THE P–HH NORMS ON CARTESIAN POWERS AND SEQUENCE SPACES

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Abstract. A new family of norms is defined on the Cartesian product of n copies of a given normed space. The new norms are related to the hypergeometric means but are not restricted to the positive real numbers. Quantitative comparisons with the usual p-norms are given. The reflexivity, convexity and smoothness of the norms are shown to be closely related to the corresponding property of the underlying space. Using a limit of isometric embeddings, the norms are extended to spaces of bounded sequences that include all summable sequences. Examples are given to show that the new sequence spaces have very different properties than the usual spaces of p-summable sequences.

1. Introduction

Let $(\mathbf{X}, \|\cdot\|)$ be a normed linear space, and consider the Cartesian product space $\mathbf{X}^n$ for a fixed positive integer n. Under the usual addition and scalar multiplication, it becomes a normed space when equipped with any of the following norms (the so-called p-norms):

$$\|\mathbf{x}\|_p = \begin{cases} (\|x_1\|^p + \cdots + \|x_n\|^p)^{1/p}, & 1 \leq p < \infty; \\ \max\{\|x_1\|, \ldots, \|x_n\|\}, & p = \infty, \end{cases}$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$. All p-norms are equivalent in $\mathbf{X}^n$, as

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p \leq n^{1/p}\|\mathbf{x}\|_\infty$$

for $1 \leq p < \infty$. Each of these norms extends in a natural way to a norm on a space of sequences in $\mathbf{X}$, giving the familiar $\ell^p(\mathbf{X})$ spaces. Despite their equivalence on $\mathbf{X}^n$ the norms on the $\ell^p(\mathbf{X})$ spaces are all inequivalent, and the $\ell^p(\mathbf{X})$ spaces are all different for different values of p.

In this paper, another family of norms on $\mathbf{X}^n$ is defined and studied. They are based on the $p - HH$ norms introduced by Kikianty and Dragomir in [11]. The study of these norms is motivated by the Hermite-Hadamard inequality and the close connection they have to the hypergeometric R-function of [2] and [3].

The classical means, exemplified by ℓ^p above, extend from means on $[0, \infty)$ to means in a normed vector space $\mathbf{X}$ in an unfortunately simple fashion; one evaluates the norms of n vectors in $\mathbf{X}$ and then calculates the mean of the resulting n real numbers. Consequently, these means depend on the original vectors only through their norms. This process does give a norm on $\mathbf{X}^n$, but one that is relatively insensitive to the geometry of $\mathbf{X}^n$. The weighted arithmetic means (as distinct from weighted ℓ^1 norms) are exceptional in this regard because one first computes, *within* $\mathbf{X}$, a fixed linear combination of the original vectors, and then evaluates the $\mathbf{X}$-norm of the result. This preserves more of the structure of $\mathbf{X}^n$. However, a

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weighted arithmetic mean of non-zero vectors can be zero so it does not give us a norm on $\mathbf{X}^n$.

To calculate the hypergeometric mean of n vectors in $\mathbf{X}$ one evaluates a number of different weighted arithmetic means, indexed by the points of an $(n-1)$-simplex, and then finds the L^p norm of this collection of means by integrating over the simplex. Theorem 1 shows that for each $p \geq 1$ this procedure does give a norm on $\mathbf{X}^n$, called the $p-HH$ norm. The $p-HH$ norms retain the sensitivity of the arithmetic means to the geometry of $\mathbf{X}^n$; they depend on the relative positions of the n original vectors in the space $\mathbf{X}$, not just on the size of each vector. Example 1 shows one concrete way that a change in the "shape" of the space $\mathbf{X}$ affects the $p-HH$ norms.

Spaces of sequences with entries in a normed space $\mathbf{X}$ can be normed using classical means in much the same way as the space $\mathbf{X}^n$ can be, provided one is willing to restrict the sequence space to ensure finiteness of the norm. Here again, the norm of the sequence depends only on the norms of the entries. Extending the $p-HH$ norms, and hence the hypergeometric means, to sequence spaces $h^p[\mathbf{X}]$ is done in Section 4. The sensitivity of these norms to the geometry of $\mathbf{X}$ is markedly different than, for instance, the spaces $\ell^p(\mathbf{X})$. A simple example of this is provided by Remark 4 and Example 3. These prove that although $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots)$ and $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ are both in ℓ^2, the first is in $h^2[\mathbb{R}]$ but the second is not. The reason for this is that, even though the entries of the two sequences are the same size, the first sequence is spread out around zero and so has significantly smaller weighted arithmetic means than the second, which is concentrated on one side of zero. A more persuasive example comes from Harmonic Analysis. Consider the sequence of terms of the trigonometric polynomial

$$f(x) = \sum_{n=-N}^{N} a_n e^{inx}.$$

Its ℓ^2 norm does not depend on x. Indeed, for any x,

$$\|(a_n e^{inx})_{n=-N}^{N}\|_{\ell^2(\mathbb{C})} = \|f\|_{L^2(-\pi,\pi)}.$$

However, Theorem 5 shows that its $2-HH$ norm does depend on x. The formula is quite straightforward;

$$\|(a_n e^{inx})_{n=-N}^{N}\|_{2-HH} = \left(\frac{\|f\|_{L^2(-\pi,\pi)} + |f(x)|^2}{(2N+1)(2N+2)} \right)^{1/2}.$$

Letting $N \to \infty$ we can, at least formally, apply Theorem 11 (for two-sided sequences) to get

$$\|(a_n e^{inx})_{n=-\infty}^{\infty}\|_{h^2[\mathbb{C}]} = \tfrac{1}{\sqrt{2}} \left(\|f\|_{L^2(-\pi,\pi)} + |f(x)|^2 \right)^{1/2}.$$

This norm may be different, may be finite or infinite, for different x depending on the pointwise convergence of the trigonometric polynomials as $N \to \infty$. This is not the case with the $\ell^2(\mathbb{C})$ norm.

It would be interesting to investigate in what precise sense the series for $f(x)$ must converge for the above formula to hold, and to explore the differences between the spaces $\ell^2(\mathbb{C})$ and $h^2[\mathbb{C}]$ but our task in this paper is to introduce the $p-HH$ norms and the spaces $h^p[\mathbf{X}]$, and to establish some basic properties.

After the $p-HH$ norms on $\mathbf{X}^n$ are defined in the next section, quantitative comparisons are made between the $p-HH$ norms and the p-norms. In particular, these prove Conjecture 1 of [11], establishing the best constant in an inequality relating the p-norms and the $p-HH$ norms in $\mathbf{X}^2$. The strict analogue of Conjecture 1 fails when $n > 2$ but a substitute is given that is also sharp. Together with an

n-dimensional Hermite-Hadamard inequality, these results prove the equivalence of the $p-HH$ norms and the p-norms.

A brief examination of the smoothness and convexity properties of the $p-HH$ norms on $\mathbf{X}^n$ follows. In keeping with the methods of [11], an isometric embedding of $\mathbf{X}^n$ into a Lebesgue-Bochner space is given. This embedding facilitates the proofs of several of the geometrical results. A formula for the semi-inner products is also presented and is used to prove that (Gâteaux) smoothness of the space $\mathbf{X}^n$ is inherited from $\mathbf{X}$.

Extending the $p-HH$ norm from $\mathbf{X}^n$ to a suitable space of sequences reveals fundamental differences between the $p-HH$ norms and the p-norms. Although the resulting sequence spaces all lie between $\ell^1(\mathbf{X})$ and $\ell^\infty(\mathbf{X})$, it seems that the resemblance to $\ell^p(\mathbf{X})$ ends there. Examples are given, in the case $\mathbf{X}=\mathbb{R}$, to show that the $2-HH$ norm extends to a sequence space that strictly contains ℓ^1, that these sequence spaces need not be lattices, they need not be complete spaces, and they need not even be closed under a permutation of the terms of the sequence.

2. The $p-HH$ norm on $\mathbf{X}^n$

Let $(\mathbf{X},\|\cdot\|)$ be a normed space. In [11], Kikianty and Dragomir introduced the $p-HH$ norm on the vector space $\mathbf{X}^2$ by defining

$$\|\mathbf{x}\|_{p-HH}=\left(\int_0^1\|(1-t)x_1+tx_2\|^p\,dt\right)^{1/p}$$

for all $\mathbf{x}=(x_1,x_2)\in\mathbf{X}^2$. Here $1\le p<\infty$. In this section we extend the definition of the $p-HH$ norm to $\mathbf{X}^n$ for $n>2$ and investigate upper and lower bounds for this new norm, in terms of p-norm.

For the upper bound, we apply the unweighted case of the n-dimensional Hermite-Hadamard inequality. The general case is Theorem 5.20 of [15] but we provide an elementary proof of the special case that we use.

