TRANSFERRING MONOTONICITY 
IN WEIGHTED NORM INEQUALITIES

GORD SINNAMON

The University of Western Ontario

November 11, 2002

Abstract. Certain weighted norm inequalities for integral operators with non-
negative, monotone kernels are shown to remain valid when the weight is replaced by
a monotone majorant or minorant of the original weight. A similar result holds for
operators with quasi-concave kernels. To prove these results a careful investigation
of the functional properties of the monotone envelopes of a non-negative function
is carried out. Applications are made to function space embeddings of the cones of
monotone functions and quasi-concave functions.

Under weaker partial orders on non-negative functions, monotone envelopes are
re-examined and the level function is recognized as a monotone envelope in two
ways. Using the level function, monotonicity can be transferred from the kernel to
the weight in inequalities restricted to a cone of monotone functions.

1. Introduction

This paper is a contribution to the theory of weighted norm inequalities for
positive integral operators but will also be of interest to those studying function
spaces. We show that the monotonicity of the kernel of an integral operator can be
transferred to the weight in a norm inequality. The result is applied to embeddings
of the cone of non-increasing functions between general rearrangement-invariant
spaces and to embeddings of the cone of quasi-concave functions between weighted
Lebesgue spaces.

The usual partial order on non-negative functions is pointwise, that is, \( u \leq v \)
provided \( u(x) \leq v(x) \) for (almost) all \( x \). For non-negative, \( \lambda \)-measurable functions

\[ \text{2000 Mathematics Subject Classification. Primary 26D15; Secondary 46E30.} \]
\[ \text{Key words and phrases. Inequalities, Weights, Monotonicity, Quasi-concave functions.} \]

Support from the Natural Sciences and Engineering Research Council of Canada is gratefully
acknowledged.
on $\mathbb{R}$ we can look at the weaker order relation $u \leq \downarrow v$ given by

$$\int_{(\infty, x]} u \, d\lambda \leq \int_{(\infty, x]} v \, d\lambda, \quad x \in \mathbb{R}.$$ 

This relation is important in the study of monotone functions because $u \leq \downarrow v$ if and only if

$$\int_{\mathbb{R}} f u \, d\lambda \leq \int_{\mathbb{R}} f v \, d\lambda$$

for all non-negative, non-increasing functions $f$. (See Corollary 1.3 below.) Naturally, there is a corresponding relation $u \leq \uparrow v$ defined by

$$\int_{[x, \infty)} u \, d\lambda \leq \int_{[x, \infty)} v \, d\lambda, \quad x \in \mathbb{R}$$

and satisfying $u \leq \uparrow v$ if and only if

$$\int_{\mathbb{R}} f u \, d\lambda \leq \int_{\mathbb{R}} f v \, d\lambda$$

for all non-negative, non-decreasing functions $f$. Although the relations $\leq \downarrow$ and $\leq \uparrow$ are reflexive and transitive on the set of all non-negative functions they are not partial orders on this large domain because antisymmetry may fail when the integrals used to define them are infinite.

It is essential to understand the interplay between these order relations when working with monotone functions. To further this understanding we examine, in Section 2, the four monotone envelopes of a non-negative function $u$: The least non-increasing majorant, the greatest non-increasing minorant, the least non-decreasing majorant, and the greatest non-decreasing minorant. There are not just four monotone envelopes, however, as we can change order relations and thereby change our notions of least, greatest, majorant and minorant. Looking at monotone envelopes with respect to the order relations $\leq \downarrow$ and $\leq \uparrow$ leads to some surprising and useful results. For instance, the level function of $u$ makes its appearance as both a least non-increasing majorant of $u$ with respect to $\leq \downarrow$ and a greatest non-increasing minorant of $u$ with respect to $\leq \uparrow$.

