REFINING JENSEN'S INEQUALITY

Shoshana Abramovich, Graham Jameson and Gord Sinnamon

University of Haifa, Lancaster University and University of Western Ontario

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Abstract. A refinement of Jensen's inequality is presented. An extra term makes the inequality tighter when the convex function is "superquadratic," a strong convexity-type condition introduced here. This condition is shown to be necessary and sufficient for the refined inequality. It is also shown to be strictly intermediate between two points of the scale of convexity from [2]. The refined Jensen's inequality is used to prove a Minkowski inequality with upper and lower estimates.

1. Introduction

Jensen's inequality states that if $\varphi : [0, \infty) \to \mathbf{R}$ is convex then

$$\varphi\left(\int f\,d\mu\right) \le \int \varphi(f(s))\,d\mu(s) \tag{1.1}$$

for all probability measures μ and all non-negative, μ-integrable functions f. If φ is concave the inequality is reversed. In particular, Jensen's inequality reduces to equality when $\varphi(x) = x$ is a line. In this paper we investigate the inequality

$$\varphi\left(\int f\,d\mu\right) \le \int \varphi(f(s)) - \varphi\left(\left|f(s) - \textstyle\int f\,d\mu\right|\right)\,d\mu(s). \tag{1.2}$$

this inequality reduces to equality when $\varphi(x) = x^2$ is a parabola rather than a line. Our interest in this refinement of Jensen's inequality began with a corresponding refinement of Hölder's inequality.

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Proposition 1.1([4, Theorem 1.1]). *Let $p \geq 2$ and define p' by $1/p + 1/p' = 1$. Then for any two non-negative ν-measurable functions f and g*

$$\int fg\,d\nu \leq \left(\int f^p\,d\nu - \int \left| f - g^{p'-1} \int fg\,d\nu \Big/ \int g^{p'}\,d\nu \right|^p d\nu \right)^{1/p} \left(\int g^{p'}\,d\nu \right)^{1/p'}.$$

If ν is a probability measure μ, $f \geq 0$, and $g \equiv 1$ this reduces to

$$t^p \leq \int f(s)^p - |f(s) - t|^p\,d\mu(s)$$

where $t = \int f\,d\mu$ and $p \geq 2$. Since this is equality when $p = 2$ we have

$$\frac{t^p - t^2}{p-2} \leq \int \frac{f(s)^p - f(s)^2}{p-2} - \frac{|f(s)-t|^p - |f(s)-t|^2}{p-2}\,d\mu(s)$$

and letting $p \to 2^+$ yields

$$t^2 \log(t) \leq \int f(s)^2 \log(f(s)) - |f(s) - t|^2 \log(|f(s) - t|)\,d\mu(s).$$

That is, (1.2) holds for the function $\varphi(x) = x^2 \log(x)$. This new φ is more interesting than a simple power function: It is not positive. It is not monotone. It is not even convex. A proper understanding of (1.2) must include functions φ of this sort.

For non-negative functions φ, inequality (1.2) implies Jensen's inequality (1.1) so the function φ is necessarily convex. We will show that (1.2) holds for functions φ that are "more convex" than a parabola in a suitable sense. Our analysis also admits negative φ as well as functions φ that change sign. Such functions φ need not be convex.

This work is a development of the refined Hölder and Minkowski inequalities given in [4] and, in particular, provides much simpler proofs of the results of that paper.

In the next section we introduce a new convexity-type condition on φ and show that it is necessary and sufficient for the inequality (1.2) to hold for all probability measures μ and all non-negative, μ-integrable functions f. In Section 3, we look at a scale of convexity introduced in [2] and place this new condition precisely within that scale. This makes it easy to verify in practice whether or not φ is "convex enough" for (1.2).

Our final section is devoted to an application of the refined Jensen inequality to a new refinement of Minkowski's inequality having both upper and lower estimates. We hope that this will be the first of many applications of the refined Jensen inequality and we encourage interested researchers to continue the investigation of superquadratic functions.

