# MAPPING PROPERTIES OF INTEGRAL AVERAGING OPERATORS

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Abstract. Characterizations are obtained for those pairs of weight functions $u$ and $v$ for which the operators $Tf(x) = \int_{a(x)}^{b(x)} f(t)\,dt$ with $a$ and $b$ certain non-negative functions are bounded from $L^p_u(0,\infty)$ to $L^q_v(0,\infty)$, $0 < p, q < \infty$, $p \geq 1$. Sufficient conditions are given for $T$ to be bounded on the cones of monotone functions.

The results are applied to give a weighted inequality comparing differences and derivatives as well as a weight characterization for the Steklov operator.

## 1. Introduction

In this paper we study mapping properties of the operator

$$Tf(x) = \int_{a(x)}^{b(x)} f(t)\,dt, \quad f \geq 0, \tag{1.1}$$

where $a$ and $b$ are increasing, differentiable functions satisfying $a(0) = b(0) = 0$, $a(x) < b(x)$ for $x \in (0,\infty)$ and $a(\infty) = b(\infty) = \infty$. Specifically, conditions on the

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weight functions $u$ and $v$ are given which are equivalent to

$$\left(\int_0^\infty \left(\int_{a(x)}^{b(x)} f\right)^q v(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f^p u\right)^{1/p}, \quad 0<p,q<\infty. \tag{1.2}$$

For example (see Theorem 2.2) if $1<p\le q<\infty$ then (1.2) holds if and only if

$$\sup \left(\int_{a(x)}^{b(t)} u^{1-p'}\right)^{1/p'} \left(\int_t^x v\right)^{1/q} = K < \infty, \tag{1.3}$$

where the supremum is taken over all $x, t$ such that $t<x$ and $a(x)<b(t)$. Moreover, the least constant $C$ in (1.2) is comparable to $K$, $K \le C \le 2p^{1/q}(p')^{1/p'}K$.

Weight characterizations for the corresponding Hardy inequality

$$\left(\int_0^\infty \left(\int_0^{b(x)} f\right)^q v(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f^p u\right)^{1/p} \tag{1.4}$$

follow easily from well known results ([3], [10]). If $1<p\le q<\infty$, (1.4) holds if and only if

$$\sup_{t>0} \left(\int_0^{b(t)} u^{1-p'}\right)^{1/p'} \left(\int_t^\infty v\right)^{1/q} < \infty. \tag{1.5}$$

Of course, this condition implies (1.3), but is actually more restrictive. For example if $b(x)=2a(x)=x$, $v(x)=x^\beta$, and $u(x)=x^\alpha$ then (1.3) is satisfied ([12]) if and only if $\frac{\alpha}{p}=\frac{\beta+1}{q}+\frac{1}{p'}$, while (1.5) if and only if this equality holds with $\beta<-1$.

In another direction, let $f \in C^{(1)}(\mathbf{R}^+)$ with $f(0)=f(\infty)=0$, then it is known (Gurka [10, Ex. 8.6], Grisvard [6], Jakovlev [8]) that for $1<p<\infty$

$$\begin{aligned} \int_0^\infty |f(x)|^p x^{-\lambda p}\,dx &\le C\int_0^\infty |f'(x)|^p x^{(1-\lambda)p}\,dx \\ \int_0^\infty |f(x)|^p x^{-\lambda p}\,dx &\le C\int_0^\infty\int_0^\infty \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda p}}\,dx\,dy \end{aligned} \tag{1.6}$$

where $\lambda \neq 1/p$. It is natural to ask which of the right sides of (1.6) is larger, the one involving the derivative of $f$ or the one involving differences of $f$. On applying the weight characterizations of (a special case of) the operator $Tf$, we answer this question by showing that for $C^{(1)}$-functions the inequality

$$\int_0^\infty \int_0^\infty \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda p}}\,dx\,dy \leq C\int_0^\infty |f'(x)|^p x^{(1-\lambda)p}\,dx, \quad 1<p<\infty, \tag{1.7}$$

is satisfied if $0<\lambda<1$.

The paper is divided into three sections. Section 2 contains the main results, namely the weight characterizations for the operator $T$ given by (1.1), in the case $1 \leq p \leq q < \infty$ (Theorem 2.2) and the case $0 < q < p$, $p > 1$ (Theorem 2.5). Corollaries yield a result of Sawyer ([12]) and a weight characterization of the related Steklov operator studied by Batuev and Stepanov ([2]). Although both are characterizations, our weight conditions have a somewhat different form than the ones given in [2].

There is considerable current interest in mapping properties of the Hardy operator defined on the cones of monotone functions. In Section 3 we provide simple sufficient conditions on weight functions under which the operator $T$ of (1.1) defined on a cone of monotone functions is bounded on weighted Lebesgue spaces. In addition we give weighted extensions of (1.7), complementing the weighted results for the inequalities in (1.6), (see [4],[7],[10, p. 99]). In particular, we establish (1.7).

The notation is standard. $\chi_E$ denotes the characteristic function of the set $E$; if $0<q<\infty$, then $q'$ denotes the conjugate exponent of $q$ defined by $1/q+1/q'=1$, and similarly for $p$. Expressions of the form $0/0$, $\infty/\infty$, and $0 \cdot \infty$ are taken to be zero, and $A \approx B$ means that $A/B$ is bounded above and below by positive constants. Let $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{R}^+$ denote the sets of integers, real numbers, and positive real numbers respectively, while $C^{(n)}(E)$ denotes the space of functions on $E$ whose $n$th derivative is continuous. Finally, inequalities (such as (1.2)) are interpreted to mean that if the right hand side is finite, so is the left hand side and the inequality holds.

## 2. Main Results

Throughout, $a$ and $b$ are taken to be increasing differentiable functions on $\mathbf{R}^+$, satisfying $a(0) = b(0) = 0$, $a(x) < b(x)$ for $x > 0$, and $a(\infty) = b(\infty) = \infty$. Since $a^{-1}$ and $b^{-1}$ exist and are increasing we may define the sequence $\{m_k\}_{k \in \mathbf{Z}}$ recursively as follows: Fix $m > 0$ and define

$$(2.1) \quad m_0 = m,\ m_{k+1} = a^{-1}(b(m_k)),\ \text{if } k \geq 0 \text{ and } m_k = b^{-1}(a(m_{k+1})),\ \text{if } k < 0.$$

Clearly $a(m_{k+1}) = b(m_k)$ for all $k \in \mathbf{Z}$.

**Lemma 2.1.** *Fix $m$ and let $\{m_k\}_{k \in \mathbf{Z}}$ be defined by* (2.1). *Then $m_k < m_{k+1}$ for $k \in \mathbf{Z}$, $\lim_{k\to\infty} m_k = \infty$, and $\lim_{k\to-\infty} m_k = 0$.*

*Proof.* Since $a(m_k) < b(m_k) = a(m_{k+1})$ and $a^{-1}$ is increasing, $m_k < m_{k+1}$ for all $k \in \mathbf{Z}$. Also the monotonicity of $\{m_k\}$ ensures the existence of $M^- \in [0, \infty)$ and $M^+ \in (0, \infty]$ such that $m_k \to M^-$ as $k \to -\infty$ and $m_k \to M^+$ as $k \to \infty$. Since $a$

and $b$ are continuous

$$b(M^-) = \lim_{k\to-\infty} b(m_k) = \lim_{k\to-\infty} a(m_{k+1}) = a(M^-)$$

and similarly $b(M^+) = a(M^+)$. But $a(x) < b(x)$ for all $x \in (0,\infty)$ so $M^- = 0$ and $M^+ = \infty$ as required.

