1}^{m} \frac{x^T A_i x}{y^T A_i y} + 2 \sum_{i=1}^{m} \sum_{j \neq i} \frac{x^T A_i y}{y^T A_i y} \frac{x^T A_j y}{y^T A_j y} \geq 0 \tag{1.1}$$

for all $x, y \in \mathbf{R^n}$ with $y \neq 0$. (When $y = 0$, $\nabla^2 f(0) = 0$ is positive semi-definite for any choice of $A_1, \dots, A_m$.)

In Theorem 3.6 of [14], Zhao gave an explicit formula for the LF-conjugate of f, provided f is known to be convex. So it is important to have simple, easily verified conditions that ensure the convexity of f. Zhao obtained the following sufficient condition for the convexity of f.

Proposition 1.1. [14] *Let $A_i \succ 0$, $i = 1, \cdots, m$ be $n \times n$ matrices. If*

$$\kappa(A_j^{-1/2} A_i A_j^{-1/2}) \leq \frac{\sqrt{4m-2}+2}{\sqrt{4m-2}-2} \quad \text{for all} \quad i, j = 1, \cdots, m, i \neq j,$$

then the product of m quadratic forms $f = \prod_{i=1}^{m} q_{A_i}$ is convex.

As a consequence of our main result, Theorem 2.3, we give the following improvement of Proposition 1.1. The proof will be given in the next section.

Theorem 1.2. *Let $A_i \succ 0$, $i = 1, \cdots, m$ be $n \times n$ matrices. If*

$$\kappa(A_j^{-1/2} A_i A_j^{-1/2}) \leq \left(\frac{\sqrt{2m-2}+1}{\sqrt{2m-2}-1} \right)^2 \quad \text{for all} \quad i, j = 1, \cdots, m, i \neq j, \tag{1.2}$$

then the product of m quadratic forms $f = \prod_{i=1}^{m} q_{A_i}$ is convex. If $m = 2$ the condition (1.2) is also necessary for the convexity of f.

Remark. For $m \geq 2$,

$$2m - 1 < \sqrt{(2m-1)^2 + [4(m-1)^2 - 1]} = \sqrt{4m-2}\sqrt{2m-2},$$

so

$$\left(\frac{\sqrt{2m-2}+1}{\sqrt{2m-2}-1} \right)^2 = \frac{2m-1+2\sqrt{2m-2}}{2m-1-2\sqrt{2m-2}} > \frac{\sqrt{4m-2}+2}{\sqrt{4m-2}-2}.$$

This shows that (1.2) is strictly weaker than the hypothesis of Proposition 1.1. When $m = 2$, the upper bound in Theorem 1.2, i.e., $17 + 12\sqrt{2}$, was already known to be the greatest possible right-hand-side value such that (1.2) could ensure the convexity of the product of two positive definite quadratic forms. See Remark 2.7 in [14].

COROLLARY 1.3. *If $A \succ 0$ is an $n \times n$ matrix, then the Kantorovich function $(x^T Ax)(x^T A^{-1}x)$, where $x \in \mathbf{R^n}$, is convex if and only if $\kappa(A) \le 3 + 2\sqrt{2}$.*

Proof. Let $m = 2$, $A_1 = A$ and $A_2 = A^{-1}$ in Theorem 1.2. The condition $\kappa(A_2^{-1/2} A_1 A_2^{-1/2}) \le (3 + 2\sqrt{2})^2$ is equivalent to $\kappa(A^2) \le (3 + 2\sqrt{2})^2$, i.e., $\kappa(A) \le 3 + 2\sqrt{2}$. ∎

The result of the corollary in the case $n = 2$, as well as the necessity of the condition on κ for general n, was given in [15].

2. Main Results. We start with a simple but useful lemma. It may be viewed as a sharp version of Theorem 1.2 in the case of two 2×2 matrices.

LEMMA 2.1. *If $\kappa \ge 1$ and $\eta = ((\sqrt{\kappa} - 1)/(\sqrt{\kappa} + 1))^2$ then*

$$\eta(\kappa + s^2)(1 + t^2) + \eta(\kappa + t^2)(1 + s^2) + 2(\kappa + st)(1 + st) \ge 0$$

for all $s, t \in \mathbf{R}$. Equality holds if and only if $s = -t = \pm\kappa^{1/4}$ or $\kappa = 1$ and $st = -1$.