2. Superquadratic Functions and the Refined Jensen's Inequality

The definition of a superquadratic function is a simple modification of the geometrical notion of a convex function: A convex function has a tangent line at each point and lies above each of its tangent lines. That is, for each x there exists a slope C_x such that

$$\varphi(y) \geq \varphi(x) + C_x(y - x)$$

for all y. (Note that if φ is differentiable at x then $C_x = \varphi'(x)$.)

For a superquadratic function we require that φ lie above its tangent line plus a translation of φ itself.

Definition 2.1. *A function $\varphi : [0, \infty) \to \mathbf{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbf{R}$ such that*

$$\varphi(y) \geq \varphi(x) + C_x(y - x) + \varphi(|y - x|) \tag{2.1}$$

for all $y \geq 0$.

We employ the absolute values above instead of explicitly extending $\varphi : [0, \infty) \to \mathbf{R}$ to be an even function on $\mathbf{R}$.

Note that if $\varphi(x) = x^2$ the condition above reduces to the identity $y^2 - x^2 - (y - x)^2 = 2x(y - x)$. We observe that if $\varphi(x)$ is superquadratic and $a, b \geq 0$ then $\varphi(x) - (ax + b)$ is also superquadratic: Since $a(|y - x| - (y - x)) + b \geq 0$ we have

$$\begin{aligned}(\varphi(y) - (ay + b)) - (\varphi(x) - (ax + b)) - (\varphi(|y - x|) - (a|y - x| + b))\\ = \varphi(y) - \varphi(x) - \varphi(|y - x|) + a(|y - x| - (y - x)) + b \geq C_x(y - x).\end{aligned}$$

At first glance, condition (2.1) appears to be stronger than convexity but if φ takes negative values then it may be considerably weaker. To emphasize just how poorly behaved superquadratic functions can be we remark that any function φ satisfying $-2 \leq \varphi(x) \leq -1$ for all x is superquadratic. Just take $C_x = 0$ in (2.1).

Non-negative superquadratic functions are much better behaved as we see next.

Lemma 2.2. *Let φ be a superquadratic function with C_x as in Definition 1.2.*
(i) Then $\varphi(0) \leq 0$.
(ii) If $\varphi(0) = \varphi'(0) = 0$, then $C_x = \varphi'(x)$ whenever φ is differentiable at $x > 0$.
(iii) If $\varphi \geq 0$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$.

Proof. The condition (2.1) with $x = y$ shows that $\varphi(0) \leq 0$ for any superquadratic function. If $x > 0$ we can write out (2.1) in the two cases $y < x$ and $y > x$ to get

$$\lim_{y \to x^-} \left(\frac{\varphi(x) - \varphi(y)}{x - y} + \frac{\varphi(x - y)}{x - y} \right) \leq C_x \leq \lim_{y \to x^+} \left(\frac{\varphi(y) - \varphi(x)}{y - x} - \frac{\varphi(y - x)}{y - x} \right).$$

If φ is differentiable at 0 and $\varphi(0) = \varphi'(0) = 0$ then this becomes $\varphi'(x) \leq C_x \leq \varphi'(x)$ whenever $\varphi'(x)$ exists.

If $\varphi \geq 0$, then $\varphi(0) \leq 0$ becomes $\varphi(0) = 0$. Also, Definition 2.1 implies that $\varphi(y) - \varphi(x) \geq C_x(y - x)$ for all $x, y \geq 0$. If $y_1 < x < y_2$ this yields

$$\frac{\varphi(x) - \varphi(y_1)}{x - y_1} \leq C_x \leq \frac{\varphi(y_2) - \varphi(x)}{y_2 - x}$$

and we have

$$\varphi(x) \leq \frac{x - y_1}{y_2 - y_1}\varphi(y_2) + \frac{y_2 - x}{y_2 - y_1}\varphi(y_1).$$

