REFINING THE HÖLDER AND MINKOWSKI INEQUALITIES

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February 28, 2000

Abstract. Refinements to the usual Hölder and Minkowski inequalities in the Lebesgue spaces $L^p_\nu$ are proved. Both are inequalities for non-negative functions and both reduce to equality in $L^2_\nu$.

1. Introduction and Main Results

The Hölder and Minkowski inequalities are fundamental to the theory of Lebesgue spaces. If $1 < p < \infty$ and $1/p + 1/p' = 1$ the first,

$$\int fg \, d\nu \leq \left( \int |f|^p \, d\nu \right)^{1/p} \left( \int |g|^{p'} \, d\nu \right)^{1/p'},$$

expresses the fact that functions in $L^p_\nu$ give rise to bounded linear functionals on $L^{p'}_\nu$. It is a sharp inequality in the sense that for any $f \in L^p_\nu$ there is a function $g \in L^{p'}_\nu$ such that the inequality becomes equality. For this reason, improvements to Hölder’s inequality must necessarily be quite delicate.

Theorem 1.1. Let $p \geq 2$ and define $p'$ by $1/p + 1/p' = 1$. Then for any two non-negative $\nu$-measurable functions $f$ and $g$

$$\int fg \, d\nu \leq \left( \int f^p \, d\nu - \int |f - g^{p'-1} f g^{p'} \, d\nu |^p \, d\nu \right)^{1/p} \left( \int g^{p'} \, d\nu \right)^{1/p'}. $$

In the case $1 < p \leq 2$ our refinement takes the form of a lower bound.

1991 Mathematics Subject Classification. Primary 26D15.
Key words and phrases. Inequalities, Hölder’s inequality, Minkowski’s inequality.

This work was supported in part by an NSERC grant.
Theorem 1.2. Let \( p \leq 2 \) and define \( p' \) by \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then for any two non-negative \( \nu \)-measurable functions \( f \) and \( g \)

\[
\left( \int f^p \, d\nu - \int |f - g|^{p'-1} f g \, d\nu \right)^{1/p} \leq \int g^p \, d\nu.
\]

The Minkowski inequality is the triangle inequality in \( L^p_\nu \): If \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \) then

\[
\left( \int |f + g|^p \, d\nu \right)^{1/p} \leq \left( \int |f|^p \, d\nu \right)^{1/p} + \left( \int |g|^p \, d\nu \right)^{1/p}.
\]

There can only be improvement in this inequality when \( f \) and \( g \) are not multiples of one another.

Theorem 1.3. Let \( p \geq 2 \) and define \( p' \) by \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then for any two non-negative \( \nu \)-measurable functions \( f \) and \( g \)

\[
\left( \int (f + g)^p \, d\nu \right)^{1/p} \leq \left( \int f^p \, d\nu - \int h^p \, d\nu \right)^{1/p} + \left( \int g^p \, d\nu - \int h^p \, d\nu \right)^{1/p}
\]

where \( h = \int |f| \, d\nu \int g (f + g)^{p-1} \, d\nu - g \int |f| (f + g)^{p-1} \, d\nu \) / \( \int (f + g)^p \, d\nu \).

Notice that the function \( h \) vanishes when \( f \) is a multiple of \( g \). Again we get a lower bound in the case \( 1 < p \leq 2 \).

Theorem 1.4. Let \( 1 < p \leq 2 \) and define \( p' \) by \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then for any two non-negative \( \nu \)-measurable functions \( f \) and \( g \)

\[
\left( \int f^p \, d\nu - \int h^p \, d\nu \right)^{1/p} + \left( \int g^p \, d\nu - \int h^p \, d\nu \right)^{1/p} \leq \left( \int (f + g)^p \, d\nu \right)^{1/p}
\]

where \( h = \int |f| \, d\nu \int g (f + g)^{p-1} \, d\nu - g \int |f| (f + g)^{p-1} \, d\nu \) / \( \int (f + g)^p \, d\nu \).

It is easy to verify directly that the inequalities given above reduce to equalities when \( p = 2 \).

The proofs of Theorems 1.1–1.4 will be given in the next section. They depend on a special case of the key inequality established in Theorem 2.3. Also in the next section we give examples to show that the inequalities may fail if the hypothesis of non-negativity is dropped.

