Problem 1. Show that for all positive sequences $\{x_i\}$ and all integers $n > 0$,

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i \leq 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k} x_j\right)^2 x_k^{-1}.$$

Solution. Interchanging the order of summation gives

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i = \sum_{j=1}^{n}(n-j+1)\sum_{i=1}^{j} x_i = \sum_{i=1}^{n}\binom{n-i+2}{2} x_i \geq \sum_{i=1}^{n}\frac{1}{2}(n-i+1)^2 x_i.$$

This observation, together with the Cauchy-Schwarz inequality, yields

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i &= \sum_{j=1}^{n}(n-j+1)\sum_{i=1}^{j} x_i \\
&= \sum_{j=1}^{n}(n-j+1)x_j^{1/2}\left(\sum_{i=1}^{j} x_i\right)x_j^{-1/2} \\
&\leq \left(\sum_{j=1}^{n}(n-j+1)^2 x_j\right)^{1/2}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{j} x_i\right)^2 x_j^{-1}\right)^{1/2} \\
&\leq \left(2\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i\right)^{1/2}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{j} x_i\right)^2 x_j^{-1}\right)^{1/2}
\end{aligned}$$

Square both sides and divide by $\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i$ to get the inequality.

Problem 2. Does the above inequality remain true without the factor 2?