Problem 1. Show that for all positive sequences \( \{x_i\} \) and all integers \( n > 0 \),

\[
\sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{i=1}^{j} x_i \leq 2 \sum_{k=1}^{n} \left( \sum_{j=1}^{k} x_j \right)^2 x_k^{-1}.
\]

Solution. Interchanging the order of summation gives

\[
\sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{i=1}^{j} x_i = \sum_{j=1}^{n} (n - j + 1) \sum_{i=1}^{j} x_i = \sum_{i=1}^{n} \left( \frac{n - i + 2}{2} \right) x_i \geq \sum_{i=1}^{n} \frac{1}{2} (n - i + 1)^2 x_i.
\]

This observation, together with the Cauchy-Schwarz inequality, yields

\[
\sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{i=1}^{j} x_i = \sum_{j=1}^{n} (n - j + 1) \sum_{i=1}^{j} x_i \\
= \sum_{j=1}^{n} (n - j + 1) x_j^{1/2} \left( \sum_{i=1}^{j} x_i \right)^{1/2} x_j^{-1/2} \\
\leq \left( \sum_{j=1}^{n} (n - j + 1)^2 x_j \right)^{1/2} \left( \sum_{j=1}^{n} \left( \sum_{i=1}^{j} x_i \right)^2 x_j^{-1} \right)^{1/2} \\
\leq \left( \frac{2}{\sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{i=1}^{j} x_i} \right)^{1/2} \left( \sum_{j=1}^{n} \left( \sum_{i=1}^{j} x_i \right)^2 x_j^{-1} \right)^{1/2}.
\]

Square both sides and divide by \( \sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{i=1}^{j} x_i \) to get the inequality.

Problem 2. Does the above inequality remain true without the factor 2?