REFINING THE HÖLDER AND MINKOWSKI INEQUALITIES

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Abstract. Refinements to the usual Hölder and Minkowski inequalities in the Lebesgue spaces L^p_μ are proved. Both are inequalities for non-negative functions and both reduce to equality in L^2_μ.

1. Introduction and Main Results

The Hölder and Minkowski inequalities are fundamental to the theory of Lebegue spaces. If $1 < p < \infty$ and $1/p + 1/p' = 1$ the first,

$$\int fg\,d\nu \le \left(\int |f|^p\,d\nu\right)^{1/p}\left(\int |g|^{p'}\,d\nu\right)^{1/p'},$$

expresses the fact that functions in $L^{p'}_\nu$ give rise to bounded linear functionals on L^p_ν. It is a sharp inequality in the sense that for any $f \in L^p_\nu$ there is a function $g \in L^{p'}_\nu$ such that the inequality becomes equality. For this reason, improvements to Hölder's inequality must necessarily be quite delicate.

Theorem 1.1. *Let $p \ge 2$ and define p' by $1/p + 1/p' = 1$. Then for any two non-negative ν-measurable functions f and g*

$$\int fg\,d\nu \le \left(\int f^p\,d\nu - \int \left| f - g^{p'-1} \int fg\,d\nu \Big/ \int g^{p'}\,d\nu \right|^p\,d\nu\right)^{1/p}\left(\int g^{p'}\,d\nu\right)^{1/p'}.$$

In the case $1 < p \le 2$ our refinement takes the form of a lower bound.

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Theorem 1.2. *Let $p \leq 2$ and define p' by $1/p + 1/p' = 1$. Then for any two non-negative ν-measurable functions f and g*

$$\left(\int f^p\, d\nu - \int \left| f - g^{p'-1} \int fg\, d\nu \Big/ \int g^{p'}\, d\nu \right|^p d\nu\right)^{1/p} \left(\int g^{p'}\, d\nu\right)^{1/p'} \leq \int fg\, d\nu.$$

The Minkowski inequality is the triangle inequality in L^p_ν: If $1 < p < \infty$ and $1/p + 1/p' = 1$ then

$$\left(\int |f+g|^p\, d\nu\right)^{1/p} \leq \left(\int |f|^p\, d\nu\right)^{1/p} + \left(\int |g|^p\, d\nu\right)^{1/p}.$$

There can only be improvement in this inequality when f and g are not multiples of one another.

Theorem 1.3. *Let $p \geq 2$ and define p' by $1/p + 1/p' = 1$. Then for any two non-negative ν-measurable functions f and g*

$$\left(\int (f+g)^p\, d\nu\right)^{1/p} \leq \left(\int f^p\, d\nu - \int h^p\, d\nu\right)^{1/p} + \left(\int g^p\, d\nu - \int h^p\, d\nu\right)^{1/p}$$

where $h = \left| f \int g(f+g)^{p-1}\, d\nu - g \int f(f+g)^{p-1}\, d\nu \right| / \int (f+g)^p\, d\nu$.

Notice that the function h vanishes when f is a multiple of g. Again we get a lower bound in the case $1 < p \leq 2$.

Theorem 1.4. *Let $1 < p \leq 2$ and define p' by $1/p + 1/p' = 1$. Then for any two non-negative ν-measurable functions f and g*

$$\left(\int f^p\, d\nu - \int h^p\, d\nu\right)^{1/p} + \left(\int g^p\, d\nu - \int h^p\, d\nu\right)^{1/p} \leq \left(\int (f+g)^p\, d\nu\right)^{1/p}$$

where $h = \left| f \int g(f+g)^{p-1}\, d\nu - g \int f(f+g)^{p-1}\, d\nu \right| / \int (f+g)^p\, d\nu$.

It is easy to verify directly that the inequalities given above reduce to equalities when $p = 2$.

The proofs of Theorems 1.1–1.4 will be given in the next section. They depend on a special case of the key inequality established in Theorem 2.3. Also in the next section we give examples to show that the inequalities may fail if the hypothesis of non-negativity is dropped.

We assume throughout that $1 < p < \infty$ and $1/p + 1/p' = 1$. Also, ν will denote an arbitrary σ-finite measure while μ will denote a probability measure, that is, a measure with total measure one. The function $\operatorname{sgn}(x)$ is defined to be 1 when $x > 0$, 0 when $x = 0$, and -1 when $x < 0$.