Studying the various monotone envelopes in Section 2 leads to our main results, in Section 3, for transferring monotonicity from the kernel to the weight in certain norm inequalities. Theorems 4.1 and 4.2 are concerned with transferring quasi-concavity. Applications extending embedding theorems for monotone functions are given in Theorem 3.7 and Corollary 3.8 and for quasi-concave functions in Theorem 4.3 and Corollary 4.4. Section 5 is devoted to proving Theorem 2.1 by a series of lemmas that set out the principle of “pushing mass.” In Section 6, we expose the simple structure of the level function, a result that was previously known only for bounded functions.
In the remainder of this section we introduce notation, prove some basic results in their natural generality, and recall those properties of the level function that we will require in the sequel. For notation and background in Banach Function Spaces we refer to \[1\].

Let \(\lambda\) be a \(\sigma\)-finite measure on the real line. In order for monotone functions to be \(\lambda\)-measurable we assume that all Borel sets are \(\lambda\)-measurable. Let \(L^+_{\lambda}\) be the collection of all non-negative, \(\lambda\)-measurable functions on \(\mathbb{R}\) and let \(L^+_{\lambda}(S)\) denote those functions in \(L^+_{\lambda}\) which vanish off \(S \subset \mathbb{R}\). We denote the collections of monotone functions on \(\mathbb{R}\) by

\[
L^1_{\lambda} = \{ f \in L^+_{\lambda} : f \text{ is non-increasing} \}
\]

and

\[
L^1_{\lambda} = \{ f \in L^+_{\lambda} : f \text{ is non-decreasing} \}.
\]

The two operators of integration we will need are \(I\) and \(I^*\) defined by

\[
If(x) = \int_{(-\infty,x]} f \, d\lambda \quad \text{and} \quad I^*f(x) = \int_{[x,\infty)} f \, d\lambda.
\]

Note that for all \(u, v \in L^+_{\lambda}\) we have

\[
\int_{\mathbb{R}} (Iu)v \, d\lambda = \int_{\mathbb{R}} u(I^*v) \, d\lambda.
\]

Now that we have defined the operators \(I\) and \(I^*\) we prefer to write \(Iu \leq Iv\) rather than the equivalent \(u \leq \downarrow v\) and to write \(I^*u \leq I^*v\) rather than the equivalent \(u \leq \uparrow v\).

It is clear that the operator \(I\) takes non-negative functions to non-decreasing functions so \(I(L^+_{\lambda}) \subset L^1_{\lambda}\). In Lemma 1.2 we show that the subset is quite a large one. To begin we show that \(I(L^+_{\lambda})\) has a useful lattice property.

**Lemma 1.1.** If \(u, v \in L^+_{\lambda}\) then there exists \(w \in L^+_{\lambda}\) such that \(Iw = \max(Iu, Iv)\).

**Proof.** Set \(W = \max(Iu, Iv)\) and let \(M = \sup\{ x \in \mathbb{R} : W(x) < \infty \}\). If \(M = -\infty\) the result is trivial, otherwise \(W\) is non-decreasing and right continuous on \((-\infty, M)\) and \(W(-\infty) = 0\). By \[8,\ Theorem 12, p301\] there exists a Borel measure \(\mu\) such that

\[
W(x) = \int_{(-\infty,x]} d\mu
\]

for all \(x < M\). We show that \(\mu\) is absolutely continuous with respect to \(\lambda\) on \((-\infty, M)\). Note that both \(u\lambda\) and \(v\lambda\) are finite on compact subsets of \((-\infty, M)\) and hence are Baire measures on \((-\infty, M)\). By \[8,\ Corollary 12, p340\] both \(u\lambda\) and \(v\lambda\) are regular measures on \((-\infty, M)\). Thus, if \(E \subset (-\infty, M)\) with \(\lambda(E) = 0\) and \(\varepsilon > 0\) then \(u\lambda(E) = v\lambda(E) = 0\) as well so we can find an open set \(O\) with \(E \subset O \subset (-\infty, M)\) such that

\[
\int_{O} u \, d\lambda < \varepsilon/2 \quad \text{and} \quad \int_{O} v \, d\lambda < \varepsilon/2.
\]
Now write $O = \bigcup_i (a_i, b_i)$, a union of its connected components, to get

$$
\mu(E) = \int_E d\mu \leq \int_O d\mu = \sum_i W(b_i) - W(a_i)
\leq \sum_i I_u(b_i) - I_u(a_i) + I_v(b_i) - I_v(a_i) = \int_O u \, d\lambda + \int_O v \, d\lambda < \varepsilon.
$$