We conclude that φ is convex. It is well known that a convex function is differentiable almost everywhere. Choose an $x>0$ such that $\varphi'(x)$ exists. Using (2.1) once again we have

$$\limsup_{y\to x^-}\left(\frac{\varphi(x)-\varphi(y)}{x-y}+\frac{\varphi(x-y)}{x-y}\right)\le C_x\le\limsup_{y\to x^+}\left(\frac{\varphi(y)-\varphi(x)}{y-x}-\frac{\varphi(y-x)}{y-x}\right)$$

and it follows that

$$\limsup_{t\to 0^+}\frac{\varphi(t)}{t}\le 0.$$

Since φ is non-negative we have

$$0\le\liminf_{t\to 0^+}\frac{\varphi(t)}{t}\le\limsup_{t\to 0^+}\frac{\varphi(t)}{t}\le 0$$

so the (one-sided) derivative at zero exists and $\varphi'(0)=0$. This completes the proof.

Theorem 2.3. *The inequality* (1.2) *holds for all probability measures μ and all non-negative, μ-integrable functions f if and only if φ is superquadratic.*

Proof. Suppose first that φ is superquadratic. Fix a probability measure μ and a non-negative, μ-integrable function f. Set $x=\int f\,d\mu$ and let C_x be the constant of Definition 2.1. Then

$$\int\varphi(f(s))-\varphi(x)-\varphi(|f(s)-x|)\,d\mu(s)\ge C_x\int(f(s)-x)\,d\mu(s)=0$$

which may be rearranged to yield

$$\varphi\left(\int f\,d\mu\right)\le\int\varphi(f(s))-\varphi\left(\left|f(s)-\textstyle\int f\,d\mu\right|\right)\,d\mu(s)$$

as required.

For the converse, suppose that (1.2) holds and set $C_0=0$. If $x=0$ or $y=x$, condition (2.1) reduces to $-\varphi(0)\ge 0$ which follows from (1.2) by taking f to be the zero function. In the remaining case we have $x>0$ and $y\ne x$. Suppose $0\le y_1<x<y_2$ and let μ be the probability measure on $\{0,1\}$ with $\mu(0)=(x-y_1)/(y_2-y_1)$ and $\mu(1)=(y_2-x)/(y_2-y_1)$. With $f(0)=y_2$ and $f(1)=y_1$ we have $\int f\,d\mu=x$ so the inequality (1.2) becomes

$$\varphi(x)\le\frac{x-y_1}{y_2-y_1}(\varphi(y_2)-\varphi(y_2-x))+\frac{y_2-x}{y_2-y_1}(\varphi(y_1)-\varphi(x-y_1)).$$

We can rewrite this as

$$\frac{\varphi(y_1)-\varphi(x)-\varphi(x-y_1)}{y_1-x}\le\frac{\varphi(y_2)-\varphi(x)-\varphi(y_2-x)}{y_2-x}.$$

By fixing a $y_1 \in (0, x)$ we obtain a lower bound that shows that

$$C_x \equiv \inf_{y_2 > x} \frac{\varphi(y_2) - \varphi(x) - \varphi(y_2 - x)}{y_2 - x}$$

exists. Now we take $y_2 = y$ to see that

$$\varphi(y) - \varphi(x) - \varphi(y - x) \geq C_x(y - x)$$

for all $y > x$ and take $y_1 = y$ to get

$$\varphi(y) - \varphi(x) - \varphi(x - y) \geq C_x(y - x)$$

for all $y < x$. Thus φ is superquadraticand the proof is complete.

If φ is convex then Jensen's inequality and Slater's companion inequality [5] (as generalized by Pečarić in [3]) show that

$$\varphi(m) \leq \int \varphi(f(s))\, d\mu(s) \leq \varphi(M), \tag{2.2}$$

where

$$m = \int f(s)\, d\mu(s) \quad \text{and} \quad M = \frac{\int f(s) C_{f(s)}\, d\mu(s)}{\int C_{f(s)}\, d\mu(s)}. \tag{2.3}$$

The function C should satisfy $\varphi'_-(x) \leq C_x \leq \varphi'_+(x)$ where φ'_- and φ'_+ are the left and right derivatives of φ, well-known to exist for any convex function φ.