2. The Key Inequality

The power function $x \mapsto x^\alpha$, $x > 0$, is convex when $\alpha > 1$ and concave when $0 < \alpha < 1$. We will use this fact in the following form. If a and b are non-negative real numbers then

$$(a+b)^\alpha \geq a^\alpha + b^\alpha \text{ when } \alpha > 1 \text{ and } (a+b)^\alpha \leq a^\alpha + b^\alpha \text{ when } 0 < \alpha < 1. \tag{2.1}$$

Equality holds only if $\alpha = 1$, $a = 0$, or $b = 0$.

Lemma 2.1. *Suppose* $1 < p \neq 2$ *and* $t > 0$. *If* $x > 0$, $y > t$ *and*

$$x^{p-1} - |x-t|^{p-1} \operatorname{sgn}(x-t) = y^{p-1} - |y-t|^{p-1} \operatorname{sgn}(y-t)$$

then $x = y$.

Proof. Let $\varphi(x) = x^{p-1} - |x-t|^{p-1} \operatorname{sgn}(x-t)$. Since $y > t$ we have $\varphi(y) = y^{p-1} - (y-t)^{p-1}$. Inequality (2.1) shows that $\varphi(y) > t^{p-1}$ when $p > 2$ and $\varphi(y) < t^{p-1}$ when $p < 2$.

If $x \leq t$ then $\varphi(x) = x^{p-1} + (t-x)^{p-1}$ so (2.1) yields $\varphi(x) \leq t^{p-1}$ when $p > 2$ and $\varphi(x) \geq t^{p-1}$ when $p < 2$. This contradicts the hypothesis $\varphi(x) = \varphi(y)$ so we must have $x > t$. Notice that for $x > t$, $\varphi'(x) = (p-1)x^{p-2} - (p-1)(x-t)^{p-2}$ does not change sign. Hence φ is monotone and therefore one-to-one on (t, ∞). We conclude that $x = y$ as required.

We begin by proving a discrete version of our key inequality.

Theorem 2.2. *Suppose* $p > 2$, n *is a positive integer,* $x_1, x_2, \ldots, x_n$ *are non-negative, and* $0 < t \leq \frac{1}{n}\sum_{j=1}^n x_j$. *Then*

$$\frac{1}{n}\sum_{j=1}^n x_j^p \geq t^p \left(\frac{2}{nt} \sum_{j=1}^n x_j - 1 \right) + \frac{1}{n}\sum_{j=1}^n |x_j - t|^p.$$

The reverse inequality holds when $1 < p < 2$.

Proof. Let

$$M_n = \sum_{j=1}^n x_j^p - t^p \left(\frac{2}{t} \sum_{j=1}^n x_j - n \right) - \sum_{j=1}^n |x_j - t|^p.$$

We will show by induction that M_n is non-negative when $p > 2$. If $n = 1$, and $0 < t \leq x = x_1$ then $M_1 = x^p - t^p(2x/t - 1) - (x-t)^p$. Fix t and consider M_1 as a function of x. At $x = t$, the function vanishes and for $x \geq t$ its derivative is $px^{p-1} - 2t^{p-1} - p(x-t)^{p-1}$ which is not less than $px^{p-1} - pt^{p-1} - p(x-t)^{p-1} \geq 0$ by (2.1). It follows that M_1 is non-negative for $x \geq t$.

Suppose now that for some $n > 1$, $M_{n-1} \geq 0$. To show that $M_n \geq 0$ we fix t and show that for all $x \geq t$, M_n is non-negative on the compact set

$$K_x \equiv \{(x_1, x_2, \ldots, x_n) \in [0,\infty)^n : \textstyle\sum_{j=1}^n x_j = nx\}.$$

First we show that M_n is non-negative on the boundary of K_x considered as a subset of the hyperplane defined by $\sum_{j=1}^n x_j = nx$. That is, that $M_n \geq 0$ when at least one of $x_1, x_2, \dots, x_n$ is zero. By symmetry we may assume that $x_n = 0$. We have

$$0 < t \leq x = \frac{1}{n}\sum_{j=1}^{n-1} x_j \leq \frac{1}{n-1}\sum_{j=1}^{n-1} x_j$$

and so, by the inductive hypothesis,

$$M_n = \sum_{j=1}^{n-1} x_j^p - t^p\left(\frac{2}{t}\sum_{j=1}^{n-1} x_j - n\right) - \sum_{j=1}^{n-1} |x_j - t|^p - t^p = M_{n-1} \geq 0.$$