We assume throughout that \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). Also, \( \nu \) will denote an arbitrary \( \sigma \)-finite measure while \( \mu \) will denote a probability measure, that is, a measure with total measure one. The function \( \text{sgn}(x) \) is defined to be 1 when \( x > 0 \), 0 when \( x = 0 \), and \(-1\) when \( x < 0 \).
2. The Key Inequality

The power function $x \mapsto x^\alpha$, $x > 0$, is convex when $\alpha > 1$ and concave when $0 < \alpha < 1$. We will use this fact in the following form. If $a$ and $b$ are non-negative real numbers then

\[(a + b)^\alpha \geq a^\alpha + b^\alpha \text{ when } \alpha > 1 \quad \text{and} \quad (a + b)^\alpha \leq a^\alpha + b^\alpha \text{ when } 0 < \alpha < 1.\]

Equality holds only if $\alpha = 1$, $a = 0$, or $b = 0$.

**Lemma 2.1.** Suppose $1 < p \neq 2$ and $t > 0$. If $x > 0$, $y > t$ and

\[x^{p-1} - |x-t|^{p-1}\text{sgn}(x-t) = y^{p-1} - |y-t|^{p-1}\text{sgn}(y-t)\]

then $x = y$.

**Proof.** Let $\varphi(x) = x^{p-1} - |x-t|^{p-1}\text{sgn}(x-t)$. Since $y > t$ we have $\varphi(y) = y^{p-1} - (y-t)^{p-1}$. Inequality (2.1) shows that $\varphi(y) > t^{p-1}$ when $p > 2$ and $\varphi(y) < t^{p-1}$ when $p < 2$.

If $x \leq t$ then $\varphi(x) = x^{p-1} + (t-x)^{p-1}$ so (2.1) yields $\varphi(x) \leq t^{p-1}$ when $p > 2$ and $\varphi(x) \geq t^{p-1}$ when $p < 2$. This contradicts the hypothesis $\varphi(x) = \varphi(y)$ so we must have $x > t$. Notice that for $x > t$, $\varphi'(x) = (p-1)x^{p-2} - (p-1)(x-t)^{p-2}$ does not change sign. Hence $\varphi$ is monotone and therefore one-to-one on $(t, \infty)$. We conclude that $x = y$ as required.

We begin by proving a discrete version of our key inequality.

**Theorem 2.2.** Suppose $p > 2$, $n$ is a positive integer, $x_1, x_2, \ldots, x_n$ are non-negative, and $0 < t \leq \frac{1}{n} \sum_{j=1}^{n} x_j$. Then

\[
\frac{1}{n} \sum_{j=1}^{n} x_j^p \geq \frac{2}{nt} \left( \sum_{j=1}^{n} x_j - 1 \right) + \frac{1}{n} \sum_{j=1}^{n} |x_j - t|^p.
\]

The reverse inequality holds when $1 < p < 2$.

**Proof.** Let

\[M_n = \sum_{j=1}^{n} x_j^p - t^p \left( \sum_{j=1}^{n} x_j - n \right) - \sum_{j=1}^{n} |x_j - t|^p.
\]

We will show by induction that $M_n$ is non-negative when $p > 2$. If $n = 1$, and $0 < t \leq x = x_1$ then $M_1 = x^p - t^p(2x/t - 1) - (x-t)^p$. Fix $t$ and consider $M_1$ as a function of $x$. At $x = t$, the function vanishes and for $x \geq t$ its derivative is $px^{p-1} - 2t^{p-1} - p(x-t)^{p-1}$ which is not less than $px^{p-1} - pt^{p-1} - p(x-t)^{p-1} \geq 0$ by (2.1). It follows that $M_1$ is non-negative for $x \geq t$.

Suppose now that for some $n > 1$, $M_{n-1} \geq 0$. To show that $M_n \geq 0$ we fix $t$ and show that for all $x \geq t$, $M_n$ is non-negative on the compact set

\[K_x = \{(x_1, x_2, \ldots, x_n) \in [0, \infty)^n : \sum_{j=1}^{n} x_j = nx \}.
\]
First we show that $M_n$ is non-negative on the boundary of $K_x$ considered as a subset of the hyperplane defined by $\sum_{j=1}^n x_j = nx$. That is, that $M_n \geq 0$ when at least one of $x_1, x_2, \ldots, x_n$ is zero. By symmetry we may assume that $x_n = 0$. We have

$$0 < t \leq x = \frac{1}{n} \sum_{j=1}^{n-1} x_j \leq \frac{1}{n-1} \sum_{j=1}^{n-1} x_j$$

and so, by the inductive hypothesis,

$$M_n = \sum_{j=1}^{n-1} x_j^p - t^p \left( \frac{2}{7} \sum_{j=1}^{n-1} x_j - n \right) - \sum_{j=1}^{n-1} |x_j - t|^p - t^p = M_{n-1} \geq 0.$$