In order to study weighted norm inequalities for the operator $T$ of (1.1), that is

$$\left(\int_0^\infty \left(\int_{a(x)}^{b(x)} f\right)^q v(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f^p u\right)^{1/p}, \quad f \ge 0, \tag{2.2}$$

it is convenient to consider the equivalent inequality

$$\left(\int_0^\infty \left(\int_{a(x)}^{b(x)} fw\right)^q v(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f^p w\right)^{1/p}, \quad f \ge 0, \tag{2.3}$$

where $w = u^{1-p'}$.

We also write $v_a(y) = v(a^{-1}(y))(a^{-1})'(y)$ so that $v_a(y)\,dy = v(x)\,dx$ if $y = a(x)$. $v_b$ is defined similarly.

**Theorem 2.2.** *Let $u$ and $v$ be weight functions, then there is a constant $C$ such that* (2.2) *holds for $1 < p \le q < \infty$ if and only if*

$$K \equiv \sup \left(\int_{a(x)}^{b(t)} u^{1-p'}\right)^{1/p'} \left(\int_t^x v\right)^{1/q} < \infty, \tag{2.4}$$

*where the supremum is taken over all $x$ and $t$ such that $t \le x$ and $a(x) \le b(t)$. Moreover, the best constant $C$ in* (1) *satisfies*

$$K \le C \le 2p^{1/q}(p')^{1/p'}K.$$

*Proof.* Suppose first that (2.4) is satisfied. If $t$ is fixed and $y = a(x)$ in (2.4) then with $w = u^{1-p'}$

$$\sup_{a(t)\le y\le b(t)} \left(\int_y^{b(t)} w\right)^{1/p'} \left(\int_{a(t)}^y v_a\right)^{1/q} = K < \infty,$$

and it follows from [3, Theorem 2] that for all $f \ge 0$

$$\left(\int_{a(t)}^{b(t)} \left(\int_y^{b(t)} fw\right)^q v_a(y)\,dy\right)^{1/q} \le CK \left(\int_{a(t)}^{b(t)} f^p w\right)^{1/p}, \tag{2.5}$$

where $C = p^{1/q}(p')^{1/p'}$. Similarly, if $x$ is fixed and $y = b(t)$ in (2.4) then we have

$$\sup_{a(x)\le y\le b(x)} \left(\int_{a(x)}^y w\right)^{1/p'} \left(\int_y^{b(x)} v_b\right)^{1/q} \le K$$

and it follows from [3, Theorem 1] that

$$\left(\int_{a(x)}^{b(x)} \left(\int_{a(x)}^y fw\right)^q v_b(y)\,dy\right)^{1/q} \le CK \left(\int_{a(x)}^{b(x)} f^p w\right)^{1/p}. \tag{2.6}$$

Fix $m \in (0,\infty)$ and let $\{m_k\}_{k\in\mathbf{Z}}$ be the sequence defined in (2.1). If $m_k < x < m_{k+1}$, then $a(x) < a(m_{k+1}) = b(m_k) \le b(x)$. Writing $a_k = a(m_k)$, $b_k = b(m_k)$, and $E_k = (m_k, m_{k+1})$ we get, using Minkowski's inequality,

$$\begin{aligned}
&\left(\int_0^\infty \left(\int_{a(x)}^{b(x)} fw\right)^q v(x)\,dx\right)^{1/q}\\
&= \left(\int_0^\infty \left(\sum_{k\in\mathbf{Z}} \chi_{E_k}(x) \int_{a(x)}^{b_k} fw + \sum_{k\in\mathbf{Z}} \chi_{E_k}(x) \int_{a_{k+1}}^{b(x)} fw\right)^q v(x)\,dx\right)^{1/q}\\
&\le \left(\int_0^\infty \left(\sum_k \chi_{E_k}(x) \int_{a(x)}^{b_k} fw\right)^q v(x)\,dx\right)^{1/q}\\
&\quad + \left(\int_0^\infty \left(\sum_k \chi_{E_k}(x) \int_{a_{k+1}}^{b(x)} fw\right)^q v(x)\,dx\right)^{1/q}\\
&\equiv I_1 + I_2,
\end{aligned}$$

respectively. But since for each $x$ only one term of the sum can be non-zero

$$
\begin{aligned}
I_1 &= \left(\int_0^\infty \sum_k \chi_{E_k}(x) \left(\int_{a(x)}^{b_k} fw\right)^q v(x)\,dx\right)^{1/q} \\
&= \left(\sum_k \int_{m_k}^{m_{k+1}} \left(\int_{a(x)}^{b_k} fw\right)^q v(x)\,dx\right)^{1/q} \\
&= \left(\sum_k \int_{a_k}^{b_k} \left(\int_y^{b_k} fw\right)^q v_a(y)\,dy\right)^{1/q},
\end{aligned}
$$

where the last equality follows from the change of variable $y = a(x)$. Applying (2.5) and using the fact that $1 \le q/p$ it follows that

$$
\begin{aligned}
I_1 \le &\left(\sum_k C^q K^q \left(\int_{a_k}^{b_k} f^p w\right)^{q/p}\right)^{1/q} \le CK \left(\left(\sum_k \int_{a_k}^{b_k} f^p w\right)^{q/p}\right)^{1/q} \\
=&CK \left(\int_0^\infty f^p w\right)^{1/p}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
I_2 &= \left(\int_0^\infty \sum_k \chi_{E_k}(x) \left(\int_{a_{k+1}}^{b(x)} fw\right)^q v(x)\,dx\right)^{1/q} \\
&= \left(\sum_k \int_{m_k}^{m_{k+1}} \left(\int_{a_{k+1}}^{b(x)} fw\right)^q v(x)\,dx\right)^{1/q} \\
&= \left(\sum_k \int_{a_{k+1}}^{b_{k+1}} \left(\int_{a_{k+1}}^{y} fw\right)^q v_b(y)\,dy\right)^{1/q} \\
&\le \left(\sum_k C^q K^q \left(\int_{a_{k+1}}^{b_{k+1}} f^p w\right)^{q/p}\right)^{1/q} \\
&\le CK \left(\left(\sum_k \int_{a_{k+1}}^{b_{k+1}} f^p w\right)^{q/p}\right)^{1/q} \\
&= CK \left(\int_0^\infty f^p w\right)^{1/p}
\end{aligned}
$$

where we have made the change of variable $y = b(x)$ and applied (2.6).

From these two estimates, (2.3) and hence (2.2) follows with constant $2CK$.