In Theorem 3 below we verify Conjecture 1 of [11] by proving a sharp lower bound for the $p-HH$ norm in terms of the p-norm on $\mathbf{X}^2$. We observe, moreover, that the best constant in this lower bound is the same for every normed space $\mathbf{X}$. This is not the case when $n>2$. Example 1 shows that when $n>2$ the sharp lower bound for the $p-HH$ norm in terms of the p-norm on $\mathbf{X}^n$ may genuinely depend on the norm of the underlying space $\mathbf{X}$. As a substitute for sharp lower bound obtained when $n=2$, we provide a sharp lower bound for the $p-HH$ norm in terms of the ∞-norm,

$$\|\mathbf{x}\|_\infty=\max\{\|x_1\|,\dots,\|x_n\|\}.$$

In this result the best constant does not depend on the space $\mathbf{X}$.

Definition 1. *Let $(\mathbf{X},\|\cdot\|)$ be a normed space, $n\ge 2$ be an integer, and $1\le p<\infty$. Set*

$$E_n=\{(u_1,\dots,u_{n-1})\in(0,1)^{n-1}:u_1+\dots+u_{n-1}<1\}.$$

When $(u_1,\dots,u_{n-1})\in E_n$ set $u_n=1-u_1-\dots-u_{n-1}$, and $du'=du_{n-1}\dots du_1$. For $\mathbf{x}=(x_1,\dots,x_n)\in\mathbf{X}^n$,

$$\|\mathbf{x}\|_{p-HH}=\left(\frac{1}{|E_n|}\int_{E_n}\|u_1x_1+\dots+u_nx_n\|^p\,du'\right)^{1/p}.$$

Here $|E_n|=\int_{E_n}du'$ is the measure of the set E_n.

Note that when $n=2$ this definition agrees with the one given in [11]. When $n=1$ it is convenient to set $\|\mathbf{x}\|_{p-HH}=\|x_1\|$ for $\mathbf{x}=(x_1)\in\mathbf{X}^1$.

There is a natural definition of the $p-HH$ norm when $p=\infty$ but it does not give a new norm. Indeed, for $\mathbf{x}=(x_1,\dots,x_n)\in\mathbf{X}^n$,

$$\|\mathbf{x}\|_{\infty-HH}=\sup_{(u_1,\dots,u_{n-1})\in E_n}\|u_1x_1+\dots+u_nx_n\|=\max\{\|x_1\|,\dots,\|x_n\|\}=\|\mathbf{x}\|_\infty.$$

Theorem 1. *Suppose $(\mathbf{X},\|\cdot\|)$ is a normed space, n is a positive integer, and $1\le p<\infty$. Then $\|\cdot\|_{p-HH}$ is a norm on $\mathbf{X}^n$.*

Proof. The triangle inequality in $\mathbf{X}$ shows that

$$(u_1,\dots,u_{n-1})\mapsto\|u_1x_1+\dots+u_nx_n\|^p$$

defines a continuous function on the closure of E_n, a compact set of finite measure. It follows that integral defining the $p-HH$ norm is finite. The norm is clearly non-negative and homogeneous. The triangle inequality follows readily from the triangle inequality in $\mathbf{X}$ and the Minkowski inequality. Now suppose that $\|\mathbf{x}\|_{p-HH}=0$. Then, $\|u_1x_1+\dots+u_nx_n\|^p=0$ for almost every $(u_1,\cdots,u_{n-1})\in E_n$. By continuity it is identically zero on E_n. In particular, it vanishes at the points $(1,0,0,\cdots,0)$, $(0,1,0,\cdots,0)$, $\cdots$, $(0,0,\cdots,0,1)$, and $(0,0,\cdots,0)$. This shows that $x_1=x_2=\dots=x_n=0$, and completes the proof. $\square$

When $\mathbf{X}=\mathbb{R}$ and $\mathbf{x}=(x_1,\dots,x_n)$ is a vector of positive real numbers, the $p-HH$ norm of $\mathbf{x}$ is the pth-hypergeometric mean of $(x_1,\dots,x_n)$, which is constructed from the unweighted hypergeometric R-function evaluated at $(x_1,\dots,x_n)$. See [1, p. 366-367] and [3, 32-33].

The $p-HH$ norms enjoy a simple relationship with each other and with the p-norms on $\mathbf{X}^n$. Since the integral defining the $p-HH$ norm is an average, Hölder's inequality shows that the $p-HH$ norm is increasing as a function of p on $[1,\infty)$. So for $1<p\le q<\infty$ we have

$$\|\mathbf{x}\|_{1-HH}\le\|\mathbf{x}\|_{p-HH}\le\|\mathbf{x}\|_{q-HH}\le\|\mathbf{x}\|_\infty\le\|\mathbf{x}\|_q\le\|\mathbf{x}\|_p\le\|\mathbf{x}\|_1.$$

It is interesting to compare this observation with Theorem 10 in Section 4.

To work effectively with the $p-HH$ norms we often need to make calculations involving integration over the simplex E_n. To assist with such calculations we offer the following useful changes of variable. Their proofs are left as exercises in multivariable Calculus.

Lemma 1. *Let n be a positive integer and $f:(0,1)^n\to\mathbb{R}$ be integrable. For $(u_1,\dots,u_{n-1})\in E_n$, set $u_n=1-u_1-\dots-u_{n-1}$ and $du'=du_{n-1}\dots du_1$. If σ is a permutation of $\{1,2,\dots,n\}$, then*

$$\int_{E_n}f(u_1,\dots,u_n)\,du'=\int_{E_n}f(u_{\sigma(1)},\dots,u_{\sigma(n)})\,du'. \tag{2.1}$$

Lemma 2. *Let $m\ge2$ and $n\ge2$ be integers and $f:(0,1)^{m+n}\to\mathbb{R}$ be integrable. For $(u_1,\dots,u_{n-1})\in E_n$, $(v_1,\dots,v_{m-1})\in E_m$, and $(w_1,\dots,w_{m+n-1})\in E_{m+n}$ set*

$$u_n=1-u_1-\dots-u_{n-1},\quad du'=du_{n-1}\dots du_1,$$
$$v_m=1-v_1-\dots-v_{m-1},\quad dv'=dv_{m-1}\dots dv_1,$$
$$w_{m+n}=1-w_1-\dots-w_{m+n-1},\quad dw'=dw_{m+n-1}\dots dw_1.$$

Then,

$$\begin{aligned}(2.2)\quad &\int_{E_{m+n}}f(w_1,\dots,w_{m+n})\,dw'\\ &=\int_0^1\int_{E_m}\int_{E_n}f(tv_1,\dots,tv_m,(1-t)u_1,\dots,(1-t)u_n)\,du'\,dv't^{m-1}(1-t)^{n-1}\,dt,\end{aligned}$$

(2.3)
$$\int_{E_{n+1}} f(w_1, \ldots, w_{n+1})\, dw' = \int_0^1 \int_{E_n} f(t, (1-t)u_1, \ldots, (1-t)u_n)\, du'(1-t)^{n-1}\, dt$$

and

$$\int_{E_{m+1}} f(w_1, \ldots, w_{m+1})\, dw' = \int_0^1 \int_{E_m} f(tv_1, \ldots, tv_m, 1-t)\, dv' t^{m-1}\, dt. \tag{2.4}$$

With $f \equiv 1$ equation (2.3) becomes

$$|E_{n+1}| = \int_{E_{n+1}} dw' = \int_0^1 \int_{E_n} du'(1-t)^{n-1}\, dt = \frac{1}{n}|E_n|$$

and by induction we find that $|E_n| = 1/(n-1)!$.

With these in hand we can easily prove the following (unweighted) n-dimensional Hermite-Hadamard inequality.

Theorem 2. *Suppose* $\mathbf{X}$ *is a vector space,* $n \geq 2$ *is an integer, and* $f : \mathbf{X} \to \mathbb{R}$ *is convex. If* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$*, then*

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{1}{|E_n|} \int_{E_n} f(u_1 x_1 + \cdots + u_n x_n)\, du' \leq \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

Proof. Let S_n denote the collection of all permutations of $\{1, \ldots, n\}$ and note that S_n has $n!$ elements. Let $(u_1, \ldots, u_{n-1}) \in E_n$ and set $u_n = 1 - u_1 - \cdots - u_{n-1}$. For each i,

$$\sum_{\sigma \in S_n} u_{\sigma(i)} = (n-1)!$$

because each of $u_1, \ldots, u_n$ occurs exactly $(n-1)!$ times in the sum and $u_1 + \cdots + u_n = 1$.