(Here we have used the observation that if $A \geq C$ and $B \geq D$ then $\max(A, B) - \max(C, D) \leq A - C + B - D$.) Since $\varepsilon$ was arbitrary, $\mu(E) = 0$ so $\mu$ is absolutely continuous with respect to $\lambda$. By the Radon-Nikodym Theorem there is a $w \in L^+\lambda$ such that $\mu = w\lambda$ and we have

$$
W(x) = \int_{(-\infty, x]} w \, d\lambda
$$

for $x < M$. If $M = \infty$ we are done. If $M$ is an atom for $\lambda$ then it is a simple matter to choose a value for $w(M)$ so that

$$
W(M) = \int_{(-\infty, M]} w \, d\lambda.
$$

For $x > M$, $W(x) = \infty$ so we may set $w = \max(u, v)$ on $(M, \infty)$ to complete the proof.

**Lemma 1.2.** If $f \in L^\uparrow\lambda$ then there exist $u_n \in L^+\lambda$ such that the sequence $Iu_n$ is non-decreasing and converges to $f$ pointwise, $\lambda$-almost everywhere.

**Proof.** We begin by replacing $f(x)$ by

$$
\text{ess sup}_{t \leq x} f(t).
$$

Since $f$ is non-decreasing, the two functions agree $\lambda$-almost everywhere and therefore this new $f$ satisfies

$$
f(x) = \text{ess sup}_{t \leq x} f(t) = \sup\{y : \lambda\{t \leq x : f(t) > y\} > 0\}.
$$

Let $f_j = \min(f - 1/j, j)$ and note that $f_j$ increases to $f$ pointwise as $j \to \infty$. Since for all $a$,

$$
f_j(a) < f(a) = \sup\{y : \lambda\{t \leq a : f(t) > y\} > 0\}
$$

the set

$$
\{t \leq a : f(t) > f_j(a)\}
$$
has positive $\lambda$-measure and since $\lambda$ is $\sigma$-finite we can choose a subset $E_{a,j}$ of finite, positive $\lambda$-measure. Let

$$v_{a,j} = f_j(a)\lambda(E_{a,j})^{-1}\chi_{E_{a,j}}.$$  

It is easy to check that $Iv_{a,j} \leq f(x)$ for $x \in \mathbb{R}$ and that $Iv_{a,j}(a) = f_j(a)$.

Since $\lambda$ is $\sigma$-finite, it has at most countably many atoms. Hence we can choose a countable dense subset $\{a_i\}$ of $\mathbb{R}$ which contains all the atoms of $\lambda$. Induction applied to Lemma 1.1 shows that for each positive integer $n$ there exists a $u_n \in L^+_\lambda$ such that

$$Iu_n = \max_{i=1,\ldots,n} \{ Iv_{a_i,j} \}.$$  

It is evident that $Iu_n$ is a non-decreasing sequence and that $Iu_n \leq f$ for each $n$. It remains to show that

$$\lim_{n \to \infty} Iv_{a_i}(x) = f(x) $$  

for $\lambda$-almost every $x$. For each $a_i$ and each $n \geq i$ we have

$$f(a_i) \geq Iv_{a_i,n}(a_i) \geq Iv_{a_i,j}(a_i) = f_j(a_i)$$  

so (1.1) holds for $x = a_i$. If $x$ is not one of the $a_i$ then for each $a_i < x$ we have

$$f(a_i) = \lim_{n \to \infty} Iv_{a_i,n}(a_i) \leq \lim_{n \to \infty} Iv_{a_i,j}(x) \leq f(x).$$  

Since the $a_i$ are dense this implies that

$$f(x-) \leq \lim_{n \to \infty} Iv_{a_i,j}(x) \leq f(x).$$  

In particular, if $x$ is a point of continuity of $f$ then (1.1) holds. Since $f$ is non-decreasing it has at most countably many points of discontinuity so the set

$$\{ x \in \mathbb{R} \setminus \{ a_i \} : f(x-) \neq f(x) \}$$  

is countable and contains no atoms of $\lambda$. Therefore it has zero $\lambda$-measure. This completes the proof.