Conversely, if (2.2) (or equivalently (2.3)) holds for some $C$, let $x$ and $t$ satisfy $t \le x$ and $a(x) \le b(t)$. Let $w_0$ be an $L^1$ weight such that $w_0 < w$ and define $f = \chi_{(a(x),b(t))} w_0/w$. If $t \le s \le x$ then $a(s) \le a(x) \le b(t) \le b(s)$ and therefore

$$\begin{aligned}\left(\int_{a(x)}^{b(t)} w_0\right)\left(\int_t^x v(s)\,ds\right)^{1/q} &= \left(\int_t^x \left(\int_{a(x)}^{b(t)} fw\right)^q v(s)\,ds\right)^{1/q} \\ \le \left(\int_0^\infty \left(\int_{a(s)}^{b(s)} fw\right)^q v(s)\,ds\right)^{1/q} &\le C\left(\int_0^\infty f^p w\right)^{1/p} \\ = C\left(\int_{a(x)}^{b(t)} w_0^p w^{1-p}\right)^{1/p} &\le C\left(\int_{a(x)}^{b(t)} w_0\right)^{1/p}.\end{aligned}$$

But since $w_0 \in L^1$, the last integral is finite and on dividing we get

$$\left(\int_{a(x)}^{b(t)} w_0\right)^{1/p'}\left(\int_t^x v(s)\,ds\right)^{1/q} \le C.$$

Let $w_0 \uparrow w$, then the Monotone Convergence Theorem implies that

$$\left(\int_{a(x)}^{b(t)} w\right)^{1/p'}\left(\int_t^x v(s)\,ds\right)^{1/q} \le C.$$

Finally, taking the supremum over all $x, t$ with $t \le x$ and $a(x) \le b(t)$ we obtain (2.4) with $w = u^{1-p'}$ and $K \le C$.

*Remark 2.3.* The case $1 = p \le q$ of Theorem 2.2 also holds and follows from [9, p. 316]. If $1 = p \le q < \infty$, then (2.2) holds if and only if

$$\operatorname*{ess\,sup}_{t>0} u(t)^{-1}\left(\int_{b^{-1}(t)}^{a^{-1}(t)} v(x)\,dx\right)^{1/q} < \infty.$$

We now give the weight characterization when $0 < q < p$, $p > 1$. First we require the following:

**Proposition 2.4.** *Suppose $w$ and $v$ are weights, $a$ and $b$ as before and $v_a$ and $v_b$ defined by*

$$v_a(y) = v(a^{-1}(y))(a^{-1})'(y), \quad v_b(y) = v(b^{-1}(y))(b^{-1})'(y). \tag{2.8}$$

*Let $0 < q < p$, $1 < p < \infty$, $1/r = 1/q - 1/p$, and for $m > 0$, $C(m)$ and $C^*(m)$ be the best constants in*

$$\left(\int_{a(m)}^{b(m)} \left(\int_{a(m)}^{y} f(t)w(t)\,dt\right)^q v_b(y)\,dy\right)^{1/q} \le C \left(\int_{a(m)}^{b(m)} f(t)^p w(t)\,dt\right)^{1/p}$$

*and*

$$\left(\int_{a(m)}^{b(m)} \left(\int_{y}^{b(m)} f(t)w(t)\,dt\right)^q v_a(y)\,dy\right)^{1/q} \le C^* \left(\int_{a(m)}^{b(m)} f(t)^p w(t)\,dt\right)^{1/p}$$

*respectively. Then $D(m) \approx C(m)$ and $D^*(m) \approx C^*(m)$, where*

(2.9)

$$D(m) = \left(\int_{a(m)}^{b(m)} \left(\int_{a(m)}^{y} w\right)^{r/p'} \left(\int_{y}^{b(m)} v_b\right)^{r/p} v_b(y)\,dy\right)^{1/r}, \quad m \in (0,\infty).$$

$$D^*(m) = \left(\int_{a(m)}^{b(m)} \left(\int_{y}^{b(m)} w\right)^{r/p'} \left(\int_{a(m)}^{y} v_a\right)^{r/p} v_a(y)\,dy\right)^{1/r}, \quad m \in (0,\infty).$$

*Proof.* Writing the first inequality and $D(m)$ in the forms

$$\left(\int_0^\infty \left(\int_0^y fw\chi_{(a(m),b(m))}\right)^q v_b(y)\chi_{(a(m),b(m))}(y)\,dy\right)^{1/q} \le C(m)\left(\int_0^\infty f^p w\chi_{(a(m),b(m))}\right)^{1/p},$$

respectively,

$$\left(\int_0^\infty \left(\int_0^y w\chi_{(a(m),b(m))}\right)^{r/p'} \left(\int_y^\infty v_b\chi_{(a(m),b(m))}\right)^{r/p} v_b(y)\chi_{(a(m),b(m))}(y)\,dy\right)^{1/r},$$

then by [13, Theorem 2.4]

$$(p')^{1/p'} q^{1/p}(1-q/p)D(m) \le C(m) \le (r/q)^{1/r} p^{1/p} p'^{1/p'} D(m).$$

The estimate for $C^*(m)$ follows in the same way, only now we use the dual of [13, Theorem 2.4].

**Theorem 2.5.** *Suppose $v$ and $w = u^{1-p'}$ are weights and $0 < q < p$, $1 < p < \infty$, then (2.2) (or equivalently (2.3)) is satisfied if and only if*

$$\left(\int_0^\infty \int_{b^{-1}(a(t))}^t \left(\int_{a(t)}^{b(x)} w\right)^{r/p'} \left(\int_x^t v\right)^{r/p} v(x)\,dx\,\sigma(t)dt\right)^{1/r} < \infty, \tag{2.10}$$

*and*

$$\left(\int_0^\infty \int_t^{a^{-1}(b(t))} \left(\int_{a(x)}^{b(t)} w\right)^{r/p'} \left(\int_t^x v\right)^{r/p} v(x)\,dx\,\sigma(t)dt\right)^{1/r} < \infty. \tag{2.11}$$

*Here the "normalizing function" $\sigma$ is defined by*

$$\sigma(t) = \sum_{k\in\mathbf{Z}} \chi_{(M_k, M_{k+1})}(t) \frac{d}{dt}(b^{-1}\circ a)^k(t)$$

*where $(b^{-1}\circ a)^k$ denotes $k$ times repeated composition and $\{M_k\}$ is constructed as $\{m_k\}$ (see (2.1)), but with $M_0 = b^{-1}(1)$.*

*Proof.* (Necessity.) Suppose first that (2.3) is satisfied. Let $v_0$ and $w_0$ be weights in $L^1$ such that $v_0 < v$ and $w_0 < w$. If $v_{b,0}(y) = v_0(b^{-1}(y))(b^{-1})'(y)$ then $v_{b,0} < v_b$, where $v_b$ is given in (2.8). Fix $m > 0$ and let $\{m_k\}_{k\in\mathbf{Z}}$ be the sequence constructed in (2.1). If $D$ and $D^*$ are the functions given by (2.9), let $\mathcal{D}(m) = \left(\sum_{k\in\mathbf{Z}} D(m_k)^r\right)^{1/r}$ and $\mathcal{D}^*(m) = \left(\sum_{k\in\mathbf{Z}} D^*(m_k)^r\right)^{1/r}$.