By Lemma 1,

$$\begin{aligned}
& \frac{1}{|E_n|} \int_{E_n} f(u_1 x_1 + \cdots + u_n x_n)\, du' \\
= \; & \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{|E_n|} \int_{E_n} f(u_{\sigma(1)} x_1 + \cdots + u_{\sigma(n)} x_n)\, du' \\
= \; & \frac{1}{|E_n|} \int_{E_n} \frac{1}{n!} \sum_{\sigma \in S_n} f(u_{\sigma(1)} x_1 + \cdots + u_{\sigma(n)} x_n)\, du'. \qquad (2.5)
\end{aligned}$$

Since f is convex and $u_{\sigma(1)} + \cdots + u_{\sigma(n)} = 1$ for all $\sigma \in S_n$,

$$\begin{aligned}
& \frac{1}{n!} \sum_{\sigma \in S_n} f(u_{\sigma(1)} x_1 + \cdots + u_{\sigma(n)} x_n) \\
\leq \; & \frac{1}{n!} \sum_{\sigma \in S_n} \big(u_{\sigma(1)} f(x_1) + \cdots + u_{\sigma(n)} f(x_n)\big) \\
= \; & \left(\frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)}\right) f(x_1) + \cdots + \left(\frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(n)}\right) f(x_n) \\
= \; & \frac{f(x_1) + \cdots + f(x_n)}{n}.
\end{aligned}$$

On the other hand, the convexity of f also yields

$$\begin{aligned}
&\frac{1}{n!}\sum_{\sigma\in S_n} f(u_{\sigma(1)}x_1+\cdots+u_{\sigma(n)}x_n)\\
\geq\ & f\left(\frac{1}{n!}\sum_{\sigma\in S_n}(u_{\sigma(1)}x_1+\cdots+u_{\sigma(n)}x_n)\right)\\
=\ & f\left(\left(\frac{1}{n!}\sum_{\sigma\in S_n}u_{\sigma(1)}\right)x_1+\cdots+\left(\frac{1}{n!}\sum_{\sigma\in S_n}u_{\sigma(n)}\right)x_n\right)\\
=\ & f\left(\frac{x_1+\cdots+x_n}{n}\right).
\end{aligned}$$

Using these upper and lower bounds for the integrand in (2.5) completes the proof. □

The following corollary gives a sharp upper bound for the $p-HH$ norm in terms of the p-norm on $\mathbf{X}^n$.

Corollary 1. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, n a positive integer, and $1 \leq p < \infty$. For $\mathbf{x} = (x_1,\dots,x_n) \in \mathbf{X}^n$,*

$$\left\|\frac{x_1+\cdots+x_n}{n}\right\| \leq \|\mathbf{x}\|_{p-HH} \leq n^{-1/p}\|\mathbf{x}\|_p. \tag{2.6}$$

The inequalities reduce to equality when $x_1 = \cdots = x_n$.

Proof. If $n = 1$ the statement holds trivially. If $n \geq 2$, note that $f(x) = \|x\|^p$ is a convex function on $\mathbf{X}$. With this f, the conclusion of the previous theorem easily implies (2.6); just take pth roots.

When $\mathbf{x} = (x,\dots,x)$ for some $x \in \mathbf{X}$,

$$\begin{aligned}
\left\|\frac{x_1+\cdots+x_n}{n}\right\| &= \|x\|,\\
\|\mathbf{x}\|_{p-HH} &= \left(\frac{1}{|E_n|}\int_{E_n}\|x\|^p\,du'\right)^{1/p} = \|x\|, \quad \text{and}\\
n^{-1/p}\|\mathbf{x}\|_p &= n^{-1/p}\left(\|x\|^p+\cdots+\|x\|^p\right)^{1/p} = \|x\|.
\end{aligned}$$
□

Obtaining a lower bound for the $p-HH$ norm in terms of the p-norm is more delicate. We begin with the case $n = 2$, giving the best constant conjectured in [11] as the lower bound.

Theorem 3. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $1 \leq p < \infty$. For any $\mathbf{x} \in \mathbf{X}^2$*

$$(2p+2)^{-1/p}\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_{p-HH}. \tag{2.7}$$

Equality holds if and only if $\mathbf{x} = (x, -x)$ for some $x \in \mathbf{X}$.

Proof. Let $\mathbf{x} = (x_1, x_2) \in \mathbf{X}^2$. For any $t \in (0,1)$,

$$(1-2t)x_1 = (1-t)((1-t)x_1+tx_2)+t(-(1-t)x_2-tx_1).$$

Since the the map $x \to \|x\|^p$ is convex on $\mathbf{X}$,

$$|1-2t|^p\|x_1\|^p \leq (1-t)\|(1-t)x_1+tx_2\|^p + t\|(1-t)x_2+tx_1\|^p.$$

Adding this inequality to the one obtained by interchanging x_1 and x_2 yields,

$$|1-2t|^p\|\mathbf{x}\|_p^p \leq \|(1-t)x_1+tx_2\|^p + \|(1-t)x_2+tx_1\|^p. \tag{2.8}$$

Integrating this from $t = 0$ to $t = 1$ and using the fact that $\|(x_1, x_2)\|_{p-HH} = \|(x_2, x_1)\|_{p-HH}$ gives

$$\frac{1}{p+1}\|\mathbf{x}\|_p^p \leq 2\|\mathbf{x}\|_{p-HH}^p.$$

Dividing by 2 and taking pth roots gives the desired inequality.

If $x_1 = -x_2$, then all the inequalities in the above argument reduce to equations. Conversely, if (2.7) holds with equality, then (2.8) is equality for almost every t. By continuity, (2.8) is also equality when $t = 1/2$, which implies that $\|x_1 + x_2\| = 0$. Thus, equality holds in (2.7) only if $x_1 = -x_2$. □

In view of this result it is natural to ask for the best (greatest) constant c in the inequality

$$c\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_{p-HH} \tag{2.9}$$

for $\mathbf{x} \in \mathbf{X}^n$. However, as the next example shows, the constant c may be different for different spaces $\mathbf{X}$. As Theorem 3 showed, this cannot happen when $n = 2$.

Example 1. *Let $n = 3$ and $p = 2$. If $\mathbf{X} = \mathbb{R}$, the best constant for which (2.9) holds is $c = 1/\sqrt{12}$. However, if $\mathbf{X} = \mathbb{R}^2_\infty$ then (2.9) fails with $c = 1/\sqrt{12}$.*

Proof. First take $\mathbf{X} = \mathbb{R}$. For $\mathbf{x} = (x_1, x_2, x_3)$, a straightforward calculation shows that

$$\|\mathbf{x}\|_{2-HH}^2 = \tfrac{1}{6}(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1).$$

Since $\|\mathbf{x}\|_2^2 = x_1^2 + x_2^2 + x_3^2$, we see that

$$0 \leq (x_1 + x_2 + x_3)^2 = 12\|\mathbf{x}\|_{2-HH}^2 - \|\mathbf{x}\|_2^2,$$

which proves (2.9) with $c = 1/\sqrt{12}$. Take $\mathbf{x} = (1, -1, 0)$ to see that no larger value of c will do.

Now let $\mathbf{X} = \mathbb{R}^2_\infty$, that is, $\mathbf{X} = \mathbb{R}^2$ with norm $\|(t_1, t_2)\| = \max\{|t_1|, |t_2|\}$. Set $\mathbf{x} = (x_1, x_2, x_3)$, where $x_1 = (-1, 2)$, $x_2 = (-1, -2)$, and $x_3 = (2, 0)$. Calculations show that

$$\|\mathbf{x}\|_2^2 = 12 \quad \text{and} \quad \|\mathbf{x}\|_{2-HH}^2 = 437/450.$$

For (2.9) to hold for this vector $\mathbf{x}$ we must have $12c^2 \leq 437/450$ so (2.9) fails with $c = 1/\sqrt{12}$. □

Rather than continuing to pursue a lower bound involving the p-norm directly, we turn our attention to the ∞-norm and get a lower bound for the $p - HH$ norm in which the same constant is sharp for each normed space $\mathbf{X}$. Since the p-norm and the ∞-norm are equivalent, this approach gives, indirectly, a lower bound for the $p - HH$ norm in terms of the p-norm.

Theorem 4. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $n \geq 2$ and integer, and $1 \leq p < \infty$. The inequality*

$$\|\mathbf{x}\|_{p-HH} \geq c\|\mathbf{x}\|_\infty \tag{2.10}$$

holds for all $\mathbf{x} \in \mathbf{X}^n$, where

$$c^p = \inf_{1\leq s\leq 2}(n-1)\int_0^1 |1 - ts|^p t^{n-2}\, dt.$$

The constant c is strictly positive and best possible.

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$. The identity (2.1) implies that the $p - HH$ norm is invariant under permutations of $x_1, \ldots, x_n$ so we may permute $x_1, \ldots, x_n$ without changing either side of the inequality above. Therefore we may suppose without loss of generality that $\|x_1\| = \max\{\|x_1\|, \ldots, \|x_n\|\}$. Set $\bar{x} = (x_2 + \cdots + x_n)/(n-1)$ and note that $\|\bar{x}\| \leq \|x_1\|$.