**Corollary 1.3.** Suppose $v, w \in L^+_\lambda$. If $f \in L^1_\lambda$ and $I^*v \leq I^*w$ or if $f \in L^1_\lambda$ and $Iv \leq Iw$ then

$$\int_{\mathbb{R}} fv d\lambda \leq \int_{\mathbb{R}} fw d\lambda.$$  

**Proof.** Suppose $f \in L^1_\lambda$ and $I^*v \leq I^*w$. By Lemma 1.2 we have

$$\int_{\mathbb{R}} fv d\lambda = \sup_{Iu \leq f} \int_{\mathbb{R}} (Iu)v d\lambda = \sup_{Iu \leq f} \int_{\mathbb{R}} u(I^*v) d\lambda$$  

$$\leq \sup_{Iu \leq f} \int_{\mathbb{R}} u(I^*w) d\lambda = \sup_{Iu \leq f} \int_{\mathbb{R}} (Iu)w d\lambda \leq \int_{\mathbb{R}} fw d\lambda.$$
The second part follows from the first by the change of variable $x \to -x$.

To work with the level function we introduce a class of averaging operators. Suppose $\{J\}$ is a countable (or finite) collection of disjoint intervals each of finite, positive $\lambda$-measure and define the operator $A$ by

$$Af(x) = \begin{cases} \frac{1}{\lambda(J)} \int_J f \, d\lambda, & x \in J \in \{J\} \\ f(x), & x \notin \cup\{J\} \end{cases}. $$

We denote the collection of all such operators $A$ by $\mathcal{A}$.

**Proposition 1.4.** Suppose that $A \in \mathcal{A}$. Then

i) $A$ is formally self-adjoint, that is, for all $f, g \in L_+^\lambda$,

$$\int_\mathbb{R} (Af)g \, d\lambda = \int_\mathbb{R} f(Ag) \, d\lambda. $$

ii) If $f \in L_+^\lambda$ then $Af \in L_+^\lambda$ and $IAf \leq If$.

iii) If $f \in L_+^\lambda$ then $Af \in L_+^\lambda$ and $If \leq IAf$.

**Proof.** Suppose $f, g \in L_+^\lambda$. Then

$$\int_\mathbb{R} (Af)g \, d\lambda = \int_{\mathbb{R} \setminus \cup J} fg \, d\lambda + \sum_j \int_J \left( \frac{1}{\lambda(J)} \int_J f \, d\lambda \right) g \, d\lambda $$

$$= \int_{\mathbb{R} \setminus \cup J} fg \, d\lambda + \sum_j \int_J f \left( \frac{1}{\lambda(J)} \int_J g \, d\lambda \right) \, d\lambda = \int_\mathbb{R} f(Ag) \, d\lambda. $$

This is (i).

Replacing a function by its average on an interval preserves monotonicity so the first statements of both (ii) and (iii) are clear.

For the second statements of (ii) and (iii), fix $y \in \mathbb{R}$ and set $\chi = \chi(-\infty, y]$. Replacing $\chi$ by its average on an interval has no effect when $\chi$ is constant on the interval so applying $A$ to $\chi$ is easy: If $y \in J$ for some $J$ then

$$A\chi(t) = \begin{cases} 1, & t \leq y, t \notin J \\ \frac{\lambda((-\infty, y] \cap J)}{\lambda(J)}, & t \in J \\ 0, & \text{otherwise.} \end{cases} $$