We shall show that

$$\max\left(\sup_{m>0}\mathcal{D}(m),\sup_{m>0}\mathcal{D}^*(m)\right)<\infty. \tag{2.12}$$

Again write $a_k = a(m_k)$, $b_k = b(m_k)$, and let $\chi = \chi_(a_k, b_k)$. Since $r/(pq') + 1 = r/(qp')$, integration yields

$$\begin{aligned}
&\left(\sum_{k\in\mathbf{Z}}\int_{a_k}^{b_k}\left(\int_{a_k}^{y}w_0\right)^{r/p'}\left(\int_{y}^{b_k}v_{b,0}\right)^{r/p}v_{b,0}(y)\,dy\right)^{1/q}\\
=&\frac{r}{p'q}\left(\sum_{k}\int_{a_k}^{b_k}\left(\int_{a_k}^{y}\left(\int_{a_k}^{t}w_0\right)^{r/pq'}w_0(t)\,dt\right)^{q}\left(\int_{y}^{b_k}v_{b,0}\right)^{r/p}v_{b,0}(y)\,dy\right)^{1/q}\\
\le&\frac{r}{p'q}\left(\sum_{k}\int_{a_k}^{b_k}\left(\int_{a_k}^{y}\left(\int_{a_k}^{t}w_0\right)^{r/pq'}\left(\int_{t}^{b_k}v_{b,0}\right)^{r/pq}w_0(t)\,dt\right)^{q}v_{b,0}(y)\,dy\right)^{1/q}
\end{aligned}$$

since $t < y$. Now for each $k$, $y$ satisfies $a_k < y < b_k$, and $t$ satisfies $a_k < t < y$ and therefore $a_k < t < b_k$. Hence the last expression is equal to

$$\begin{aligned}
&\frac{r}{p'q}\left(\int_0^\infty\sum_k\chi(y)\left(\int_{a_k}^{y}\left(\int_{a_k}^{t}w_0\right)^{r/pq'}\times\right.\right.\\
&\qquad\qquad\qquad\left.\left.\left(\int_{t}^{b_k}v_{b,0}\right)^{r/pq}\chi(t)w_0(t)\,dt\right)^{q}v_{b,0}(y)\,dy\right)^{1/q}\\
=&\frac{r}{p'q}\left(\int_0^\infty\left(\sum_k\chi(y)\int_{a_k}^{y}\left(\int_{a_k}^{t}w_0\right)^{r/pq'}\times\right.\right.\\
&\qquad\qquad\qquad\left.\left.\left(\int_{t}^{b_k}v_{b,0}\right)^{r/pq}\chi(t)w_0(t)\,dt\right)^{q}v_{b,0}(y)\,dy\right)^{1/q}.
\end{aligned}$$

Here we used the fact that for each $y$ only one term of the sum can be non-zero. Now if $y < b_k$, that is $b^{-1}(y) < m_k$, then $a(b^{-1}(y)) < a_k$ and hence we may increase the interval of integration from $(a_k, y)$ to $(a(b^{-1}(y)), y)$. Moreover, replacing $\chi(y)$

by 1, the last expression is not larger than

$$\frac{r}{p'q}\Big(\int_0^\infty\Big(\sum_k\int_{a(b^{-1}(y))}^y\Big(\int_{a_k}^t w_0\Big)^{r/pq'}\times \\ \Big(\int_t^{b_k} v_{b,0}\Big)^{r/pq}\chi(t)w_0(t)\,dt\Big)^q v_{b,0}(y)\,dy\Big)^{1/q}$$

$$= \frac{r}{p'q}\Big(\int_0^\infty\Big(\int_{a(b^{-1}(y))}^y\Big(\sum_k\Big(\int_{a_k}^t w_0\Big)^{r/q'}\times \\ \Big(\int_t^{b_k} v_{b,0}\Big)^{r/q}\chi(t)\Big)^{1/p} w_0(t)\,dt\Big)^q v_{b,0}(y)\,dy\Big)^{1/q}.$$

Again we used the fact that only one term of the sum can be non-zero. If we take

$$f(t) = \left(\sum_k\Big(\int_{a_k}^t w_0\Big)^{r/q'}\Big(\int_t^{b_k} v_{b,0}\Big)^{r/q}\chi(t)\right)^{1/p} w_0(t)/w(t)$$

and make the change of variable $y = b(x)$, this is equal to

$$\frac{r}{p'q}\Big(\int_0^\infty\Big(\int_{a(x)}^{b(x)}\Big(\sum_k\Big(\int_{a_k}^t w_0\Big)^{r/q'}\times \\ \Big(\int_t^{b_k} v_{b,0}\Big)^{r/q}\chi(t)\Big)^{1/p} w_0(t)\,dt\Big)^q v_0(x)\,dx\Big)^{1/q}.$$

$$\begin{aligned} &=\frac{r}{p'q}\left(\int_0^\infty\left(\int_{a(x)}^{b(x)} f(t)w(t)\,dt\right)^q v_0(x)\,dx\right)^{1/q} \\ &\le\frac{r}{p'q}\left(\int_0^\infty\left(\int_{a(x)}^{b(x)} f(t)w(t)\,dt\right)^q v(x)\,dx\right)^{1/q} \le \frac{Cr}{p'q}\left(\int_0^\infty f^p w\right)^{1/p} \end{aligned}$$

by (2.3). Now since $w_0 < w$

$$
\begin{aligned}
&\left(\int_0^\infty f^p w\right)^{1/p} \\
&= \left(\int_0^\infty \sum_k \left(\int_{a_k}^t w_0\right)^{r/q'} \left(\int_t^{b_k} v_{b,0}\right)^{r/q} \chi(t) w_0(t)^p w(t)^{1-p}\, dt\right)^{1/p} \\
&\le \left(\int_0^\infty \sum_k \left(\int_{a_k}^t w_0\right)^{r/q'} \left(\int_t^{b_k} v_{b,0}\right)^{r/q} \chi(t) w_0(t)\, dt\right)^{1/p} \\
&= \left(\sum_k \int_{a_k}^{b_k} \left(\int_{a_k}^t w_0\right)^{r/q'} \left(\int_t^{b_k} v_{b,0}\right)^{r/q} w_0(t)\, dt\right)^{1/p} \\
&= \left(\frac{p'}{q}\right)^{1/p} \left(\sum_k \int_{a_k}^{b_k} \left(\int_{a_k}^y w_0\right)^{r/p'} \left(\int_y^{b_k} v_{b,0}\right)^{r/p} v_{b,0}(y)\, dy\right)^{1/p}.
\end{aligned}
$$

where the last equality is obtained on integrating by parts. Since $v_{b,0}$ and $w_0$ are in $L^1$, the sum is finite and on dividing we obtain

$$
\left(\sum_k \int_{a_k}^{b_k} \left(\int_{a_k}^y w_0\right)^{r/p'} \left(\int_y^{b_k} v_{b,0}\right)^{r/p} v_{b,0}(y)\, dy\right)^{1/r} \le \frac{Cr}{p'q} \left(\frac{p'}{q}\right)^{1/p}.
$$

Since $v_{b,0} < v_b$ and $w_0 < w$ the Monotone Convergence Theorem implies that this inequality also holds with $v_{b,0}$ and $w_0$ replaced by $v_b$ and $w$ respectively. In particular (see (2.9)) we obtain that $\sup_{m>0}(\sum_k \mathcal{D}(m_k)^r)^{1/r} < \infty$. The same argument, with minor modifications, shows that $\sup_{m>0}(\sum_k \mathcal{D}^*(m_k)^r)^{1/r} < \infty$ and we have proved (2.12).