To complete the proof we use a Lagrange Multiplier argument to show that if the minimum value of M_n occurs in the interior of K_x (considered as a subset of the hyperplane) then it is non-negative. Note that since $p > 1$, M_n has continuous first partial derivatives with respect to each of $x_1, x_2, \dots, x_n$. Thus it will suffice to show that the value of M_n is non-negative at critical points of

$$M_n - \lambda\left(\sum_{j=1}^{n} x_j - nx\right),$$

considered as a function of $x_1, x_2, \dots, x_n, \lambda$ with x and t still fixed. At critical points we have $\sum_{j=1}^n x_j = nx$ and for each j

$$px_j^{p-1} - 2t^{p-1} - p|x_j - t|^{p-1}\operatorname{sgn}(x_j - t) - \lambda = 0.$$

It follows that $x_j^{p-1} - |x_j - t|^{p-1}\operatorname{sgn}(x_j - t)$ takes the same value for each j. Since t is no greater than the average of $x_1, x_2, \dots, x_n$, either $x_1 = x_2 = \cdots = x_n = x = t$ or at least one x_j is greater than t. In the latter case, Lemma 2.1 applies and we conclude that $x_1 = x_2 = \cdots = x_n = x$. In either case we have

$$M_n = n(x^p - t^p(2x/t - 1) - (x - t)^p)$$

which is non-negative as we have seen in the case $n = 1$. This completes the proof in the case $p > 2$.

The proof that $M_n \leq 0$ in the case $1 < p < 2$ proceeds similarly.

The key inequality is presented next. It is more general than Theorem 2.2 and will readily imply Theorems 1.1–1.4.

Theorem 2.3. *Suppose $p \geq 2$ and μ is a probability measure. If $f \geq 0$ is a μ-measurable function then*

$$\int f^p \, d\mu \geq t^p\left(\frac{2}{t}\int f \, d\mu - 1\right) + \int |f - t|^p \, d\mu \tag{2.2}$$

whenever $0 < t \leq \int f\,d\mu$*. The reverse inequality holds when* $1 < p \leq 2$*.*

Proof. It is a simple matter to show that (2) holds with equality when $p = 2$. When $p > 2$ we argue as follows.

If f is not in L^p_μ then both sides of (2.2) are infinite so there is nothing to prove. Fix $f \in L^p_\mu$, and t with $0 < t < \int f\,d\mu$. Let f^* denote the non-increasing rearrangement of f with respect to μ. We view f^* as a Lebesgue measurable function on $[0, 1]$. Since f is non-negative, f and f^* are equimeasurable, f^p and f^{*p} are equimeasurable, and $|f - t|^p$ and $|f^* - t|^p$ are equimeasurable. Thus (2.2) becomes

$$\int_0^1 f^{*p} \geq t^p \left(\frac{2}{t} \int_0^1 f^* - 1 \right) + \int_0^1 |f^* - t|^p. \tag{2.3}$$

For each positive integer n define the function f_n on $[0, 1]$ by

$$f_n(s) = \sum_{j=1}^{n} f^*(j/n)\chi_{((j-1)/n, j/n)}(s)$$

and note that since f^* is non-increasing, $f^*(s + 1/n) \leq f_n(s) \leq f^*(s)$ for $0 < s \leq 1$. Clearly, the sequence $\{f_n\}$ converges to f^* in $L^p[0, 1]$. It follows that $\int_0^1 f_n$ converges to $\int_0^1 f^*$ so for sufficiently large n we have $0 < t < \int_0^1 f_n$. By the Lebesgue Dominated Convergence Theorem, (2.3) will follow provided we establish

$$\int_0^1 f_n^p \geq t^p \left(\frac{2}{t} \int_0^1 f_n - 1 \right) + \int_0^1 |f_n - t|^p. \tag{2.4}$$

for sufficiently large n. If we set $x_j = f^*(j/n)$ then (2.4) becomes

$$\frac{1}{n} \sum_{j=1}^{n} x_j^p \geq t^p \left(\frac{2}{nt} \sum_{j=1}^{n} x_j - 1 \right) + \frac{1}{n} \sum_{j=1}^{n} |x_j - t|^p$$

which holds by Theorem 2.2 when n is large enough that $t \leq \int_0^1 f_n$.