To complete the proof we use a Lagrange Multiplier argument to show that if the minimum value of $M_n$ occurs in the interior of $K_x$ (considered as a subset of the hyperplane) then it is non-negative. Note that since $p > 1$, $M_n$ has continuous first partial derivatives with respect to each of $x_1, x_2, \ldots, x_n$. Thus it will suffice to show that the value of $M_n$ is non-negative at critical points of

$$M_n - \lambda \left( \sum_{j=1}^n x_j - nx \right),$$

considered as a function of $x_1, x_2, \ldots, x_n, \lambda$ with $x$ and $t$ still fixed. At critical points we have $\sum_{j=1}^n x_j = nx$ and for each $j$

$$px_j^{p-1} - 2t^{p-1} - p|x_j - t|^{p-1} \text{sgn}(x_j - t) - \lambda = 0.$$

It follows that $x_j^{p-1} - |x_j - t|^{p-1} \text{sgn}(x_j - t)$ takes the same value for each $j$. Since $t$ is no greater than the average of $x_1, x_2, \ldots, x_n$, either $x_1 = x_2 = \cdots = x_n = x = t$ or at least one $x_j$ is greater than $t$. In the latter case, Lemma 2.1 applies and we conclude that $x_1 = x_2 = \cdots = x_n = x$. In either case we have

$$M_n = n(x^p - t^p(2x/t - 1) - (x - t)^p)$$

which is non-negative as we have seen in the case $n = 1$. This completes the proof in the case $p > 2$.

The proof that $M_n \leq 0$ in the case $1 < p < 2$ proceeds similarly.

The key inequality is presented next. It is more general than Theorem 2.2 and will readily imply Theorems 1.1–1.4.

**Theorem 2.3.** Suppose $p \geq 2$ and $\mu$ is a probability measure. If $f \geq 0$ is a $\mu$-measurable function then

$$\int f^p \, d\mu \geq t^p \left( \frac{2}{7} \int f \, d\mu - 1 \right) + \int |f - t|^p \, d\mu$$

(2.2)
whenever $0 < t \leq \int f \, d\mu$. The reverse inequality holds when $1 < p \leq 2$.

Proof. It is a simple matter to show that (2) holds with equality when $p = 2$. When $p > 2$ we argue as follows.

If $f$ is not in $L^p_\mu$, then both sides of (2.2) are infinite so there is nothing to prove. Fix $f \in L^p_\mu$, and $t$ with $0 < t < \int f \, d\mu$. Let $f^*$ denote the non-increasing rearrangement of $f$ with respect to $\mu$. We view $f^*$ as a Lebesgue measurable function on $[0,1]$. Since $f$ is non-negative, $f$ and $f^*$ are equimeasurable, $f^p$ and $f^{*p}$ are equimeasurable, and $|f - t|^p$ and $|f^* - t|^p$ are equimeasurable. Thus (2.2) becomes

\[
\int_0^1 f^* p \geq t^p \left( \frac{2}{t} \int_0^1 f^* - 1 \right) + \int_0^1 |f^* - t|^p.
\]

For each positive integer $n$ define the function $f_n$ on $[0,1]$ by

\[
f_n(s) = \sum_{j=1}^n f^*(j/n)\chi((j-1)/n,j/n)(s)
\]

and note that since $f^*$ is non-increasing, $f^*(s + 1/n) \leq f_n(s) \leq f^*(s)$ for $0 < s \leq 1$. Clearly, the sequence $\{f_n\}$ converges to $f^*$ in $L^p[0,1]$. It follows that $\int_0^1 f_n$ converges to $\int_0^1 f^*$ so for sufficiently large $n$ we have $0 < t < \int_0^1 f_n$. By the Lebesgue Dominated Convergence Theorem, (2.3) will follow provided we establish

\[
\int_0^1 f^* p \geq t^p \left( \frac{2}{t} \int_0^1 f_n - 1 \right) + \int_0^1 |f_n - t|^p.
\]

for sufficiently large $n$. If we set $x_j = f^*(j/n)$ then (2.4) becomes

\[
\frac{1}{n} \sum_{j=1}^n x_j^p \geq t^p \left( \frac{2}{nt} \sum_{j=1}^n x_j - 1 \right) + \frac{1}{n} \sum_{j=1}^n |x_j - t|^p
\]

which holds by Theorem 2.2 when $n$ is large enough that $t \leq \int_0^1 f_n$.