Let $\sigma$ be the normalizing function defined above, then

(2.13)
$$\begin{aligned}
&\left(\int_0^\infty \int_{a(t)}^{b(t)} \left(\int_{a(t)}^y w\right)^{r/p'} \left(\int_y^{b(t)} v_b\right)^{r/p} v_b(y)\sigma(t)\,dy\,dt\right)^{1/r}\\
&=\left(\sum_k \int_{M_k}^{M_{k+1}}\int_{a(t)}^{b(t)} \left(\int_{a(t)}^y w\right)^{r/p'}\left(\int_y^{b(t)} v_b\right)^{r/p} v_b(y)\,dy \left(\frac{d}{dt}(b^{-1}\circ a)^k(t)\right)\,dt\right)^{1/r}\\
&=\left(\sum_k \int_{b^{-1}(1)}^{a^{-1}(1)} \int_{a_k}^{b_k} \left(\int_{a_k}^y w\right)^{r/p'} \left(\int_y^{b_k} v_b\right)^{r/p} v_b(y)\,dy\,dm\right)^{1/r}\\
&=\left(\int_{b^{-1}(1)}^{a^{-1}(1)} \sum_k D(m_k)^r\,dm\right)^{1/r}\\
&\le(a^{-1}(1)-b^{-1}(1))^{1/r}\sup_{m>0}\left(\sum_k D(m_k)^r\right)^{1/r} < \infty.
\end{aligned}$$

Here we made the change of variable $t=(a^{-1}\circ b)^k(m)=m_k$ so that when $t=M_{k+1}$ we have $(a^{-1}\circ b)^k(m)=(a^{-1}\circ b)^{k+1}(M_0)$ which implies $m=a^{-1}(1)$ and when $t=M_k$ we have $(a^{-1}\circ b)^k(m)=(a^{-1}\circ b)^k(M_0)$ which implies $m=b^{-1}(1)$. In the same way, using $\sup_{m>0}(\sum_k \mathcal{D}^*(m_k)^r)^{1/r}<\infty$ one shows that

$$\left(\int_0^\infty \int_{a(t)}^{b(t)} \left(\int_y^{b(t)} w\right)^{r/p'} \left(\int_{a(t)}^y v_a\right)^{r/p} v_a(y)\sigma(t)\,dy\,dt\right)^{1/r} < \infty. \tag{2.14}$$

The changes of variable $y=b(x)$ in (2.13) and $y=a(x)$ in (2.14) yield (2.10) and (2.11), respectively.

(Sufficiency.) To prove sufficiency, we first show that for some $m>0$, both $\mathcal{D}(m)$ and $\mathcal{D}^*(m)$ are finite. Since (2.10) and (2.11) are satisfied therefore (2.13) and 2.14 are finite. As we have just seen this means that

$$\left(\int_{b^{-1}(1)}^{a^{-1}(1)} \mathcal{D}(m)^r\,dm\right)^{1/r} \quad\text{and}\quad \left(\int_{b^{-1}(1)}^{a^{-1}(1)} \mathcal{D}^*(m)^r\,dm\right)^{1/r}$$

are finite. Thus $\mathcal{D}(m)$ and $\mathcal{D}^*(m)$ are finite almost everywhere in $(b^{-1}(1), a^{-1}(1))$ and so there is an $m \in (b^{-1}(1), a^{-1}(1))$ where both $\mathcal{D}(m)$ and $\mathcal{D}^*(m)$ are finite.

Next construct $\{m_k\}$ from this $m$. If $m_k < x < m_{k+1}$ then $a(x) < a(m_{k+1}) = b(m_k) < b(x)$ so (with $a_k = a(m_k)$ and $b_k = b(m_k)$)

$$\begin{aligned}
&\int_0^\infty \left(\int_{a(x)}^{b(x)} fw\right)^q v(x)\,dx = \sum_{k\in\mathbf{Z}} \int_{m_k}^{m_{k+1}} \left(\int_{a(x)}^{b_k} fw + \int_{a_{k+1}}^{b(x)} fw\right)^q v(x)\,dx \\
\leq& \tilde{C}\left(\sum_k \int_{m_k}^{m_{k+1}} \left(\int_{a(x)}^{b_k} fw\right)^q v(x)\,dx + \sum_k \int_{m_k}^{m_{k+1}} \left(\int_{a_{k+1}}^{b(x)} fw\right)^q v(x)\,dx\right) \\
\equiv& \tilde{C}\left(S_1 + S_2\right),
\end{aligned}$$

where $\tilde{C} = \max(1, 2^q - 1)$. In $S_1$ make the change of variable $y = a(x)$, apply Proposition 2.4 and then Hölders inequality with indices $r/q$ and $p/q$ to get

$$\begin{aligned}
S_1 =& \sum_k \int_{a_k}^{b_k} \left(\int_y^{b_k} fw\right)^q v_a(y)\,dy \\
\leq& \sum_k C^*(m_k)^q \left(\int_{a_k}^{b_k} f^p w\right)^{q/p} \\
\leq& \left(\sum_k C^*(m_k)^r\right)^{q/r} \left(\sum_k \int_{a_k}^{b_k} f^p w\right)^{q/p} \\
\leq& C_1^q \left(\sum_k D^*(m_k)^r\right)^{q/r} \left(\int_0^\infty f^p w\right)^{q/p} = C_1^q \mathcal{D}^*(m)^q \left(\int_0^\infty f^p w\right)^{q/p}.
\end{aligned}$$

Here $C_1$ is the constant from Proposition 2.4.

In $S_2$ make the change of variable $y = b(x)$, apply Proposition 2.4 and then

Hölder's inequality to get

$$
\begin{aligned}
S_2 =& \sum_k \int_{a_{k+1}}^{b_{k+1}} \left( \int_{a_{k+1}}^{y} fw \right)^q v_b(y)\, dy \\
\le& \sum_k C(m_{k+1})^q \left( \int_{a_{k+1}}^{b_{k+1}} f^p w \right)^{q/p} \\
\le& \left( \sum_k C(m_{k+1})^r \right)^{q/r} \left( \sum_k \int_{a_{k+1}}^{b_{k+1}} f^p w \right)^{q/p} \\
\le& C_1^q \left( \sum_k D(m_{k+1})^r \right)^{q/r} \left( \int_0^\infty f^p w \right)^{q/p} = C_1^q \mathcal{D}(m)^q \left( \int_0^\infty f^p w \right)^{q/p}.
\end{aligned}
$$

Therefore

$$
\left( \int_0^\infty \left( \int_{a(x)}^{b(x)} fw \right)^q v(x)\, dx \right)^{1/q} \le \tilde{C}^{1/q} C_1 \left( \mathcal{D}(m)^q + \mathcal{D}^*(m)^q \right)^{1/q} \left( \int_0^\infty f^p w \right)^{1/p}.
$$

This completes the proof of the theorem.

**Corollary 2.6.** *Let $u$ and $v$ be weight functions and $A$ and $B$ be real numbers such that $0 < A < B$. Then there is a constant $C > 0$ such that*

$$
\left( \int_0^\infty \left( \int_{Ax}^{Bx} f \right)^q v(x)\, dx \right)^{1/q} \le C \left( \int_0^\infty f^p u \right)^{1/p} \tag{2.15}
$$

*for all $f \ge 0$ if and only if*

*i) for $1 < p \le q < \infty$,*

$$
K \equiv \sup_{t \le x \le Bt/A} \left( \int_{Ax}^{Bt} u^{1-p'} \right)^{1/p'} \left( \int_t^x v \right)^{1/q} < \infty;
$$

*ii) for $0 < q < p$, $1 < p < \infty$, $\max(K_1, K_2) < \infty$ where*

$$
K_1 = \left( \int_0^\infty \frac{1}{t} \int_{At/B}^{t} \left( \int_{At}^{Bx} u^{1-p'} \right)^{r/p'} \left( \int_x^t v \right)^{r/p} v(x)\, dx\, dt \right)^{1/r}
$$

*and*

$$K_2 = \left( \int_0^\infty \frac{1}{t} \int_t^{Bt/A} \left( \int_{Ax}^{Bt} u^{1-p'} \right)^{r/p'} \left( \int_t^x v \right)^{r/p} v(x)\, dx\, dt \right)^{1/r},$$

*where* $1/r = 1/q - 1/p$.