For superquadratic φ we can prove a corresponding inequality which, in the event that $\varphi \geq 0$, tightens both the upper and lower bounds in (2.2).

Theorem 2.4. *Suppose φ is superquadratic and C is as in Definition 2.1. If μ is a probability measure, f is a non-negative μ-measurable function, $\int C_{f(s)}\, d\mu(s) \neq 0$, and m and M are as defined by* (2.3) *then*

$$\varphi(m) + \int \varphi(|f(s) - m|)\, d\mu(s) \leq \int \varphi(f(s))\, d\mu(s) \leq \varphi(M) - \int \varphi(|f(s) - M|)\, d\mu(s).$$

Proof. The first inequality was proved in Theorem 2.2. For the second we replace x by $f(s)$ and y by M in Definition 2.1 to get

$$\begin{aligned}
\int \varphi(f(s))\, d\mu(s) - \varphi(M) &= \int \varphi(f(s)) - \varphi(M)\, d\mu(s) \\
&\leq \int C(f(s))(f(s) - M)\, d\mu(s) - \int \varphi(|f(s) - M|)\, d\mu \\
&= -\int \varphi(|f(s) - M|)\, d\mu.
\end{aligned}$$

Adding $\varphi(M)$ to both sides completes the proof.

3. Superquadratic Functions and the Scale of Convexity

The six conditions given below constitute the scale of convexity introduced in [2] and studied in [1]. A function $h : [0,\infty) \to \mathbf{R}$ is *superadditive* provided $h(x+y) \geq h(x) + h(y)$ for all $x, y \geq 0$.

(K1) $\varphi'(x)$ convex,

(K2) $\varphi(x)/x$ convex,

(K3) $\varphi'(x)/x$ non-decreasing,

(K4) $\varphi'(x)$ superadditive,

(K5) $\varphi(x)/x^2$ non-decreasing,

(K6) $\varphi(x)/x$ superadditive.

For a continuously differentiable function φ satisfying $\varphi(0) = \varphi'(0) = 0$ it was shown in [2] that each of these conditions implies the next and that no two are equivalent. (In that paper these conditions were viewed as conditions on the function φ' rather than φ.)

Among continuously differentiable functions φ satisfying $\varphi(0) = \varphi'(0) = 0$ the superquadratic ones fall strictly between (K4) and (K5). Lemmas 3.1 and 3.2 provide the two inclusions and the examples that follow show that neither inclusion is strict.

In addition to (K4), Lemma 3.1 shows that functions satisfying (K3) are superquadratic. In view of Theorem 2.3, each of these conditions is sufficient for the inequality (1.2). For another sufficient condition see Lemma 4.1.

Lemma 3.1. *Suppose $\varphi : [0,\infty) \to \mathbf{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\varphi'(x)/x$ is non-decreasing, then φ is superquadratic.*

Proof. If φ' is superadditive and $x \leq y$ then

$$0 \leq \int_x^y \varphi'(t) - \varphi'(x) - \varphi'(t-x)\,dt = \varphi(y) - \varphi(x) - (y-x)\varphi'(x) - \varphi(y-x) + \varphi(0)$$

and if $y \leq x$ then

$$0 \leq \int_y^x \varphi'(x) - \varphi'(x-t) - \varphi'(t)\,dt = (x-y)\varphi'(x) + \varphi(0) - \varphi(x-y) - \varphi(x) + \varphi(y).$$

Together these show that for any $x, y \geq 0$,

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq \varphi'(x)(y-x) - \varphi(0) \geq \varphi'(x)(y-x).$$

Taking $C_x = \varphi'(x)$ we see that φ is superquadratic.

We reduce the second statement to the first by observing that if φ is continuously differentiable and $\varphi'(x)/x$ is non-decreasing then

$$\varphi'(x+y) = \frac{x\varphi'(x+y)}{x+y} + \frac{y\varphi'(x+y)}{x+y} \geq \varphi'(x) + \varphi'(y)$$

so φ' is superadditive. This completes the proof.