This proves the theorem for $p > 2$ in the case $0 \leq t < \int f\,d\mu$. The case $t = \int f\,d\mu$ follows by an easy limiting argument.

The same argument yields the reverse inequality when $1 < p < 2$.

Corollary 2.4. *Suppose* $p \geq 2$*,* μ *is a probability measure, and* f *is a non-negative,* μ*-measurable function. Then*

$$\int f\,d\mu \leq \left(\int f^p\,d\mu - \int |f - \textstyle\int f\,d\mu|^p\,d\mu \right)^{1/p}$$

The reverse inequality holds when $1 < p < 2$*.*

Proof. Take $t = \int f\,d\mu$ in Theorem 2.3, rearrange the result and take p-th roots.

Proofs of Theorems 1.1–1.4. To prove Theorems 1.1 and 1.2 we fix non-negative ν-measurable functions f and g and apply Corollary 2.4 with $fg^{1-p'}$ in place of f and $d\mu = g^{p'}\,d\nu/\int g^{p'}\,d\nu$.

Theorems 1.3 follows from Theorems 1.1 in the same way that Minkowski's inequality follows from Hölder's. Fix non-negative ν-measurable functions f and g and define h by

$$h = \left| f \int g(f+g)^{p-1}\,d\nu - g \int f(f+g)^{p-1}\,d\nu \right| \Big/ \int (f+g)^p\,d\nu\ .$$

Let $p \geq 2$ and apply Theorem 1.1 with g replaced by $(f+g)^{p-1}$ to get

$$\int f(f+g)^{p-1}\,d\nu \leq \left(\int f^p\,d\nu - \int h^p\,d\nu \right)^{1/p} \left(\int (f+g)^p\,d\nu \right)^{1/p'}.$$

Interchanging the roles of f and g yields

$$\int g(f+g)^{p-1}\,d\nu \leq \left(\int g^p\,d\nu - \int h^p\,d\nu \right)^{1/p} \left(\int (f+g)^p\,d\nu \right)^{1/p'}.$$

Adding the last two inequalities gives Theorem 1.3.

Theorem 1.4 follows from Theorem 1.2 by a similar argument.

Example 2.5. *The hypothesis that f be non-negative cannot be dropped in Corollary 2.4. That is, it is not necessarily true that*

$$\left| \int f\,d\mu \right| \leq \left(\int |f|^p\,d\mu - \int |f - \textstyle\int f\,d\mu|^p\,d\mu \right)^{1/p}$$

when $p > 2$. The reverse inequality may also fail when $p < 2$ if f takes negative values.

Proof. Take $p = 3$ and let $f = \chi_{[0,7/8)} - \chi_{(7/8,1]}$. Here μ is Lebesgue measure on $[0,1]$. The left hand side is $3/4$ while the right hand side evaluates to $(3/4)^{(4/3)}$.

To show that the reverse inequality may fail it suffices to take $p = 15/8$ and $f = \chi_{[0,1/32)} - \chi_{(1/32,1]}$. We omit the calculations.

Example 2.5 also shows that Theorems 1.1 and 1.2 may fail if f is allowed to take negative values. Just take $g \equiv 1$.

Theorems 1.3 and 1.4 may fail for simpler reasons. They may fail to make sense. When f and g are non-negative the function h is always less than each of them in L^p_ν-norm. This may not be true if f and g take negative values.

Example 2.6. *Let ν be Lebesgue measure on $[0,1]$ and suppose $p > 2$. Set $f \equiv 1/2$ and $g = (1/2)(\chi_{[0,1/2)} - \chi_{(1/2,1]})$. The function h of Theorems 1.3 and 1.4 satisfies*

$$\int h^p\,d\nu > \int |f|^p\,d\nu \ \text{ and } \int h^p\,d\nu > \int |g|^p\,d\nu.$$

Proof. $f + g = \chi_{[0,1/2)}$ so $h = \chi_{(1/2,1]}$. Thus $\int h^p\,d\nu = 1/2$ while both $\int |f|^p\,d\nu$ and $\int |g|^p\,d\nu$ are $(1/2)^p$.

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