Let σ be the $(n-1)$-cycle $(2 \ldots n)$ and apply σ to $x_1, \ldots, x_n$ repeatedly to get

$$\begin{aligned} \|(x_1, x_2, \ldots, x_n)\|_{p-HH} &= \tfrac{1}{n-1} \left(\|(x_1, x_2, \ldots, x_n)\|_{p-HH} \right. \\ &+ \|(x_1, x_3, \ldots, x_n, x_2)\|_{p-HH} \\ &+ \cdots + \left. \|(x_1, x_n, x_2 \ldots, x_{n-1})\|_{p-HH} \right) \\ &\geq \|(x_1, \bar{x}, \ldots, \bar{x})\|_{p-HH}^p. \end{aligned}$$

The last inequality above is the triangle inequality in the $p-HH$ norm.

If $n \geq 3$, (2.3) implies that

$$\begin{aligned} \|(x_1, \bar{x}, \ldots, \bar{x})\|_{p-HH}^p &= \frac{1}{|E_n|} \int_{E_n} \|w_1 x_1 + (1-w_1)\bar{x}\|^p \, dw' \\ &= \frac{1}{|E_n|} \int_0^1 \int_{E_{n-1}} \|t x_1 + (1-t)\bar{x}\|^p \, du' (1-t)^{n-2} \, dt \\ &= (n-1) \int_0^1 \|(1-t)x_1 + t\bar{x}\|^p t^{n-2} \, dt. \end{aligned}$$

It is straightforward to check that this equation also holds when $n = 2$. Putting these together and applying the triangle inequality in $\mathbf{X}$ shows that

$$\begin{aligned} \|(x_1, x_2, \ldots, x_n)\|_{p-HH}^p &\geq (n-1) \int_0^1 |(1-t)\|x_1\| - t\|\bar{x}\||^p t^{n-2} \, dt \\ &= (n-1) \int_0^1 |1 - t(1 + \|\bar{x}\|/\|x_1\|)|^p t^{n-2} \, dt \|x_1\|^p \\ &\geq c^p \|x_1\|^p. \end{aligned}$$

Observe that $\int_0^1 |1 - ts|^p t^{n-2} \, dt$ is a strictly positive, continuous function of s on $[1, 2]$. The infimum of such a function is strictly positive so c is strictly positive.

To complete the proof we show that c is the best possible constant in (2.10). If $1 \leq s \leq 2$ and $x \neq 0$, set

$$\mathbf{x} = (x, (1-s)x, \ldots, (1-s)x) \in \mathbf{X}^n$$

and note that $\|\mathbf{x}\|_\infty = \|x\|$. On the other hand, if $n \geq 3$ then (2.3) implies

$$\begin{aligned} \|\mathbf{x}\|_{p-HH}^p &= \frac{1}{|E_n|} \int_{E_n} \|w_1 x + (1-w_1)(1-s)x\|^p \, dw' \\ &= \frac{\|x\|^p}{|E_n|} \int_0^1 \int_{E_{n-1}} |t + (1-t)(1-s)|^p \, du' (1-t)^{n-2} \, dt \\ &= (n-1)\|x\|^p \int_0^1 |1 - ts|^p t^{n-2} \, dt. \end{aligned}$$

It is straightforward to check that this equation also holds when $n = 2$.

If follows that (2.10) fails for any constant larger than c so c is best possible. $\square$

Corollary 2. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, n a positive integer, and $1 \leq p < \infty$. Then the $p-HH$ norm is equivalent to the p-norm on $\mathbf{X}^n$. If $\mathbf{X}$ is a Banach space then $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is a Banach space. If $\mathbf{X}$ is reflexive then $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is also reflexive.*

Proof. The Hermite-Hadamard inequality gives an upper bound for the $p-HH$ norm in terms of the the p-norm and the previous theorem gives a lower bound for the $p-HH$ norm in terms of the ∞-norm. Since the ∞-norm is equivalent to the p-norm there is a lower bound for the $p-HH$ norm in terms of the p-norm and so the two norms are equivalent.

It is well known that if $\mathbf{X}$ is complete then $(\mathbf{X}^n, \|\cdot\|_p)$ is also complete. Since the $p-HH$ norm is equivalent to the p-norm, $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is complete as well.

It is an exercise to show that

$$(\mathbf{X}^n, \|\cdot\|_p)^* = ((\mathbf{X}^*)^n, \|\cdot\|_{p'}),$$

or, strictly speaking, to exhibit a natural isometric isomorphism between the two normed spaces. It follows that if $\mathbf{X}$ is reflexive then $(\mathbf{X}^n, \|\cdot\|_p)$ is also reflexive. The equivalence of the p-norm and the $p-HH$ norm implies that $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is reflexive whenever $\mathbf{X}$ is. □

To end this section we point out that if the norm in $\mathbf{X}$ is induced by a (real) inner product then both the $2-$norm and the $2-HH$ norm in $\mathbf{X}^n$ are also induced by inner products. It is easy to verify that the inner product that induces the 2-norm is

$$\langle \mathbf{x}, \mathbf{y} \rangle_2 = \langle x_1, y_1 \rangle + \cdots + \langle x_n, y_n \rangle$$

where $\mathbf{x} = (x_1, \dots, x_n)$, and $\mathbf{y} = (y_1, \dots, y_n)$ are in $\mathbf{X}^n$; and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{X}$. For convenience in expressing the formula for the inner product that induces the $2-HH$ norm we define $s(\mathbf{x}) = x_1 + \cdots + x_n$ for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{X}^n$.

Theorem 5. *Suppose $(\mathbf{X}, \langle \cdot, \cdot \rangle)$ is an inner product space and $n \geq 2$ is an integer. Then $\mathbf{X}^n$ is an inner product space with respect to the inner product*

$$\langle \mathbf{x}, \mathbf{y} \rangle_{2-HH} = \frac{1}{n(n+1)} \Big(\langle \mathbf{x}, \mathbf{y} \rangle_2 + \langle s(\mathbf{x}), s(\mathbf{y}) \rangle \Big)$$

and $\|\mathbf{x}\|_{2-HH}^2 = \langle \mathbf{x}, \mathbf{x} \rangle_{2-HH}$ for all $\mathbf{x} \in \mathbf{X}^n$.

Proof. It is a simple matter to check that the formula for $\langle \cdot, \cdot \rangle_{2-HH}$ given above does define an inner product. To verify that it induces the $2-HH$ norm suppose $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{X}^n$. Then

$$\begin{aligned}
\|\mathbf{x}\|_{2-HH}^2 &= \frac{1}{|E_n|} \int_{E_n} |u_1 x_1 + \cdots + u_n x_n|^2 \, du' \\
&= (n-1)! \int_{E_n} \langle u_1 x_1 + \cdots + u_n x_n, u_1 x_1 + \cdots + u_n x_n \rangle \, du' \\
&= \sum_{j=1}^n \sum_{k=1}^n (n-1)! \int_{E_n} u_j u_k \, du' \langle x_j, x_k \rangle \quad \text{and} \\
\langle \mathbf{x}, \mathbf{x} \rangle_{2-HH} &= \frac{1}{n(n+1)} \Big(\langle \mathbf{x}, \mathbf{x} \rangle_2 + \langle s(\mathbf{x}), s(\mathbf{x}) \rangle \Big) \\
&= \sum_{j=1}^n \sum_{k=1}^n \frac{\delta_{j\,k} + 1}{n(n+1)} \langle x_j, x_k \rangle
\end{aligned}$$

where $\delta_{j\,k}$ is 1 when $j = k$ and 0 otherwise.

It remains to show that

$$(n+1)! \int_{E_n} u_j u_k \, du' = \delta_{j\,k} + 1$$

for all j, k. By (2.1) it is enough to show that

$$(n+1)! \int_{E_n} u_1 u_2 \, du' = 1 \quad \text{and} \quad (n+1)! \int_{E_n} u_1^2 \, du' = 2.$$

A pair of straightforward calculations using (2.2) and (2.3) completes the proof. □

3. Convexity and smoothness

Although the $p-HH$ norm on $\mathbf{X}^n$ is equivalent to the p-norm, it is not identical. Geometrical properties such as convexity and smoothness are not preserved under equivalence of norms. In this section we investigate the extent to which geometrical properties of $\mathbf{X}$ are inherited by $\mathbf{X}^n$ when it is given the $p-HH$ norm. In addition, we give simple formulas for the semi-inner products on $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ in terms of the semi-inner products on $\mathbf{X}$.

Our approach follows the method of [11], giving an isometric embedding of $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ into a much larger space, the Lebesgue-Bochner space $L^p(E_n, \mathbf{X})$, which is known to inherit geometric properties of $\mathbf{X}$. Before introducing this space we recall the definitions of smoothness, Fréchet smoothness, strict convexity, and uniform convexity.