Note that if $p > 1$ and $\varphi(x) = \operatorname{sgn}(p-2)x^p$ then $\varphi'(x)/x$ is non-decreasing. By Lemma 3.1, φ is superquadratic so by Theorem 2.3 inequality (1.2) holds. That is,

$$\left(\int f\, d\mu\right)^p \leq \int f(s)^p - \left|f(s) - \textstyle\int f\, d\mu\right|^p d\mu(s)$$

when $p \geq 2$ and the reverse inequality when $1 < p \leq 2$. This is a much simpler proof of Corollary 2.4 of [4] than was given there as it avoids the Lagrange multiplier argument and the cumbersome discrete approximation step. The main results of [4] follow readily from Corollary 2.4.

In the introduction the function $\varphi(x) = x^2 \log(x)$ was shown to satisfy (1.2) so by Theorem 2.3 it must be superquadratic. Lemma 3.1 gives a simple way to see this directly by checking that $\varphi'(x)/x$ is non-decreasing.

Lemma 3.2. *Suppose φ is differentiable and $\varphi(0) = \varphi'(0) = 0$. If φ is superquadratic, then $\varphi(x)/x^2$ is non-decreasing on $(0, \infty)$.*

Proof. According to Lemma 2.2, the constant C_x in Definition 2.1 is necessarily $\varphi'(x)$. Using this, we take $y = 0$ in (2.1) to get

$$\varphi(0) - \varphi(x) - \varphi(x) \geq \varphi'(x)(0 - x)$$

or simply

$$x\varphi'(x) \geq 2\varphi(x).$$

Now

$$\frac{d}{dx}\frac{\varphi(x)}{x^2} = \frac{x\varphi'(x) - 2\varphi(x)}{x^3} \geq 0$$

and it follows that $\varphi(x)/x^2$ is non-decreasing.

This justifies the term “superquadratic” at least for non-negative, differentiable functions. If φ is non-negative and superquadratic then Lemma 2.2 gives $\varphi(0) = \varphi'(0) = 0$. By Lemma 3.2, $\varphi(x)/x^2$ is non-decreasing which is to say $\varphi(x)$ is increasing as fast or faster than a quadratic function.

Example 3.3. *A non-negative continuously differentiable superquadratic function need not satisfy* (K4)*: Define φ by $\varphi(0) = 0$ and*

$$\varphi'(x) = \begin{cases} 0, & x \leq 1 \\ 1 + (x-2)^3, & x \geq 1. \end{cases}$$

Proof. It is easy to check that φ' is not superadditive:

$$\varphi'(8/3) - \varphi'(4/3) - \varphi'(4/3) = -1/9 < 0.$$

To show that φ is superquadratic we show that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) - (y-x)\varphi'(x) \geq 0$$

by looking at two cases.

Case 1: $x \leq y$. With $t = y - x$ our task is to show that

$$h(x,t) \equiv \varphi(x+t) - \varphi(x) - \varphi(t) - t\varphi'(x) = \int_0^t \varphi'(x+s) - \varphi'(s) - \varphi'(x)\, ds$$

is non-negative for all $x, t \geq 0$. If $x \leq 1$ then $\varphi'(x) = 0$ and the integrand is non-negative because φ' is non-decreasing. If $t \leq 1$ then $\varphi'(s) = 0$ when $0 \leq s \leq t$ so again the integrand is non-negative. When $x \geq 1$ and $t \geq 1$, we have explicitly

$$h(x,t) = \tfrac{3}{2}t^2x^2 + t^2(t-6)x + 7t - \tfrac{11}{4}.$$

For fixed t, this quadratic in x takes its least value when $x = 2 - t/3$. If $t > 3$, this implies that $h(x,t) \geq h(1,t) \geq 0$, while if $t \leq 3$, the least value is

$$F(t) =: -\tfrac{1}{6}t^4 + 2t^3 - 6t^2 + 7t - \tfrac{11}{4},$$

and one verifies by elementary calculus that $F(t) > 0$ on $[1,3]$.