Proof. For any s, t, and z we may factor out $z^2 + 1$ and complete the square on z to get,

$$\begin{aligned}&(z-1)^2(z^2+s^2)(1+t^2) + (z-1)^2(z^2+t^2)(1+s^2) + 2(z+1)^2(z^2+st)(1+st) \\ &= (z^2+1)(4+(s+t)^2)\left(\left(z - \frac{(s-t)^2}{4+(s+t)^2}\right)^2 + \frac{4(s+t)^2(1+st)^2}{(4+(s+t)^2)^2}\right).\end{aligned}$$

The second expression is non-negative and vanishes if and only if either $s + t = 0$ and $z = s^2$, or $st = -1$ and $z = 1$. In the first expression, divide through by $(z + 1)^2$ and take $z = \sqrt{\kappa}$ to obtain the conclusion of the lemma. ∎

The next lemma essentially gives a reduction of the case of two $n \times n$ matrices to the case of two 2×2 matrices, and then applies the previous result.

LEMMA 2.2. *Suppose $A, B \succ 0$ are $n \times n$ matrices and let $\kappa = \kappa(A^{-1/2}BA^{-1/2})$. Then for $x, y \in \mathbf{R^n}$, with $y \ne 0$, we have*

$$2\frac{x^T Ay}{y^T Ay}\frac{x^T By}{y^T By} \ge -\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^2 \left(\frac{x^T Ax}{y^T Ay} + \frac{x^T Bx}{y^T By}\right).$$

The inequality is sharp.

Proof. Since $A^{-1/2}BA^{-1/2} \succ 0$, there exists an orthogonal matrix U such that $U^T A^{-1/2}BA^{-1/2}U$ is a diagonal matrix with diagonal entries $\lambda_1 \ge \ldots \ge \lambda_n > 0$. Note that $\kappa = \lambda_1/\lambda_n$. Let $\eta = ((\sqrt{\kappa}-1)/(\sqrt{\kappa}+1))^2$. If we replace x by $A^{-1/2}Ux$ and y by $A^{-1/2}Uy$, an invertible change of variable, the statement of the lemma reduces to showing,

$$2\frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n y_i^2}\frac{\sum_{j=1}^n \lambda_j x_j y_j}{\sum_{j=1}^n \lambda_j y_j^2} \ge -\eta\left(\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n y_i^2} + \frac{\sum_{j=1}^n \lambda_j x_j^2}{\sum_{i=j}^n \lambda_j y_j^2}\right), \tag{2.1}$$

for all x and y in $\mathbf{R^n}$ with $y \ne 0$. Multiplying through to eliminate the denominators, we see that this is equivalent to showing $\sum_{j=1}^n \lambda_j r_j \ge 0$, where

$$r_j = \eta x_j^2 \sum_{i=1}^n y_i^2 + \eta y_j^2 \sum_{i=1}^n x_i^2 + 2x_j y_j \sum_{i=1}^n x_i y_i.$$

Because $\sum_{j=1}^n \lambda_j r_j$ is continuous in both x and y, it is enough to show that it is non-negative for all x and y such that x_1, x_n, y_1, y_n are all non-zero. Fix x and y satisfying that condition and partition $\{1, \ldots, n\}$ into subsets I_1 and I_2 as follows: $1 \in I_1$, $n \in I_2$ and for $2 \le j \le n-1$, $j \in I_1$ if $r_i \le 0$ and $j \in I_2$ otherwise. This ensures that $\lambda_j r_j \ge \lambda_1 r_j$ for $j \in I_1$ and $\lambda_j r_j \ge \lambda_n r_j$ for $j \in I_2$. Thus,

$$\sum_{j=1}^n \lambda_j r_j \ge \lambda_1 \sum_{j \in I_1} r_j + \lambda_n \sum_{j \in I_2} r_j.$$