A normed space $(\mathbf{X}, \|\cdot\|)$ is called *smooth* provided its norm is Gâteaux differentiable away from zero. That is, the limit

$$\lim_{t\to 0} \tfrac{1}{t}(\|y+tx\| - \|y\|)$$

exists for all $x, y \in \mathbf{X}$ with $y \neq 0$. The space is called *Fréchet smooth* provided its norm is Fréchet differentiable away from zero. This means that for each $y \in \mathbf{X}$ with $y \neq 0$ there exists a continuous linear functional G_y such that

$$\lim_{\|h\|\to 0} \frac{|\|y+h\| - \|y\| - G_y(h)|}{\|h\|} = 0.$$

For any $x, y \in \mathbf{X}$ the functions $t \mapsto \|y+tx\|$ and $t \mapsto \frac{1}{2}\|y+tx\|^2$ are convex and therefore have one-sided derivatives. We denote the right- and left-hand derivatives of the norm by

$$(\nabla_+\|\cdot\|(y))(x) = \lim_{t\to 0^+} \tfrac{1}{t}(\|y+tx\|-\|y\|) \text{ and } (\nabla_-\|\cdot\|(y))(x) = \lim_{t\to 0^-} \tfrac{1}{t}(\|y+tx\|-\|y\|)$$

and the superior and inferior semi-inner products of the norm by

$$\langle x, y\rangle_s = \lim_{t\to 0^+} \tfrac{1}{2t}(\|y+tx\|^2 - \|y\|^2) \quad \text{and} \quad \langle x, y\rangle_i = \lim_{t\to 0^-} \tfrac{1}{2t}(\|y+tx\|^2 - \|y\|^2).$$

The chain rule gives the relationships

$$\langle x, y\rangle_s = \|y\|(\nabla_+\|\cdot\|(y))(x) \quad \text{and} \quad \langle x, y\rangle_i = \|y\|(\nabla_-\|\cdot\|(y))(x) \tag{3.1}$$

from which it is evident that $\mathbf{X}$ is smooth if and only if the superior and inferior semi-inner products are equal for all $x, y \in \mathbf{X}$. See [6], [7], [8] and [9] for further properties of the semi-inner products.

Let $S_{\mathbf{X}} = \{x \in \mathbf{X} : \|x\| = 1\}$ be the unit sphere in a normed space $(\mathbf{X}, \|\cdot\|)$. The space $\mathbf{X}$ is *strictly convex* provided

$$\|\lambda x + (1-\lambda)y\| < 1$$

whenever $0 < \lambda < 1$ and $x, y \in S_{\mathbf{X}}$ with $x \neq y$. The space $\mathbf{X}$ is *uniformly convex* provided for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|\tfrac{1}{2}(x+y)\| \leq 1 - \delta$$

whenever $x, y \in S_{\mathbf{X}}$ satisfy $\|x - y\| > \varepsilon$.

Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $n \geq 2$ an integer, and $1 \leq p < \infty$. Recall that

$$E_n = \{(u_1, \ldots, x_{n-1}) \in (0,1)^{n-1} : u_1 + \cdots + u_{n-1} < 1\},$$

$du' = du_{n-1} \ldots du_1$ and $u_n = 1 - u_1 - \cdots - u_{n-1}$. The Lebesgue-Bochner space $L^p(E_n, \mathbf{X})$ is the vector space of all $f : E_n \to \mathbf{X}$ such that the function

$$(u_1, \ldots, u_{n-1}) \mapsto \|f(u_1, \ldots, u_{n-1})\|^p$$

is integrable on E_n. The norm is given by

$$\|f\|_{L^p(E_n,\mathbf{X})} = \left(\frac{1}{|E_n|}\int_{E_n} \|f(u_1,\ldots,u_{n-1})\|^p\,du'\right)^{1/p}$$

and, as usual, functions that agree almost everywhere are taken to be equal. For properties of the Lebesgue-Bochner spaces see III.3 of [10] and for applications to the geometry of Banach spaces, see [16].

For each $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbf{X}^n$ we define the function $f : E_n \to \mathbf{X}$ by $f_{\mathbf{x}}(u_1,\ldots,u_{n-1}) = u_1x_1 + \cdots + u_nx_n$. Evidently, the map $\mathbf{x} \mapsto f_{\mathbf{x}}$ is an isometry from $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ into $L^p(E_n,\mathbf{X})$.

Using this embedding we show that both types of convexity are preserved as we pass from $\mathbf{X}$ to $(\mathbf{X}^n, \|\cdot\|_{p-HH})$, although we must exclude the case $p = 1$.

Theorem 6. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $n \geq 2$ an integer, and $1 < p < \infty$. If $\mathbf{X}$ is uniformly convex then so is $(\mathbf{X}^n, \|\cdot\|_{p-HH})$. If $\mathbf{X}$ is strictly convex then so is $(\mathbf{X}^n, \|\cdot\|_{p-HH})$.*

Proof. Suppose first that $\mathbf{X}$ is uniformly convex. By Theorem 2 (and the remark on page 507) of [4], $L^p(E_n,\mathbf{X})$ is also uniformly convex. See also [13] and [17]. It is clear from the definition that any subspace of a uniformly convex space is also uniformly convex. The above embedding shows that $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is isometrically isomorphic to a subspace of $L^p(E_n,\mathbf{X})$ and therefore $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is uniformly convex.

The uniform convexity of $\mathbb{R}$ is trivial, and it follows that $L^p(E_n,\mathbb{R})$ is uniformly convex and hence strictly convex. (See Theorem 5.2.6 in [14].)

Now suppose that $\mathbf{X}$ is strictly convex. The strict convexity of $L^p(E_n,\mathbb{R})$ and Theorem 6 of [5] together imply that $L^p(E_n,\mathbf{X})$ is strictly convex. The definition of strict convexity shows that any subspace of a strictly convex space is strictly convex. Since $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is isometrically isomorphic to a subspace of $L^p(E_n,\mathbf{X})$, it is also strictly convex. □

For Fréchet smoothness we exclude the case $p = 1$ and also require that $\mathbf{X}$ be complete.

Theorem 7. *Let $(\mathbf{X}, \|\cdot\|)$ be a Banach space, $n \geq 2$ an integer, and $1 < p < \infty$. If $\mathbf{X}$ is Fréchet smooth then so is $(\mathbf{X}^n, \|\cdot\|_{p-HH})$.*

Proof. The norm in the Banach space $\mathbf{X}$ is Fréchet differentiable away from zero so, according to Theorem 2.5 of [12], the norm in $L^p(E_n,\mathbf{X})$ is also Fréchet differentiable away from zero. In particular, the norm in $L^p(E_n,\mathbf{X})$ is Fréchet differentiable at each non-zero point of the isometric image of $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ in $L^p(E_n,\mathbf{X})$. It follows that $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is Fréchet smooth. □

The next result gives formulas for the one-sided derivatives and the semi-inner products for the $p - HH$ norm. In a slight abuse of notation we let

$$\mathbf{u}\cdot\mathbf{x} = u_1x_1 + \cdots + u_nx_n$$

where $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbf{X}^n$ and $(u_1,\ldots,u_{n-1}) \in E_n$, with $u_n = 1 - u_1 - \cdots - u_{n-1}$ as usual.

Theorem 8. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $n \geq 2$ an integer, and $1 \leq p < \infty$. For any $\mathbf{x},\mathbf{y} \in \mathbf{X}^n$ with $\mathbf{y} \neq 0$,*

$$(\nabla_+\|\cdot\|_{p-HH}(\mathbf{y}))(\mathbf{x}) = \|\mathbf{y}\|_{p-HH}^{1-p}\frac{1}{|E_n|}\int_{E_n}\|\mathbf{u}\cdot\mathbf{y}\|^{p-1}(\nabla_+\|\cdot\|(\mathbf{u}\cdot\mathbf{y}))(\mathbf{u}\cdot\mathbf{x})\,du'$$

and

$$\langle\mathbf{x},\mathbf{y}\rangle_{p-HH,s} = \|\mathbf{y}\|_{p-HH}^{2-p}\frac{1}{|E_n|}\int_{E_n}\|\mathbf{u}\cdot\mathbf{y}\|^{p-2}\langle\mathbf{u}\cdot\mathbf{x},\mathbf{u}\cdot\mathbf{y}\rangle_s\,du'.$$

Corresponding formulas hold for the left-hand derivative and the inferior semi-inner product.

Proof. First, observe that if $\mathbf{y} \neq 0$ then the set

$$\{(u_1, \ldots, u_{n-1}) \in E_n : \mathbf{u} \cdot \mathbf{y} = 0\}$$

is a section of an affine set of dimension $n-2$ and is therefore of measure zero in the $(n-1)$-dimensional set E_n. This ensures that the expressions $\|\mathbf{u} \cdot \mathbf{y}\|^{p-1}$ and $\|\mathbf{u} \cdot \mathbf{y}\|^{p-2}$ apearing above are well-defined and finite almost everywhere.