*Moreover, if* $C$ *is the best constant in* (2.15) *then for* $1 < p \leq q < \infty$, $K \leq C \leq 2p^{1/q}(p')^{1/p'} K$ *and in the case* $0 < q < p$, $1 < p < \infty$, $\max(K_1, K_2) \approx C$.

*Proof.* Let $a(x) = Ax$ and $b(x) = Bx$ in Theorem 2.2 and $(i)$ follows. With the same choice of $a$ and $b$ in Theorem 2.5 it is easy to see that $(b^{-1} \circ a)^k(t) = (A/B)^k t$ so $\frac{d}{dt}(b^{-1} \circ a)^k(t) = (A/B)^k$. Now with $M_0 = 1/B$ we obtain $M_k = B^{k-1}/A^k$ for all $k \in \mathbf{Z}$. If $M_k < t < M_{k+1}$ the normalizing function $\sigma$ satisfies

$$1/B = M_k(A/B)^k < t\sigma(t) < M_{k+1}(A/B)^k = 1/A$$

for all $k$. Hence $\sigma(t) \approx 1/t$ and substituting this into (2.10) and (2.11) we get (ii).

The estimate of $C$ in (2.15) in terms of $\max(K_1, K_2)$ follows on tracing the proof of Theorem 2.5.

The next corollary involves the Steklov operator $SF(x) = \int_{x-1}^{x+1} F$, $F \geq 0$.

**Corollary 2.7.** *Let* $U$ *and* $V$ *be weight functions on* $\mathbf{R}$. *Then there is a constant* $C > 0$ *such that*

$$\left( \int_{-\infty}^\infty \left( \int_{y-1}^{y+1} F(s)\, ds \right)^q V(y)\, dy \right)^{1/q} \leq C \left( \int_{-\infty}^\infty F(s)^p U(s)\, ds \right)^{1/p} \tag{2.16}$$

*for all* $F \geq 0$ *if and only if*

*i) for* $1 < p \le q < \infty$*,*

$$\sup_{s \le y \le s+2} \left( \int_{y-1}^{s+1} U^{1-p'} \right)^{1/p'} \left( \int_s^y V \right)^{1/q} < \infty;$$

*ii) for* $0 < q < p$*,* $1 < p < \infty$*,* $\max(K_1, K_2) < \infty$ *where*

$$K_1 = \left( \int_{-\infty}^{\infty} \int_{s-2}^{s} \left( \int_{s-1}^{y+1} U^{1-p'} \right)^{r/p'} \left( \int_y^s V \right)^{r/p} V(y)\, dy\, ds \right)^{1/r}$$

*and*

$$K_2 = \left( \int_{-\infty}^{\infty} \int_{s}^{s+2} \left( \int_{y-1}^{s+1} U^{1-p'} \right)^{r/p'} \left( \int_s^y V \right)^{r/p} V(y)\, dy\, ds \right)^{1/r},$$

*where* $1/r = 1/q - 1/p$*.*

*Proof.* Take $A = 1/e$, $B = e$, $v(x) = (1/x)V(\log(x))$, and $u(t) = t^{p-1}U(\log(t))$ in Corollary 2.6 and make the substitutions $x = e^y$ and $t = e^s$. Since $F$ is a non-negative function on $\mathbf{R}$ if and only if $f(t) = (1/t)F(\log(t))$ is a non-negative function on $(0, \infty)$ the result follows.

*Remark 2.8.* In the case $1 < p \le q < \infty$ the result of Corollary 2.6 was obtained by Sawyer [12] while the result of Corollary 2.7 for $1 < p, q < \infty$ was given by Batuev and Stepanov ([2, Theorems 2.1 and 2.2]) with somewhat different (but equivalent) weight conditions.

## 3. Monotone functions and a weighted Hardy type inequality

Let $a$ and $b$ be as before. We now consider the operator of (1.1) where $f$ is monotone on $(0, \infty)$. The next result considers the case when $f$ is non-increasing.

**Theorem 3.1.** *Suppose $u$ and $v$ are weight functions with $\int_0^\infty u = \infty$. Then* (2.2) *is satisfied for all non-negative, non-increasing $f$ whenever*

$$\sup_{x>0}\left(\int_0^{a(x)} u\right)^{-1/p}\left(\int_0^x [b(s)-a(s)]^q v(s)\,ds\right)^{1/q} < \infty, \tag{3.1}$$

*if $1 < p \le q < \infty$, and*

$$\int_0^\infty \left(\int_0^x [b(s)-a(s)]^q v(s)\,ds\right)^{r/q}\left(\int_0^{a(x)} u\right)^{-r/q} u(a(x))a'(x)\,dx < \infty, \tag{3.2}$$

*if $1 < q < p < \infty$, $1/r = 1/q - 1/p$.*

*Proof.* It is well known ([11],[5],[14]) that for non-increasing $f$ (2.2) is equivalent to the inequality

$$\left(\int_0^\infty \left(\int_0^x T^* g\right)^{p'}\left(\int_0^x u\right)^{-p'} u(x)\,dx\right)^{1/p'} \le C\left(\int_0^\infty g^{q'}v^{1-q'}\right)^{1/q'}, \tag{3.3}$$

where $T^*$ is the adjoint of $T$, and $g \ge 0$ is arbitrary.

But since

$$(T^*g)(t) = \int_{b^{-1}(t)}^{a^{-1}(t)} g$$

we have

$$\begin{aligned}
\int_0^x (T^*g)(t)\,dt &= \int_0^x \left(\int_{b^{-1}(t)}^{a^{-1}(t)} g(s)\,ds\right) dt \\
&= \int_0^{a^{-1}(x)} \int_{b^{-1}(a(y))}^{y} g(s)\,ds a'(y)\,dy \\
&\le \int_0^{a^{-1}(x)} g(s) \int_s^{a^{-1}(b(s))} a'(y)\,dy\,ds \\
&= \int_0^{a^{-1}(x)} [b(s)-a(s)]g(s)\,ds.
\end{aligned}$$

Therefore, after the change of variable $x = a(t)$, we see that the validity of

$$\left(\int_0^\infty \left[\int_0^t [b(s)-a(s)]g(s)\,ds\right]^{p'} \left(\int_0^{a(t)} u\right)^{-p'} u(a(t))a'(t)\,dt\right)^{1/p'} \le C\left(\int_0^\infty g^{q'}v^{1-q'}\right)^{1/q'}. \tag{3.4}$$

for all non-negative $g$ is sufficient to imply (3.3) for all non-negative $g$ and hence (2.2) for all non-increasing $f$. But (3.4) is a weighted Hardy inequality which holds ([3],[10]) if and only if for $1 < p \le q < \infty$

$$\sup_{x>0}\left(\int_x^\infty \left(\int_0^{a(t)} u\right)^{-p'} u(a(t))a'(t)\,dt\right)^{1/p'} \left(\int_0^x [b(s)-a(s)]^q v(s)\,ds\right)^{1/q}$$

is finite, and for $1 < q < p < \infty$ ([10],[13, Theorem 2.5])

$$\left(\int_0^\infty \left(\int_0^x [b(s)-a(s)]^q v(s)\,ds\right)^{r/q} \left(\int_x^\infty \left(\int_0^{a(t)} u\right)^{-p'} u(a(t))a'(t)\,dt\right)^{r/q'} \times \left(\int_0^{a(x)} u\right)^{-p'} u(a(x))a'(x)\,dx\right)^{1/r}$$

is finite. Since integration yields

$$\int_x^\infty \left(\int_0^{a(t)} u\right)^{-p'} u(a(t))a'(t)\,dt = (p'-1)^{-1}\left(\int_0^{a(x)} u\right)^{1-p'},$$

and $r(1-p')/q' - p' = -r/q$, these conditions are (3.1) and (3.2).