Now for $p = 1, 2$, define u_p and v_p by,

$$u_p^2 = \left(\frac{\sum_{i \in I_p} x_i^2}{\sum_{i \in I_p} y_i^2} \right)^{1/2} \sum_{i \in I_p} x_i y_i \quad \text{and} \quad v_p^2 = \left(\frac{\sum_{i \in I_p} y_i^2}{\sum_{i \in I_p} x_i^2} \right)^{1/2} \sum_{i \in I_p} x_i y_i,$$

ensuring that $u_p \ge 0$ and choosing the sign of v_p so that $u_p v_p = \sum_{i \in I_p} x_i y_i$. The Cauchy-Schwarz inequality shows $u_p^2 \le \sum_{i \in I_p} x_i^2$ and $v_p^2 \le \sum_{i \in I_p} y_i^2$, and it follows from the definition of r_j that,

$$\sum_{j \in I_p} r_j \ge \eta u_p^2 (v_1^2 + v_2^2) + \eta v_p^2 (u_1^2 + u_2^2) + 2 u_p v_p (u_1 v_1 + u_2 v_2).$$

These estimates complete the proof, as

$$\begin{aligned}
\sum_{j=1}^n \lambda_j r_j &\ge \lambda_1 \sum_{j \in I_1} r_j + \lambda_n \sum_{j \in I_2} r_j \\
&= \lambda_n \left(\kappa \sum_{j \in I_1} r_j + \sum_{j \in I_2} r_j \right) \\
&\ge \lambda_n (\eta (\kappa u_1^2 + u_2^2)(v_1^2 + v_2^2) + \eta (\kappa v_1^2 + v_2^2)(u_1^2 + u_2^2) \\
&\quad + 2(\kappa u_1 v_1 + u_2 v_2)(u_1 v_1 + u_2 v_2)) \\
&= \lambda_n u_1^2 v_1^2 [\eta (\kappa + s^2)(1 + t^2) + \eta (\kappa + t^2)(1 + s^2) + 2(\kappa + st)(1 + st)],
\end{aligned}$$

where $s = u_2/u_1$ and $t = v_2/v_1$. The last expression is non-negative by Lemma 2.1.

To see that the inequality of the lemma is sharp it is enough to find $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ such that equality is achieved in (2.1). Since $\kappa = \lambda_1/\lambda_n$ it is routine to verify that the choice, $x_1 = 1$, $x_n = \kappa^{1/4}$, $y_1 = 1$, $y_n = -\kappa^{1/4}$ and $x_2 = \ldots = x_{n-1} = y_2 = \ldots = y_{n-1} = 0$ will suffice. $\square$

The following theorem gives the main result of the paper, a readily computed condition for a product of positive definite quadratic forms to be a convex function. The condition is expressed in terms of the condition numbers of the matrices involved.

THEOREM 2.3. *Let $A_1, A_2, \ldots, A_m$ be real symmetric positive definite $n \times n$ matrices and let $\kappa_{i,j} = \kappa(A_i^{-1/2} A_j A_i^{-1/2})$ for $i, j = 1, \ldots, m$. If*

$$\sum_{j=1}^m \left(\frac{\sqrt{\kappa_{i,j}} - 1}{\sqrt{\kappa_{i,j}} + 1} \right)^2 \le \frac{1}{2} \tag{2.2}$$

for $i = 1, 2, \ldots, m$, then $f = \prod_{i=1}^m q_{A_i}$ is convex. If $m = 2$ the condition is also necessary for the convexity of f.

Proof. First note that $\kappa_{i,i} = 1$. Next, observe that for $i \neq j$,

$$A_i^{-1/2} A_j A_i^{-1/2} = \left(A_j^{1/2} A_i^{-1/2} \right)^{-1} \left(A_j^{-1/2} A_i A_j^{-1/2} \right)^{-1} \left(A_j^{1/2} A_i^{-1/2} \right).$$

That is, $A_i^{-1/2} A_j A_i^{-1/2}$ is similar to the inverse of $A_j^{-1/2} A_i A_j^{-1/2}$. Similar matrices have the same eigenvalues and hence the same condition number. Moreover, for any matrix A with positive eigenvalues, $\lambda_{\min}(A^{-1}) = 1/\lambda_{\max}(A)$ and $\lambda_{\max}(A^{-1}) = 1/\lambda_{\min}(A)$ so $\kappa(A^{-1}) = \kappa(A)$. It follows that $\kappa_{i,j} = \kappa_{j,i}$. With this in mind, let $\eta_{i,j} = \left(\frac{\sqrt{\kappa_{i,j}}-1}{\sqrt{\kappa_{i,j}}+1} \right)^2$ and apply Lemma 2.2 to get