Fix $\mathbf{x}, \mathbf{y} \in \mathbf{X}^n$ with $\mathbf{y} \neq 0$ and define

$$f_t = f_t(u_1, \ldots, u_{n-1}) = \|\mathbf{u} \cdot (\mathbf{y} + t\mathbf{x})\|$$

for all $t \in (0,1)$ and for all $(u_1, \ldots, u_{n-1}) \in E_n$ satisfying $\mathbf{u} \cdot \mathbf{y} \neq 0$. The triangle inequality shows that $|f_t| \leq \|\mathbf{y}\|_1 + \|\mathbf{x}\|_1$ for all t and that

$$\tfrac{1}{t}(f_t - f_0) \leq \|\mathbf{u} \cdot \mathbf{x}\| \leq \|\mathbf{x}\|_1 \leq \|\mathbf{y}\|_1 + \|\mathbf{x}\|_1.$$

By the mean value theorem,

$$|\tfrac{1}{t}(f_t^p - f_0^p)| \leq p(\|\mathbf{y}\|_1 + \|\mathbf{x}\|_1)^{p-1} |\tfrac{1}{t}(f_t - f_0)| \leq p(\|\mathbf{y}\|_1 + \|\mathbf{x}\|_1)^p.$$

Thus, $\frac{1}{t}(f_t^p - f_0^p)$ is dominated by a constant independent of t and $(u_1, \ldots, u_{n-1})$.

For almost every $(u_1, \ldots, u_{n-1}) \in E_n$, $f_0 = \|\mathbf{u} \cdot \mathbf{y}\| \neq 0$ so the chain rule implies

$$\lim_{t \to 0^+} \tfrac{1}{t}(f_t^p - f_0^p) = p f_0^{p-1} (\nabla_+ \| \cdot \| (\mathbf{u} \cdot \mathbf{y}))(\mathbf{u} \cdot \mathbf{x})$$

and by Lebesgue's dominated convergence theorem,

$$\lim_{t \to 0^+} \frac{1}{t} \left(\int_{E_n} f_t^p \, du' - \int_{E_n} f_0^p \, du' \right) = \int_{E_n} p f_0^{p-1} (\nabla_+ \| \cdot \| (\mathbf{u} \cdot \mathbf{y}))(\mathbf{u} \cdot \mathbf{x}) \, du'.$$

Applying the chain rule again gives

$$\lim_{t \to 0^+} \tfrac{1}{t}(\|\mathbf{y}+t\mathbf{x}\|_{p-HH} - \|\mathbf{y}\|_{p-HH}) = \|\mathbf{y}\|_{p-HH}^{1-p} \int_{E_n} \|\mathbf{u}\cdot\mathbf{y}\|^{p-1} (\nabla_+ \|\cdot\| (\mathbf{u}\cdot\mathbf{y}))(\mathbf{u}\cdot\mathbf{x}) \, du',$$

the first formula of the theorem.

The second formula follows from the first by applying (3.1). With obvious minor modifications the proof will apply to the left-hand derivative and the inferior semi-inner product. □

These formulas imply that if the superior and inferior semi-inner products of $\mathbf{X}$ agree then the superior and inferior semi-inner products of $(\mathbf{X}^n, \| \cdot \|_{p-HH})$ agree, giving the following corollary.

Corollary 3. *Let $(\mathbf{X}, \| \cdot \|)$ be a normed space, $n \geq 2$ an integer, and $1 \leq p < \infty$. If $\mathbf{X}$ is smooth then so is $(\mathbf{X}^n, \| \cdot \|_{p-HH})$.*

Proof. Since $\mathbf{X}$ is smooth, $\langle x, y \rangle_s = \langle x, y \rangle_i$ for all $x, y \in \mathbf{X}$. It follows that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^n$ with $\mathbf{y} \neq 0$, and for almost all $(u_1, \ldots, u_{n-1}) \in E_n$,

$$\|\mathbf{u} \cdot \mathbf{y}\|^{2-p} \langle \mathbf{u} \cdot \mathbf{x}, \mathbf{u} \cdot \mathbf{y} \rangle_s = \|\mathbf{u} \cdot \mathbf{y}\|^{2-p} \langle \mathbf{u} \cdot \mathbf{x}, \mathbf{u} \cdot \mathbf{y} \rangle_i.$$

Theorem 8 implies that $\langle \mathbf{x}, \mathbf{y} \rangle_{p-HH,s} = \langle \mathbf{x}, \mathbf{y} \rangle_{p-HH,i}$ for all $\mathbf{y} \neq 0$. It also holds when $\mathbf{y} = 0$, from the definition of the semi-inner products. Equality of these two semi-inner products for the $p-HH$ norm implies that $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is smooth. □

4. The h^p spaces

In this section we introduce a space of sequences of elements of the normed space $\mathbf{X}$. The norm in this sequence space will be based on the $p-HH$ norm in $\mathbf{X}^n$. To do this we first renormalize the $p-HH$ norms so that the embedding $(x_1,\dots,x_n)\mapsto(x_1,\dots,x_n,0)$ of $\mathbf{X}^n$ into $\mathbf{X}^{n+1}$ is an isometry. For $1\le p<\infty$ and $n\ge 2$ we define the space $h_n^p=h_n^p[\mathbf{X}]$ to be $\mathbf{X}^n$ with norm

$$\|(x_1,\dots,x_n)\|_{h_n^p}=\left(\frac{\Gamma(p+n)}{\Gamma(p+1)\Gamma(n)}\right)^{1/p}\|(x_1,\dots,x_n)\|_{p-HH}.$$

For convenience we let $h_1^p[\mathbf{X}]=\mathbf{X}$, with identical norms.

Define

$$h^p=h^p[\mathbf{X}]=\left\{(x_1,x_2,\dots):\lim_{N\to\infty}\sup_{n>m\ge N}\|(x_{m+1},\dots,x_n)\|_{h_{n-m}^p}=0\right\}$$

and, for $(x_1,x_2,\dots)\in h^p$, define

$$\|(x_1,x_2,\dots)\|_{h^p}=\lim_{n\to\infty}\|(x_1,x_2,\dots,x_n)\|_{h_n^p}. \tag{4.1}$$

Some work is required before we can show that h^p is a normed space.

Theorem 9. *The embedding $(x_1,\dots,x_n)\mapsto(x_1,\dots,x_n,0)$ of h_n^p into h_{n+1}^p is an isometry for $n\ge 1$.*

Proof. If $n=1$ and $x_1\in\mathbf{X}$ we have

$$\begin{aligned}\|(x_1,0)\|_{h_2^p}^p &= \frac{\Gamma(p+2)}{\Gamma(p+1)}\int_{E_2}\|w_1x_1+(1-w_1)0\|^p\,dw'\\ &= \|x_1\|^p(p+1)\int_0^1 w_1^p\,dw_1=\|x_1\|^p=\|x_1\|_{h_1^p}^p.\end{aligned}$$

Suppose $n>1$ and $x_1,\dots x_n\in\mathbf{X}$. Applying (2.4) with m replaced by n yields

$$\begin{aligned}\|(x_1,\dots,x_n,0)\|_{h_{n+1}^p}^p &= \frac{\Gamma(p+n+1)}{\Gamma(p+1)}\int_{E_{n+1}}\|w_1x_1+\cdots+w_nx_n+w_{n+1}0\|^p\,dw'\\ &= \frac{\Gamma(p+n+1)}{\Gamma(p+1)}\int_0^1\int_{E_n}\|tv_1x_1+\cdots+tv_nx_n\|^p\,dv't^{n-1}\,dt\\ &= \frac{\Gamma(p+n+1)}{\Gamma(p+1)}\int_0^1 t^{p+n-1}\,dt\int_{E_n}\|v_1x_1+\cdots+v_nx_n\|^p\,dv'\\ &= \frac{\Gamma(p+n+1)}{\Gamma(p+n)}\frac{1}{p+n}\|(x_1,\dots,x_n)\|_{h_n^p}^p\\ &= \|(x_1,\dots,x_n)\|_{h_n^p}^p.\end{aligned}$$

This completes the proof. □

The change of variable (2.1) shows that the norm in h_n^p is invariant under permutations of $x_1,\dots,x_n$. This observation, together with the embedding lemma just given, enables us to show that the limit in (4.1) exists for every $(x_1,x_2,\dots)\in h^p$: It is enough to show that the sequence $\|(x_1,\dots,x_n)\|_{h_n^p}$ is Cauchy. If $m<n$, then

$$\begin{aligned}\|(x_1,\dots,x_n)\|_{h_n^p} &\le \|(x_1,\dots,x_m,0\dots,0)\|_{h_n^p}+\|(0,\dots,0,x_{m+1},\dots,x_n)\|_{h_n^p}\\ &= \|(x_1,\dots,x_m)\|_{h_m^p}+\|(x_{m+1},\dots,x_n)\|_{h_{n-m}^p}\end{aligned}$$

so

$$\|(x_1,\dots,x_n)\|_{h_n^p}-\|(x_1,\dots,x_m)\|_{h_m^p}\le\|(x_{m+1},\dots,x_n)\|_{h_{n-m}^p}.$$

The definition of h^p shows that the last term goes to zero as m and n go to infinity.