A result corresponding to Theorem 3.1 for non-negative, non-decreasing functions follows at once by imitating the proof of Theorem 3.1.

**Proposition 3.2.** *Suppose $u$ and $v$ are weight functions with $\int_0^\infty u = \infty$. Then inequality* (2.2) *holds for all non-negative, non-decreasing $f$ whenever*

$$\sup_{x>0}\left(\int_{b(x)}^\infty u\right)^{-1/p} \left(\int_x^\infty [b(s)-a(s)]^q v(s)\,ds\right)^{1/q} < \infty,$$

*if* $1 < p \le q < \infty$, *and*

$$\int_0^\infty \left(\int_x^\infty [b(s)-a(s)]^q v(s)\,ds\right)^{r/q} \left(\int_{b(x)}^\infty u\right)^{-r/q} u(b(x))b'(x)\,dx < \infty,$$

*if* $1 < q < p < \infty$, $1/r = 1/q - 1/p$.

In order to give a weighted generalization of (1.7) we require the following result.

**Theorem 3.3.** *Let* $u$ *and* $v$ *be weight functions and* $1 < p \le q < \infty$. *If*

$$\left(\int_0^1 \sup_{0<a<b<a/t} \left[\int_b^{a/t} w(t,x) \left(\int_a^b u^{1-p'}\right)^{q/p'} dx\right] dt\right)^{1/q} = K < \infty \tag{3.6}$$

*then*

$$\left(\int_0^1 \int_0^\infty w(t,x) \left(\int_{xt}^x g\right)^q dx\,dt\right)^{1/q} \le C \left(\int_0^\infty u(x)g(x)^p\,dx\right)^{1/p} \tag{3.7}$$

*is satisfied for all* $g \ge 0$.

*Conversely, if* (3.7) *holds for all* $g \ge 0$ *then*

$$\sup_{0<a<b<\infty} \left(\int_0^{a/b} \int_b^{a/t} w(t,x)\,dx\,dt\right)^{1/q} \left(\int_a^b u^{1-p'}\right)^{1/p'} < \infty. \tag{3.8}$$

*Proof.* For each $t \in (0,1)$ we apply Corollary 2.6 with $B = 1$, $A = t$, and $v(x) = w(t,x)$ to get

$$\begin{aligned}
&\left(\int_0^1 \left(\int_0^\infty w(t,x) \left(\int_{xt}^x g\right)^q dx\right) dt\right)^{1/q} \\
\le& 2p^{1/q}(p')^{1/p'} \left(\int_0^\infty g^p u\right)^{1/p} \times \\
&\left(\int_0^1 \sup_{0<a<b<a/t} \left[\int_b^{a/t} w(t,x) \left(\int_a^b u^{1-p'}\right)^{q/p'} dx\right]^q dt\right)^{1/q} \\
=& 2Kp^{1/q}(p')^{1/p'} \left(\int_0^\infty g^p u\right)^{1/p},
\end{aligned}$$

so (3.7) follows.

Conversely, if (3.7) is satisfied for all $g \geq 0$, fix $0 < a < b$ and define $g(x) = \chi_{(a,b)}(x)u(x)^{1-p'}$. Then

$$\begin{aligned} C\left(\int_a^b u^{1-p'}\right)^{1/p} =& C\left(\int_0^\infty u g^p\right)^{1/p} \\ &\geq \left(\int_0^1\int_0^\infty w(t,x)\left(\int_{xt}^x g\right)^q dx\,dt\right)^{1/q} \\ &\geq \left(\int_0^{a/b}\int_b^{a/t} w(t,x)\left(\int_{xt}^x u(s)^{1-p'}\chi_{(a,b)}(s)\,ds\right)^q dx\,dt\right)^{1/q} \\ &= \left(\int_0^{a/b}\int_b^{a/t} w(t,x)\,dx\,dt\right)^{1/q}\left(\int_a^b u^{1-p'}\right), \end{aligned}$$

since $b < x < a/t$ and hence $(a,b) \subset (xt,x)$. The result follows on dividing by $\left(\int_a^b u^{1-p'}\right)^{1/p}$ and taking the supremum over $a < b$.

The next two examples show that the necessary condition of Theorem 3.3 is not sufficient and the sufficient condition is not necessary. The problem of characterizing the weights for which (3.7) holds remains open.

*Example 3.4a.* Let $p = q = 2$, $u(x) = x$, and $w(t,x) = (1/xt)[\log(1/t)]^{-3}$ for $0 < t < 1$ and $x > 0$. The necessary condition (3.8) of Theorem 3.3 holds because

$$\begin{aligned} &\sup_{0<a<b}\left(\int_0^{a/b}\int_b^{a/t} w(t,s)\,ds\,dt\right)\left(\int_a^b u^{-1}\right) \\ =& \sup_{0<a<b}\left(\int_0^{a/b}\frac{1}{t}[\log(1/t)]^{-3}\log(a/bt)\,dt\right)\log(b/a) \\ \leq& \sup_{0<a<b}\left(\int_0^{a/b}\frac{1}{t}[\log(1/t)]^{-2}\,dt\right)\log(b/a) = 1. \end{aligned}$$

However, the inequality (3.7) fails. To see this set $g(s) = s^{1/2}\chi_{(0,1)}$ and notice that the right hand side of (3.7) is finite. The left hand side (squared) becomes

$$\begin{aligned}
&\int_0^1 \frac{1}{t}[\log(1/t)]^{-3} \int_0^\infty \frac{1}{x} \left( \int_{xt}^x g(s)\, ds \right)^2 dx\, dt \\
\geq &\int_0^1 \frac{1}{t}[\log(1/t)]^{-3} \int_0^1 \frac{1}{x} \left( \int_{xt}^x s^{1/2}\, ds \right)^2 dx\, dt \\
= &\frac{4}{27} \left( \int_0^1 \frac{1}{t}[\log(1/t)]^{-3} \left(1 - t^{3/2}\right)^2 dt \right).
\end{aligned}$$

This integral diverges because the integrand behaves like $1/(1-t)$ near $t = 1$. For this choice of $g$ the right hand side is finite and the left hand side is infinite so inequality (3.7) fails.