$$\begin{aligned}
&\sum_{i=1}^{m} \frac{x^T A_i x}{y^T A_i y} + 2 \sum_{i=1}^{m} \sum_{j \neq i} \frac{x^T A_i y}{y^T A_i y} \frac{x^T A_j y}{y^T A_j y} \\
&\geq \sum_{i=1}^{m} \frac{x^T A_i x}{y^T A_i y} - \sum_{i=1}^{m} \sum_{j \neq i} \eta_{i,j} \left(\frac{x^T A_i x}{y^T A_i y} + \frac{x^T A_j x}{y^T A_j y} \right) \\
&= \sum_{i=1}^{m} \frac{x^T A_i x}{y^T A_i y} - 2 \sum_{i=1}^{m} \left(\sum_{j \neq i} \eta_{i,j} \right) \frac{x^T A_i x}{y^T A_i y} \\
&= \sum_{i=1}^{m} \frac{x^T A_i x}{y^T A_i y} \left(1 - 2 \sum_{j=1}^{m} \eta_{i,j} \right) \geq 0.
\end{aligned}$$

As pointed out in (1.1) this shows that f is convex.

If $m = 2$, the convexity of f implies, via (1.1), that

$$2 \frac{x^T A_1 y}{y^T A_1 y} \frac{x^T A_2 y}{y^T A_2 y} \geq -\frac{1}{2} \left(\frac{x^T A_1 x}{y^T A_1 y} + \frac{x^T A_2 x}{y^T A_2 y} \right)$$

for all x and non-zero y. Combining this with the sharpness of the inequality of Lemma 2.2 gives,

$$\left(\frac{\sqrt{\kappa_{1,2}} - 1}{\sqrt{\kappa_{1,2}} + 1} \right)^2 \leq \frac{1}{2},$$

showing that (2.2) is necessary for convexity. $\square$

Proof. Proof of Theorem 1.2. We verify the condition of the Theorem 2.3. Recall that $\eta_{i,i} = 0$ and calculate as follows,

$$\sum_{j=1}^{m} \left(\frac{\sqrt{\kappa_{i,j}} - 1}{\sqrt{\kappa_{i,j}} + 1} \right)^2 \leq (m-1) \left(\frac{\frac{\sqrt{2m-2}+1}{\sqrt{2m-2}-1} - 1}{\frac{\sqrt{2m-2}+1}{\sqrt{2m-2}-1} + 1} \right)^2 = \frac{1}{2}.$$

So (2.2) is satisfied and therefore f is convex. If $m = 2$, an easy calculation shows that the conditions (1.2) and (2.2) coincide so (1.2) is also necessary for convexity. $\square$

Remark. The proof of Theorem 2.3 suggests the following weakening of condition (2.2). Since

$$\frac{1}{\kappa(A_i)} \frac{x^T x}{y^T y} \leq \frac{x^T A_i x}{y^T A_i y} \leq \kappa(A_i) \frac{x^T x}{y^T y},$$

if we define,

$$L = \left\{ i : \sum_{j=1}^{m} \eta_{i,j} \leq \frac{1}{2} \right\} \text{ and } G = \left\{ i : \sum_{j=1}^{m} \eta_{i,j} > \frac{1}{2} \right\}$$

then the proof goes through with condition (2.2) replaced by,

$$\sum_{i \in L} \frac{1}{\kappa(A_i)} \left(1 - 2 \sum_{j=1}^{m} \eta_{i,j} \right) + \sum_{i \in G} \kappa(A_i) \left(1 - 2 \sum_{j=1}^{m} \eta_{i,j} \right) \geq 0. \tag{2.3}$$

This condition is weaker than (2.2) and still implies that f is convex, but is complicated and rather unwieldy. It can be applied, however, as we see in the next example where it is used to show that the condition (2.2) is not necessary when $m > 2$.

Example. With $m = 3$, take A_1 and A_2 to be 2×2 identity matrices, and A_3 to be a 2×2 diagonal matrix with diagonal entries $(3 + \delta)^2$ and 1. Calculations show that for sufficiently small positive δ, (2.2) fails but (2.3) holds. (Any positive $\delta < 0.18$ will do.) Thus, the sufficient condition of Theorem 2.3 is not necessary for general m.

Acknowledgements. The first author would like to thank Dr. Yun-Bin Zhao for bringing [14] to his attention.

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