This proves the theorem for $p > 2$ in the case $0 \leq t < \int f \, d\mu$. The case $t = \int f \, d\mu$ follows by an easy limiting argument.

The same argument yields the reverse inequality when $1 < p < 2$.

**Corollary 2.4.** Suppose $p \geq 2$, $\mu$ is a probability measure, and $f$ is a non-negative, $\mu$-measurable function. Then

\[
\int f \, d\mu \leq \left( \int f^p \, d\mu - \int |f - \int f \, d\mu|^p \, d\mu \right)^{1/p}
\]

The reverse inequality holds when $1 < p < 2$.

Proof. Take $t = \int f \, d\mu$ in Theorem 2.3, rearrange the result and take $p$-th roots.
Proofs of Theorems 1.1–1.4. To prove Theorems 1.1 and 1.2 we fix non-negative \( \nu \)-measurable functions \( f \) and \( g \) and apply Corollary 2.4 with \( fg^{1-p'} \) in place of \( f \) and \( d\mu = g^p \, d\nu \).

Theorems 1.3 follows from Theorems 1.1 in the same way that Minkowski’s inequality follows from Hölder’s. Fix non-negative \( \nu \)-measurable functions \( f \) and \( g \) and define \( h \) by

\[
h = \left| f \int g(f + g)^{p-1} \, d\nu - g \int f(f + g)^{p-1} \, d\nu \right| / \int (f + g)^p \, d\nu.
\]

Let \( p \geq 2 \) and apply Theorem 1.1 with \( g \) replaced by \((f + g)^{p-1}\) to get

\[
\int f(f + g)^{p-1} \, d\nu \leq \left( \int f^p \, d\nu - \int h^p \, d\nu \right)^{1/p} \left( \int (f + g)^p \, d\nu \right)^{1/p'}.
\]

Interchanging the roles of \( f \) and \( g \) yields

\[
\int g(f + g)^{p-1} \, d\nu \leq \left( \int g^p \, d\nu - \int h^p \, d\nu \right)^{1/p} \left( \int (f + g)^p \, d\nu \right)^{1/p'}.
\]

Adding the last two inequalities gives Theorem 1.3.

Theorem 1.4 follows from Theorem 1.2 by a similar argument.

Example 2.5. The hypothesis that \( f \) be non-negative cannot be dropped in Corollary 2.4. That is, it is not necessarily true that

\[
\left| \int f \, d\mu \right| \leq \left( \int |f|^p \, d\mu - \int |f - \int f \, d\mu|^p \, d\mu \right)^{1/p}
\]

when \( p > 2 \). The reverse inequality may also fail when \( p < 2 \) if \( f \) takes negative values.

Proof. Take \( p = 3 \) and let \( f = \chi_{[0,7/8]} - \chi_{(7/8,1]} \). Here \( \mu \) is Lebesgue measure on \([0,1]\). The left hand side is 3/4 while the right hand side evaluates to \((3/4)^{1/3}\).

To show that the reverse inequality may fail it suffices to take \( p = 15/8 \) and \( f = \chi_{[0,1/32]} - \chi_{(1/32,1]} \). We omit the calculations.

Example 2.5 also shows that Theorems 1.1 and 1.2 may fail if \( f \) is allowed to take negative values. Just take \( g \equiv 1 \).

Theorems 1.3 and 1.4 may fail for simpler reasons. They may fail to make sense. When \( f \) and \( g \) are non-negative the function \( h \) is always less than each of them in \( L^p_\nu \)-norm. This may not be true if \( f \) and \( g \) take negative values.

Example 2.6. Let \( \nu \) be Lebesgue measure on \([0,1]\) and suppose \( p > 2 \). Set \( f \equiv 1/2 \) and \( g = (1/2)(\chi_{[0,1/2]} - \chi_{(1/2,1]}) \). The function \( h \) of Theorems 1.3 and 1.4 satisfies

\[
\int h^p \, d\nu > \int |f|^p \, d\nu \quad \text{and} \quad \int h^p \, d\nu > \int |g|^p \, d\nu.
\]

Proof. \( f + g = \chi_{[0,1/2]} \), so \( h = \chi_{(1/2,1]} \). Thus \( \int h^p \, d\nu = 1/2 \) while both \( \int |f|^p \, d\nu \) and \( \int |g|^p \, d\nu \) are \((1/2)^p\).