If $y \notin \cup J$ then $A\chi = \chi$. In the latter case we certainly have $IA\chi \leq I\chi$ and $I^*\chi \leq I^*A\chi$. These hold in the former case as well: Since

$$IA\chi(x) = \begin{cases} \lambda((-\infty, x]), & x \leq y, x \notin J \\ \lambda((-\infty, x] \cap J^c) + \lambda((-\infty, x] \cap J) \frac{\lambda((-\infty, y] \cap J)}{\lambda(J)}, & x \in J \\ \lambda((-\infty, y]), & \text{otherwise.} \end{cases} $$
we easily see that $IA\chi(x)$ is no greater than \(\lambda((\infty, \min(x, y))] = I\chi(x)\). Since

\[
I^*A\chi(x) = \begin{cases} 
0, & x \geq y, x \not\in J \\
\lambda([x, \infty) \cap J) \frac{\lambda((\infty, y] \cap J)}{\lambda(J)}, & x \in J \\
\lambda([x, y]), & \text{otherwise}
\end{cases}
\]

we easily see that $I^*A\chi(x)$ is no less than \(\lambda((\infty, y])\chi([x, y]) = I^\ast\chi(x)\). Now that we have $IA\chi \leq I\chi$ and $I^*\chi \leq I^*A\chi$ we can apply Corollary 1.3 to see that for any $f \in L_{\lambda}^1$,

\[
IAf(y) = \int_{\mathbb{R}} (Af)\chi d\lambda = \int_{\mathbb{R}} f(A\chi) d\lambda \leq \int_{\mathbb{R}} f\chi d\lambda = If(y)
\]

and for any $f \in L_{\lambda}^1$,

\[
IAf(y) = \int_{\mathbb{R}} (Af)\chi d\lambda = \int_{\mathbb{R}} f(A\chi) d\lambda \geq \int_{\mathbb{R}} f\chi d\lambda = If(y)
\]

Since $y$ was arbitrary these complete the proof.

The level function of $u$ depends on the underlying measure $\lambda$. To avoid technical difficulties we require that $\lambda$ satisfy

\[(1.2) \quad \lambda((\infty, x]) < \infty, \quad x \in \mathbb{R},\]

when working with the level function.

**Proposition 1.5.** Suppose that $\lambda$ satisfies (1.2). To each $u \in L_{\lambda}^1$ there corresponds a function $u^o \in L_{\lambda}^1$, called the level function of $u$ with respect to $\lambda$ such that

i) $Iu \leq Iu^o$.

ii) If $u_n \uparrow u$ then $u_n^o \uparrow u^o$. That is, if an increasing sequence of functions converges pointwise $\lambda$-almost everywhere to $u$ then the sequence of their level functions is increasing and converges pointwise $\lambda$-almost everywhere to the level function of $u$.

iii) If $u \in L_{\lambda}^1$ is bounded and vanishes on $[M, \infty)$ for some $M$ then there exists an $A_u \in \mathcal{A}$ such that $u^o = A_u u$.

iv) $I^*u^o \leq I^*u$.

**Proof.** The structure of the level function of $u$ is given in [10, Theorem 4.4, Definition 4.6, Corollary 4.8 and Theorem 4.9]. There it is shown that $u^o$ is non-negative and non-increasing and that (i) holds. Regarding part (iii), we take the intervals of $A_u$ to be the intervals $I_i$ of [10, Definition 4.6 and Corollary 4.8]. It is not assumed in [10] that $u$ is supported on $(\infty, M]$ so the possibility of an interval of infinite $\lambda$-measure is considered there. An easy argument shows that if $u$ is supported on
$(-\infty, M]$ then all the intervals are contained in $(-\infty, M]$ and hence are of finite 
$\lambda$-measure. Clearly we may discard those of zero $\lambda$-measure.

As a consequence of [10, Theorem 5.2] the property in part (ii) can be used to 
extend the level function construction by monotonicity. In [11, Theorem 5.2] it is 
shown that the extended construction retains the property.