Theorem 10. *If* $\mathbf{X}$ *is a normed space then* $h^p = h^p[\mathbf{X}]$ *is a normed space. Moreover, if* $1 \le p \le q < \infty$ *then*

$$\ell^1(\mathbf{X}) \subset h^q[\mathbf{X}] \subset h^p[\mathbf{X}] \subset \ell^\infty(\mathbf{X})$$

with continuous inclusions.

Proof. It is easy to verify that h^p is a vector space of sequences of elements of $\mathbf{X}$ and that (4.1) defines a non-negative function that is positive homogeneous and satisfies the triangle inequality. Theorem 4 may be used to show that the limit in (4.1) is zero only when $(x_1, x_2, \dots) = (0, 0, \dots)$ but first we need an estimate of the constant c for $n \ge 2$. Set

$$\varphi(s) = (n-1)\int_0^1 |1-ts|t^{n-2}\,dt$$

and split the integral at $t = 1/s$ to calculate

$$\varphi(s) = s - 1 - \frac{s}{n}\left(1 - \frac{2}{s^n}\right) \quad \text{and} \quad \varphi'(s) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{s^n}\right).$$

Since φ is decreasing on $[1, 2^{1/n}]$ and increasing on $[2^{1/n}, 2]$ its infimum is $\varphi(2^{1/n}) = 2^{1/n} - 1$. By Hölder's inequality,

$$\begin{aligned} c &= \inf_{1\le s\le 2}\left((n-1)\int_0^1 |1-ts|^p t^{n-2}\,dt\right)^{1/p} \\ &\ge \inf_{1\le s\le 2}(n-1)\int_0^1 |1-ts|t^{n-2}\,dt = 2^{1/n} - 1. \end{aligned}$$

By Theorem 4 and the definition of the norm in h_n^p

$$\|(x_1, \dots, x_n)\|_{h_n^p} \ge (2^{1/n} - 1)\left(\frac{\Gamma(p+n)}{\Gamma(p+1)\Gamma(n)}\right)^{1/p} \max\{\|x_1\|, \dots, \|x_n\|\}.$$

The limit as $n \to \infty$ of $\max\{\|x_1\|, \dots, \|x_n\|\}$ is $\|(x_1, x_2, \dots)\|_{\ell^\infty(\mathbf{X})}$ and Stirling's formula shows that

$$\lim_{n\to\infty}(2^{1/n} - 1)\left(\frac{\Gamma(p+n)}{\Gamma(p+1)\Gamma(n)}\right)^{1/p} = \frac{\log 2}{\Gamma(p+1)^{1/p}}.$$

Thus,

$$\|(x_1, x_2 \dots)\|_{h^p} \ge (\log 2)\Gamma(p+1)^{-1/p}\|(x_1, x_2, \dots)\|_{\ell^\infty(\mathbf{X})}.$$

This finishes the proof that (4.1) defines a norm by showing that only the zero vector in h^p can have zero norm. It also proves that h^p is contained in $\ell^\infty(\mathbf{X})$ with continuous inclusion.

Next we show that h^p contains $\ell^1(\mathbf{X})$. If $0 \le m < n$ then the permutation invariance of the h_n^p norm, together with the isometry of the embeddings $h_n^p \hookrightarrow h_{n+1}^p$ yields

$$\begin{aligned} \|(x_{m+1}, \dots, x_n)\|_{h_{n-m}^p} &\le \|(x_{m+1}, 0, \dots, 0)\|_{h_{n-m}^p} + \cdots + \|(0, \dots, 0, x_n)\|_{h_{n-m}^p} \\ &= \|x_{m+1}\| + \cdots + \|x_n\|. \end{aligned}$$

If $(x_1, x_2, \dots) \in \ell^1(\mathbf{X})$ then this sum tends to zero as $m, n \to \infty$ so, by definition, $(x_1, x_2, \dots) \in h^p$. Moreover, taking $m = 0$ above gives,

$$\begin{aligned} \|(x_1, x_2, \dots)\|_{h^p} &= \lim_{n\to\infty}\|(x_1, \dots, x_n)\|_{h_n^p} \\ &\le \lim_{n\to\infty}(\|x_1\| + \cdots + \|x_n\|) = \|(x_1, x_2, \dots)\|_{\ell^1(\mathbf{X})}. \end{aligned}$$

This shows that the inclusion is continuous.

As mentioned previously, the $p-HH$ norm is defined as an integral average so Hölder's inequality shows that for any $\mathbf{x} \in \mathbf{X}^n$,

$$\|\mathbf{x}\|_{p-HH} \leq \|\mathbf{x}\|_{q-HH}$$

when $p \leq q$. In terms of the h^p and h^q norms this is,

$$\|\mathbf{x}\|_{h_n^p} \leq \frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}} \left(\frac{\Gamma(p+n)}{\Gamma(n)}\right)^{1/p} \left(\frac{\Gamma(n)}{\Gamma(q+n)}\right)^{1/p} \|\mathbf{x}\|_{h_n^q}. \tag{4.2}$$

By Stirling's formula,

$$\lim_{n\to\infty} \left(\frac{\Gamma(p+n)}{\Gamma(n)}\right)^{1/p} \left(\frac{\Gamma(n)}{\Gamma(q+n)}\right)^{1/p} = 1.$$

Therefore, the constant

$$C_{p,q} = \sup_n \frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}} \left(\frac{\Gamma(p+n)}{\Gamma(n)}\right)^{1/p} \left(\frac{\Gamma(n)}{\Gamma(q+n)}\right)^{1/p},$$

is finite, independent of n, and satisfies

$$\|\mathbf{x}\|_{h_n^p} \leq C_{p,q} \|\mathbf{x}\|_{h_n^q}.$$

This implies that $h^q \subset h^p$. In addition, taking the limit in (4.2) yields

$$\|\mathbf{x}\|_{h^p} \leq \frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}} \|\mathbf{x}\|_{h^q}$$

for all $\mathbf{x} \in h^q$, showing that the inclusion is continuous. □

Remark 1. *Since h^p contains ℓ^1 it contains all sequences that are eventually zero. Theorem 9 shows that for these sequences the norm in h^p reduces to the norm in h_n^p for some n. That is,*

$$\|(x_1, x_2, \ldots, x_n, 0, 0, \ldots)\|_{h^p} = \|(x_1, x_2, \ldots, x_n)\|_{h_n^p}.$$

Remark 2. *It is important to distinguish between the spaces $h^p[\mathbf{X}]$ and $h^p[\mathbb{R}](\mathbf{X})$. The latter provides a norm on the space*

$$h^p[\mathbb{R}](\mathbf{X}) = \{(x_1, x_2, \ldots) : (\|x_1\|, \|x_2\|, \ldots) \in h^p[\mathbb{R}]\}$$

given by

$$\|(x_1, x_2, \ldots)\|_{h^p[\mathbb{R}](\mathbf{X})} = \|(\|x_1\|, \|x_2\|, \ldots)\|_{h^p[\mathbb{R}]}.$$

Even in the case $\mathbf{X} = \mathbb{R}$ the spaces $h^p[\mathbf{X}]$ and $h^p[\mathbb{R}](\mathbf{X})$ are not the same, although in this special case the two norms do coincide on vectors with non-negative entries.

The next example shows that the spaces $h^p[\mathbf{X}]$ need not be complete, even if the underlying space $\mathbf{X}$ is complete. In the example, $\mathbf{X} = \mathbb{R}$ but, since every non-trivial normed space contains an isometric copy of $\mathbb{R}$, the example is easily adapted to any $\mathbf{X}$.