Case 2. $y \leq x$. With $t = x - y$ we must show that

$$g(y,t) \equiv \varphi(y) - \varphi(y+t) - \varphi(t) + t\varphi'(y+t) = \int_0^t -\varphi'(y+t-s) - \varphi'(s) + \varphi'(y+t)\, ds$$

is non-negative for all $y, t \geq 0$. If $t \leq 1$ then $\varphi'(s) = 0$ when $0 \leq s \leq t$ and the integrand is non-negative so $g(y,t) \geq 0$. If $t \geq 1$ and $y \leq 1$ then $\varphi'(y+t-s) = 0$ for $y+t-1 \leq s \leq t$ so

$$\begin{aligned} g(y,t) &= \int_0^t -\varphi'(y+t-s) - \varphi'(s) + \varphi'(y+t)\, ds \\ &\geq \int_0^{y+t-1} -\varphi'(y+t-s) - \varphi'(s) + \varphi'(y+t)\, ds = g(1, y+t-1). \end{aligned}$$

Therefore, it is enough to show that $g(y,t) \geq 0$ for $(y,t) \in [1,\infty) \times [1,\infty)$. This can be deduced from Case 1, because one can verify that $g(y,t) = h(y + t/3, t)$ for such y and t.

Example 3.4. *A continuously differentiable function that is non-negative, zero at zero, and satisfies* (K5) *need not be superquadratic.*

$$\varphi(x) = \begin{cases} (3x - x^3)x^2, & x \leq 1 \\ 2x^2, & x > 1. \end{cases}$$

Proof. It is easy to see from the definition that $\varphi(x)/x^2$ is non-decreasing so φ satisfies (K5). Also φ is non-negative but since $\varphi''(x) = 18x - 20x^3$ is negative for $3/\sqrt{10} < x < 1$, φ is not convex. By Lemma 2.2 φ is not superadditive.

4. Application

If $-\varphi$ is superquadratic then (1.2) holds with the inequality reversed. In this section we take a look at negative superquadratic functions and use the inequality (1.2) with $\varphi(x) = -(1+x^{1/p})^p$. It is interesting to see how the cases $0 < p < 1/2$ and $p \geq 1/2$ affect the resulting refined Minkowski inequalities.

Lemma 4.1. *A non-positive, non-increasing, superadditive function is superquadratic.*

Proof. Suppose φ is non-positive, non-increasing and superadditive. If $x \leq y$ then superadditivity shows that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq 0$$

and if $y \leq x$ then $\varphi(y) \geq \varphi(x)$ so

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq -\varphi(|y-x|) \geq 0.$$

By Definition 2.1 with $C_x = 0$, φ is superquadratic.

Example 4.2. *Let*

$$\varphi_p(x) = -(1+x^{1/p})^p.$$

Then φ_p is superquadratic for $p > 0$ and $1 + \varphi_p$ is superquadratic for $p \geq 1/2$.

Proof. It is clear that $\varphi_p(x)$ is negative and non-increasing. Since

$$\frac{d}{dx}\frac{\varphi_p(x)}{x} = \frac{-(1+x^{1/p})^{p-1}x^{1/p} + (1+x^{1/p})^p}{x^2} = \frac{(1+x^{1/p})^{p-1}}{x^2} > 0$$

for $x > 0$ we see that $\varphi_p(x)/x$ is non-decreasing and so, arguing as in Lemma 3.1, $\varphi_p(x)$ is superadditive. By Lemma 4.1, φ_p is superquadratic.

For $p \geq 1/2$ we look at $1 + \varphi_p(x)$. This function is zero at zero and its derivative is $\varphi_p'(x) = -(1+x^{-1/p})^{p-1}$ so

$$\frac{d}{dx}\frac{\varphi_p'(x)}{x} = \frac{(1+x^{-1/p})^{p-2}}{x^2}\left(1 + \frac{2p-1}{p}x^{-1/p}\right) \geq 0.$$

Thus $\varphi_p'(x)/x$ is non-decreasing and so by Lemma 3.1, $1 + \varphi_p(x)$ is superquadratic.