In view of (ii) above and the Monotone Convergence Theorem we observe that it 
is enough to work with bounded $u$ vanishing on $[M, \infty)$ for some $M$ when proving 
part (iv). Since in this case we have $u^o = A_u u$ it follows that
\[
\int_{\mathbb{R}} u^o \, d\lambda = \int_{\mathbb{R}} u \, d\lambda < \infty.
\]
Now part (iv) follows from part (i): For each $x \in \mathbb{R}$,
\[
I^* u^o(x) = \int_{\mathbb{R}} u^o \, d\lambda - I u^o(x-) \leq \int_{\mathbb{R}} u \, d\lambda - I u(x-) = I^* u(x).
\]

At one point in the sequel we require an extension of the Proposition 1.5(iii)
in which the restriction to bounded functions vanishing on $[M, \infty)$ is removed.
Because of the technical nature of this result and the delicate argument it requires 
we defer its statement and proof to Section 6.

2. Monotone Envelopes

For $u \in L^+_\lambda$ we define the monotone envelopes of $u$ as follows: The least non-
increasing majorant of $u$ is
\[
u^\downarrow(x) = \operatorname{ess \sup}_{t \geq x} u(t)
\]
and the greatest non-increasing minorant of $u$ is
\[
u^\uparrow(x) = \operatorname{ess \inf}_{t \leq x} u(t).
\]
The two non-decreasing envelopes of $u$ are defined analogously,
\[
u^\uparrow(x) = \operatorname{ess \sup}_{t \leq x} u(t) \quad \text{and} \quad \nu^\downarrow(x) = \operatorname{ess \inf}_{t \geq x} u(t).
\]
A routine measure theory exercise shows that for $\lambda$-almost every $x \in \mathbb{R}$
\[
u^\downarrow(x) \leq u(x) \leq \nu^\uparrow(x) \quad \text{and} \quad \nu^\uparrow(x) \leq u(x) \leq \nu^\downarrow(x).
\]
For $f$ and $g$ in $L^+_\lambda$ the condition $Ig \leq If$ is a weaker order relation than $g \leq f$.
Consequently, the supremum in the obvious identity
\[
sup_{g \leq f} \int_{\mathbb{R}} gu \, d\lambda = \int_{\mathbb{R}} fu \, d\lambda
\]
may become larger when the condition $g \leq f$ is weakened to $Ig \leq If$. Our first 
theorem makes this observation precise.
Theorem 2.1. Suppose \( f, u \in L^+_{\lambda} \). Then

\[
(2.1) \quad \sup_{I_g \leq I_f} \int_R gu \, d\lambda = \int_R f u^\downarrow \, d\lambda
\]

and

\[
(2.2) \quad \inf_{I_g \geq I_f} \int_R gu \, d\lambda = \int_R f u^\downarrow \, d\lambda.
\]

Also

\[
(2.3) \quad \sup_{I^*g \leq I^*f} \int_R gu \, d\lambda = \int_R f u^\uparrow \, d\lambda
\]

and

\[
(2.4) \quad \inf_{I^*g \geq I^*f} \int_R gu \, d\lambda = \int_R f u^\downarrow \, d\lambda.
\]

A careful proof is given in Section 5. Here we sketch the essential idea of the proof of (2.1): The function \( u^\downarrow \) majorizes \( u \) and is non-increasing so by Corollary 1.3 we have

\[
\int_R gu \, d\lambda \leq \int_R g u^\downarrow \, d\lambda \leq \int_R f u^\downarrow \, d\lambda
\]

for each \( g \) with \( Ig \leq If \). Thus

\[
\sup_{I_g \leq I_f} \int_R gu \, d\lambda \leq \int_R f u^\downarrow \, d\lambda.
\]

For the other inequality in (2.1) we fix \( f \) and construct \( g_f \) as follows: A well-behaved function \( u \) agrees with \( u^\downarrow \) except on a collection of intervals where \( u^\downarrow \) is constant. On each such interval we push the mass of \( f \) over until it sits on the right endpoint. The result is a “function” \( g_f \) that is zero inside the intervals where \( u \neq u^\downarrow \). Thus

\[
\int_R g_f u \, d\lambda = \int_R g_f u^\downarrow \, d\lambda.
\]

Also, the mass of \( f \) has been pushed to the right to form \( g_f \) so for each \( x \)

\[
Ig_f(x) \leq If(x).
\]

Now we have

\[
\sup_{I_g \leq I_f} \int_R gu \, d\lambda \geq \int_R g_f u \, d\lambda = \int_R g_f u^\downarrow \, d\lambda = \int_R f u^\downarrow \, d\lambda.
\]
The last equality holds because the mass of \( f \) has been shifted only on intervals where \( u^\downarrow \) is constant.