Example 2. *The normed space $h^2[\mathbb{R}]$ is not complete.*

Proof. Consider the sequence $(a, \ldots, a, b, \ldots, b, 0, 0, \ldots)$ in which the first m entries equal $a \in \mathbb{R}$, the next n entries equal $b \in \mathbb{R}$ and the rest of the entries are zero. If

$m,n \geq 2$ we use (2.2) to get

$$\begin{aligned}
& \|(a,\dots,a,b,\dots,b,0,0,\dots)\|_{h^2}^2 \\
= \ & \|(a,\dots,a,b,\dots,b)\|_{h^2_{m+n}}^2 \\
= \ & \frac{(m+n+1)!}{2}\int_{E_{m+n}} |(w_1+\dots+w_m)a+(w_{m+1}+\dots+w_{m+n})b|^2\,dw' \\
= \ & \frac{(m+n+1)!}{2}\int_0^1\int_{E_m}\int_{E_n}(ta+(1-t)b)^2\,dv'\,du'\,t^{m-1}(1-t)^{n-1}\,dt \\
= \ & \frac{(m+n+1)!}{2(m-1)!(n-1)!}\int_0^1 (ta+(1-t)b)^2\,t^{m-1}(1-t)^{n-1}\,dt \\
= \ & \tfrac{1}{2}m(m+1)a^2+mnab+\tfrac{1}{2}n(n+1)b^2.
\end{aligned}$$

Similar arguments using (2.3), (2.4) show that the conclusion remains valid when $m,n \geq 0$.

In particular, if $\xi_n = (\frac{1}{n},\dots\frac{1}{n},0,0,\dots)$ is chosen to have exactly n non-zero entries then $\|\xi_n\|_{h^2}^2 = (n+1)/(2n)$. Since $\|\xi_n\|_{h^2} \to 1/\sqrt{2}$ as $n\to\infty$ the sequence $\{\xi_n\}$ does not converge to 0 in h^2. However, ξ_n does converge to 0 in ℓ^∞ so $\{\xi_n\}$ cannot have a limit at all in the smaller space h^2.

On the other hand, the above calculation shows that

$$\begin{aligned}
\|\xi_{m+n}-\xi_m\|_{h^2}^2 &= \tfrac{1}{2}m(m+1)\left(\frac{1}{m+n}-\frac{1}{m}\right)^2 \\
&+ mn\left(\frac{1}{m+n}-\frac{1}{m}\right)\left(\frac{1}{m+n}\right) \\
&+ \tfrac{1}{2}n(n+1)\left(\frac{1}{m+n}\right)^2 \\
&= \frac{n}{2m(m+n)} \leq \frac{1}{2m}.
\end{aligned}$$

Since $\|\xi_{m+n}-\xi_m\|_{h^2}\to 0$ uniformly in n as $m\to\infty$ the sequence $\{\xi_n\}$ is a Cauchy sequence in h^2. As we have seen, $\{\xi_n\}$ does not converge in h^2. Thus h^2 is not complete. $\square$

Remark 3. *The formula,*

$$\|(a,\dots,a,b,\dots,b,0,0,\dots)\|_{h^2}^2 = \tfrac{1}{2}m(m+1)a^2+mnab+\tfrac{1}{2}n(n+1)b^2$$

given above, shows that $h^2[\mathbb{R}]$ does not have the lattice property since replacing a by $-a$ may affect the norm in $h^2[\mathbb{R}]$.

When $\mathbf{X}$ is an inner product space, $h^2[\mathbf{X}]$ is too. Also, there is a simple formula relating their inner products. Recall that $s:\mathbf{X}^n\to\mathbf{X}$ was defined earlier by $s(x_1,\dots,x_n)=x_1+\dots+x_n$. By identifying $(x_1,\dots,x_n)\in\mathbf{X}^n$ with the sequence $(x_1,\dots,x_n,0,0,\dots)$ we can extend this definition to

$$s(x_1,x_2,\dots)=x_1+x_2+\dots$$

for all sequences $(x_1,x_2,\dots)$ that are eventually zero.

Theorem 11. *If $\mathbf{X}$ is a (real) inner product space, then $h^2=h^2[\mathbf{X}]\subset\ell^2(\mathbf{X})$, the operator s extends uniquely to a bounded linear operator on h^2, and h^2 is an inner product space satisfying*

$$\langle \mathbf{x},\mathbf{y}\rangle_{h^2} = \frac{1}{2}\Big(\langle \mathbf{x},\mathbf{y}\rangle_2+\langle s(\mathbf{x}),s(\mathbf{y})\rangle\Big) \tag{4.3}$$

for all $\mathbf{x},\mathbf{y}\in h^2$.

Proof. By Theorem 5, $(\mathbf{X}^n, \|\cdot\|_{2-HH})$ is an inner product space and consequently so is h^2_n. Moreover, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^n$,

$$\langle \mathbf{x}, \mathbf{y}\rangle_{h^2_n} = \tfrac{1}{2}n(n+1)\langle \mathbf{x}, \mathbf{y}\rangle_{2-HH} = \frac{1}{2}\Big(\langle \mathbf{x}, \mathbf{y}\rangle_2 + \langle s(\mathbf{x}), s(\mathbf{y})\rangle\Big).$$

Taking $\mathbf{y} = \mathbf{x}$ in this equation implies

$$\|\mathbf{x}\|_2 \le \sqrt{2}\|\mathbf{x}\|_{h^2_n} \tag{4.4}$$

and

$$\|s(\mathbf{x})\| \le \sqrt{2}\|\mathbf{x}\|_{h^2_n}. \tag{4.5}$$

For $\mathbf{x} = (x_1, x_2, \dots) \in h^2$ set $\mathbf{x}^{(n)} = (x_1, \dots, x_n) \in \mathbf{X}^n$. Inequality (4.4) shows that

$$\|\mathbf{x}\|_2 = \lim_{n\to\infty} \|\mathbf{x}^{(n)}\|_2 \le \sqrt{2} \lim_{n\to\infty} \|\mathbf{x}^{(n)}\|_{h^2} = \sqrt{2}\|\mathbf{x}\|_{h^2}.$$

Thus, $\mathbf{x} \in \ell^2(\mathbf{X})$ and we have $h^2 \subset \ell^2(\mathbf{X})$.

Inequality (4.5) shows that if $x_1, x_2, \dots$ is eventually zero, then

$$\|s(\mathbf{x})\| = \lim_{n\to\infty} \|s(\mathbf{x}^{(n)})\| \le \sqrt{2} \lim_{n\to\infty} \|x^{(n)}\|_{h^2} = \sqrt{2}\|x\|_{h^2}.$$

The definition of h^2 implies that $\mathbf{x}^{(n)} \to \mathbf{x}$ in h^2 so the space of sequences that are eventually zero is dense in h^2. We have shown that the linear operator s is densely defined and bounded on h^2. It therefore extends uniquely to a bounded linear map on h^2, which we also denote by s.

The map

$$(\mathbf{x}, \mathbf{y}) \mapsto \frac{1}{2}\Big(\langle \mathbf{x}, \mathbf{y}\rangle_2 + \langle s(\mathbf{x}), s(\mathbf{y})\rangle\Big)$$

is an inner product on h^2 and the norm it defines,

$$\mathbf{x} \mapsto \tfrac{1}{2}\left(\|\mathbf{x}\|_2^2 + \|s(\mathbf{x})\|^2\right),$$

agrees with the norm in h^2 on a dense subset. Therefore, h^2 is an inner product space and (4.3) holds for all $\mathbf{x}, \mathbf{y} \in h^2$. □

Remark 4. *If $\mathbf{X}$ is an inner product space then $\ell^2(\mathbf{X}) \ne h^2[\mathbf{X}]$. To see this, fix a unit vector $x \in \mathbf{X}$. The sequence $(x, x/2, x/3, \dots)$ is in $\ell^2(\mathbf{X})$ because the series $1^2 + (1/2)^2 + (1/3)^2 + \dots$ converges. However, for any m,*

$$\sup_{m<n} \|(x/(m+1), \dots, x/n)\|^2_{h^2_{n-m}} = \sup_{m<n} \frac{1}{2}\left(\sum_{k=m+1}^{n} \frac{1}{k^2} + \left(\sum_{k=m+1}^{n} \frac{1}{k}\right)^2\right) = \infty.$$

By definition, $(x, x/2, x/3, \dots) \notin h^2$.

In the next example we construct an element of h^2 that is not in ℓ^1, showing that the inclusion $\ell^1 \subset h^2$ is strict.

Example 3. *When $\mathbf{X} = \mathbb{R}$, $h^2 \not\subset \ell^1$.*

Proof. The sequence $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots)$ is not in ℓ^1. However, if $m < n$ then

$$\left\|\left(\frac{(-1)^m}{m+1}, \dots, \frac{(-1)^{n-1}}{n}\right)\right\|_{h^2_{n-m}} = \frac{1}{2}\left(\sum_{j=m+1}^{n} \frac{1}{j^2} + \left|\sum_{j=m+1}^{n} \frac{(-1)^{j-1}}{j}\right|^2\right)$$

can be made arbitrarily small by taking m sufficiently large. This shows that $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots)$ is in h^2. □

The permutation invariance of the p-norms carries over from finite-dimensional spaces to sequence spaces. In contrast, the permutation invariance of the norm on h_n^p may be lost in the transition to h^p. We have seen that $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots) \in h^2$ but it is a simple matter to rearrange the terms of the conditionally convergent series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ so that its partial sums are unbounded. The resulting sequence is not in h^2.

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