In the case $0 < p \leq 1$ Minkowski's inequality provides a lower bound for the L^p "norm" of the sum of two functions. We give two upper bounds, one valid for all $p > 0$ and the other valid when $p \geq 1/2$. Although both remain valid when $p > 1$ the first is weaker than Minkowski's inequality in that range. The second may be viewed as a companion to Minkowski's inequality when $p > 1$ giving a weaker or stronger estimate depending on the pair of functions involved.

Theorem 4.3. *Suppose that ν is a measure and f and g are non-negative functions such that f^p and g^p are ν-integrable. Set*

$$h = \left| g^p - f^p \frac{\int g^p \, d\nu}{\int f^p \, d\nu} \right|^{1/p}.$$

If $0 < p \le 1$ then

$$\left(\int f^p \, d\nu \right)^{1/p} + \left(\int g^p \, d\nu \right)^{1/p} \le \left(\int (f+g)^p \, d\nu \right)^{1/p}$$

and

$$\left(\int (f+g)^p \, d\nu - \int (f+h)^p \, d\nu \right)^{1/p} \le \left(\int f^p \, d\nu \right)^{1/p} + \left(\int g^p \, d\nu \right)^{1/p}.$$

If $p \ge 1/2$ then

$$\left(\int (f+g)^p \, d\nu - \int (f+h)^p \, d\nu + \int g^p \, d\nu \right)^{1/p} \le \left(\int f^p \, d\nu \right)^{1/p} + \left(\int g^p \, d\nu \right)^{1/p}.$$

Proof. The first inequality is well known and can be proved in many ways. We include a proof using only Jensen's inequality because it points the way to using the refined Jensen's inequality (1.2) to establish the other two inequalities.

Define the probability measure μ by $d\mu = f^p \, d\nu / \int f^p \, d\nu$, set $F = g^p/f^p$ and note that F is μ-integrable. Since $(1+x^{1/p})^p$ is convex for $0 < p \le 1$, Jensen's inequality (1.1) shows that

$$\left(1 + \left(\int F \, d\mu \right)^{1/p} \right)^p \le \int \left(1 + F^{1/p} \right)^p \, d\mu. \tag{4.1}$$

When $p > 0$, Lemma 4.2 and Theorem 2.3 show that the inequality (1.2) holds with $\varphi = \varphi_p$, the function of Example 4.2. Therefore,

$$-\left(1 + \left(\int F \, d\mu \right)^{1/p} \right)^p \le -\int \left(1 + F^{1/p} \right)^p \, d\mu + \int \left(1 + |F - \textstyle\int F \, d\mu|^{1/p}\right)^p \, d\mu. \tag{4.2}$$

For $p \ge 1/2$, Lemma 4.2 and Theorem 2.3 show that the inequality (1.2) holds with $\varphi = 1 + \varphi_p$. Therefore,

(4.3)

$$-\left(1 + \left(\int F \, d\mu \right)^{1/p} \right)^p + 1 \le -\int \left(1 + F^{1/p} \right)^p + 1 \, d\mu + \int \left(1 + |F - \textstyle\int F \, d\mu|^{1/p}\right)^p + 1 \, d\mu.$$

Substituting $F = g^p/f^p$ and $d\mu = f^p \, d\nu / \int f^p \, d\nu$ into (4.1), (4.2), and (4.3) and simplifying completes the proof.

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Department of Mathematics, University of Haifa, Haifa, ISRAEL; Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster, LA1 4YF, ENGLAND; and Department of Mathematics, University of Western Ontario, London, Ontario, N6A 5B7, CANADA

E-mail address: `abramos@math.haifa.ac.il, g.jameson@lancaster.ac.uk and sinnamon@uwo.ca`