*Example 3.4b.* Let $p = q = 2$, $u(x) \equiv 1$, and $w(t,x) = \chi_{(1/(x+1),1/x)}(t)$. The inequality (3.7) follows from the classical Hardy inequality:

$$\begin{aligned}
&\left( \int_0^1 \int_0^\infty w(t,x) \left( \int_{xt}^x g \right)^2 dx\, dt \right)^{1/2} \\
\leq &\left( \int_0^\infty \int_0^1 \chi_{(1/(x+1),1/x)}(t)\, dt \left( \int_0^x g \right)^2 dx \right)^{1/2} \\
\leq &\left( \int_0^\infty \left( \frac{1}{x} - \frac{1}{x+1} \right) \left( \int_0^x g \right)^2 dx \right)^{1/2} \leq \left( \int_0^\infty \left( \frac{1}{x} \int_0^x g \right)^2 dx \right)^{1/2} \\
\leq &2 \left( \int_0^\infty g^2 \right)^{1/2}.
\end{aligned}$$

On the other hand, the sufficient condition (3.6) fails in this case. If $t < 1/2$, then with $a = 1$ and $b = 1/t - 1 > 1$ we have

$$\begin{aligned}
&\sup_{a<b<a/t} \int_b^{a/t} w(t,x) \left( \int_a^b u^{-1} \right) dx \\
\geq &\int_{1/t-1}^{1/t} \chi_{(1/(x+1),1/x)}(t) \left( \int_1^{1/t-1} ds \right) dx = 1/t - 2.
\end{aligned}$$

With this estimate we see that (3.6) is not less than

$$\left(\int_0^{1/2} (1/t-2)\,dt\right)^{1/2} = \infty.$$

Since (3.7) holds but (3.6) fails, the sufficient condition (3.6) is not necessary.

**Corollary 3.5.** *Let $u$ and $v$ be weights and $1<p\le q<\infty$. Then for $f\in C^1(\mathbf{R}^+)$*

$$\left(\int_0^\infty\int_0^\infty \frac{|f(x)-f(y)|^q}{v(|x-y|)}\,dx\,dy\right)^{1/q} \le C\left(\int_0^\infty |f'(x)|^p u(x)\,dx\right)^{1/p} \tag{3.9}$$

*whenever*

$$\left(\int_0^1 \sup_{0<a<b<a/t}\left(\int_b^{a/t}\frac{x}{v(x(1-t))}\,dx\right)\left(\int_a^b u^{1-p'}\right)^{q/p'} dt\right)^{1/q} < \infty. \tag{3.10}$$

*Proof.* An interchange of the order of integration and two changes of variable show that

$$\begin{aligned}
&\left(\int_0^\infty\int_0^\infty \frac{|f(x)-f(y)|^q}{v(|x-y|)}\,dx\,dy\right)^{1/q}\\
=&\left(\int_0^\infty\int_0^x \frac{|f(x)-f(y)|^q}{v(|x-y|)}\,dy\,dx + \int_0^\infty\int_x^\infty \frac{|f(x)-f(y)|^q}{v(|x-y|)}\,dy\,dx\right)^{1/q}\\
=&\left(2\int_0^\infty\int_0^x \frac{|f(x)-f(y)|^q}{v(|x-y|)}\,dy\,dx\right)^{1/q}\\
=&2^{1/q}\left(\int_0^1\int_0^\infty \frac{x|f(x)-f(xt)|^q}{v(|x-xt|)}\,dx\,dt\right)^{1/q}\\
\le&2^{1/q}\left(\int_0^1\int_0^\infty \frac{x}{v(x(1-t))}\left(\int_{xt}^x |f'(s)|\,ds\right)^q dx\,dt\right)^{1/q}.
\end{aligned}$$

The result now follows from Theorem 3.3 with $w(t,x)=x/v(x(1-t))$.

Observe that this result extends to all weight functions $v(x,y)$ satisfying $v(x,y)=v(y,x)$.

We now derive from Corollary 3.5 the inequality (1.7) given in the introduction. That is, we show that (3.9) holds with $q = p$, $v(x) = x^{1+\lambda p}$, and $u(x) = x^{(1-\lambda)p}$ for $\lambda \in (0,1)$.

It follows from Corollary 3.5 that (1.7) is satisfied if

(3.11)
$$\int_0^1 \sup_{a<b<a/t} \left(\int_b^{a/t} x^{-\lambda p}(1-t)^{-1-\lambda p}\,dx\right)\left(\int_a^b x^{(1-\lambda)p(1-p')}\,dx\right)^{p-1} dt < \infty.$$

Now

$$\left(\int_b^{a/t} x^{-\lambda p}\,dx\right)\left(\int_a^b x^{-p'(1-\lambda)}\,dx\right)^{p-1}$$
$$= \begin{cases} \frac{(p-1)^{p-1}}{|1-\lambda p|^p}\left|(a/bt)^{-\lambda p+1}-1\right|\left|1-(a/b)^{-p'(1-\lambda)+1}\right|^{p-1} & \text{if } \lambda \neq 1/p \\ \log(a/bt)[\log(b/a)]^{p-1} & \text{if } \lambda = 1/p \end{cases}$$

Let $s = a/b$, then $t < s < 1$ and we must find the maximum of

$$g(s) = \begin{cases} \left(1-(s/t)^{1-\lambda p}\right)\left(1-s^{(\lambda p-1)/(p-1)}\right)^{p-1} & \text{if } \lambda > 1/p \\ \left((s/t)^{1-\lambda p}-1\right)\left(s^{(\lambda p-1)/(p-1)}-1\right)^{p-1} & \text{if } \lambda < 1/p \\ \log(s/t)[\log(1/s)]^{p-1} & \text{if } \lambda = 1/p. \end{cases}$$

If $\lambda > 1/p$

$$\frac{d}{ds}\log(g(s)) = \frac{-(1-\lambda p)t^{\lambda p-1}s^{-\lambda p}}{1-(t/s)^{\lambda p-1}} + \frac{(1-p)\frac{\lambda p-1}{p-1}s^{(\lambda p-p)/(p-1)}}{1-s^{(\lambda p-1)/(p-1)}},$$

and this is zero when $s = t^{1/p'}$. Similarly, we see that the maximum of $g(s)$ occurs at $s = t^{1/p'}$ when $\lambda \le 1/p$. But

$$g(t^{1/p'}) = \begin{cases} \left|1-t^{\lambda-1/p}\right|^p & \text{if } \lambda \neq 1/p \\ \frac{1}{p(p')^{p-1}}[\log(1/t)]^p & \text{if } \lambda = 1/p. \end{cases}$$

If $\lambda > 1/p$, (3.11) takes the form

$$\int_0^1 (1-t)^{-1-\lambda p}\left(1-t^{\lambda-1/p}\right)^p dt,$$

and since $\lim_{t\to 1}(1-t^\gamma)/(1-t) = \gamma$, the integral converges if $-\lambda p + p > 0$, ie. $\lambda < 1$.

If $\lambda < 1/p$, (3.11) takes the form

$$\int_0^1 (1-t)^{-1-\lambda p} t^{\lambda p - 1} \left(1 - t^{1/p-\lambda}\right)^p \, dt,$$

and this integral converges if $\lambda > 0$ and $\lambda < 1$.

Finally, if $\lambda = 1/p$ then (3.11) takes the form $(t = e^{-y})$

$$\int_0^1 (1-t)^{-2}[\log(1/t)]^p \, dt = \int_0^\infty \frac{y^p e^{-y}}{(1-e^{-y})^2} \, dy.$$

This integral converges at $\infty$ and since $\lim_{y\to 0} y/(1-e^{-y}) = 1$, the integrand behaves like $y^{p-2}$ near 0 so the integral is finite.

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