Despite the vagueness of “pushing mass” and the strong simplifying assumption on \( u \) the basic idea in this sketch survives in the proofs of Section 5.

For the next result we look at another kind of monotone envelope—the level function. We show that the level function \( u^o \) of \( u \) with respect to \( \lambda \) is a least non-increasing majorant of \( u \) when the order \( u \leq v \) is replaced by \( Iu \leq Iv \). Surprisingly, when the order \( u \leq v \) is replaced by \( I^*u \leq I^*v \) the same level function becomes a greatest non-increasing minorant of \( u \). As suggested by the use of the indefinite article above, it may happen that a function \( u \) has more than one least non-increasing majorant when the order is \( \leq \downarrow \). This is because the order relation lacks antisymmetry for very large functions. If \( Iu^o(x) < \infty \) for \( x \in \mathbb{R} \) then the level function \( u^o \) is the unique least non-increasing majorant of \( u \) with order relation \( \leq \downarrow \).

A similar comment applies to the greatest non-increasing minorant of \( u \) with order relation \( \leq \uparrow \).

As we see in Examples 2.4 and 2.6 there need be no greatest non-increasing minorant with respect to the order \( \leq \downarrow \) nor least non-increasing majorant with respect to the order \( \leq \uparrow \).

**Lemma 2.2.** The level function of \( u \) is a least non-increasing majorant of \( u \) with respect to the order relation \( \leq \downarrow \). That is, \( Iu \leq Iu^o \) and if \( v \in L^\downarrow_\lambda \) with \( Iu \leq Iv \) then \( Iu^o \leq Iv \). Also, the level function of \( u \) is a greatest non-increasing minorant of \( u \) with respect to the order relation \( \leq \downarrow \). That is, \( I^*u^o \leq I^*u \) and if \( v \in L^\downarrow_\lambda \) with \( I^*v \leq I^*u \) then \( I^*v \leq I^*u^o \).

**Proof.** In view of Proposition 1.5(ii) and the Monotone Convergence Theorem it is enough to prove the first part of the lemma for \( u \) bounded and vanishing on \([M, \infty)\) for some \( M \). Suppose we have such a \( u \) and a \( v \) with \( v \in L^\downarrow_\lambda \) and \( Iu \leq Iv \). By Proposition 1.5(i), \( Iu \leq Iu^o \) and by Proposition 1.5(iii) we can choose \( A_u \in \mathcal{A} \) so that \( u^o = A_u u \). Fix \( x \in \mathbb{R} \) and let \( \chi = \chi(-\infty, x] \). Then

\[
Iu^o(x) = \int_{\mathbb{R}} \chi u^o \ d\lambda = \int_{\mathbb{R}} \chi (A_u u) \ d\lambda = \int_{\mathbb{R}} (A_u \chi) u \ d\lambda.
\]

Since \( \chi \) is non-increasing, so is \( A_u \chi \). Therefore the hypothesis \( Iu \leq Iv \) and Corollary 1.3 show that

\[
\int_{\mathbb{R}} (A_u \chi) u \ d\lambda \leq \int_{\mathbb{R}} (A_u \chi) v \ d\lambda = \int_{\mathbb{R}} \chi (A_u v) \ d\lambda.
\]

Proposition 1.4(ii) shows that \( IA_u v \leq Iv \) and since \( \chi \in L^\downarrow_\lambda \), Corollary 1.3 implies that

\[
\int_{\mathbb{R}} \chi (A_u v) \ d\lambda \leq \int_{\mathbb{R}} \chi v \ d\lambda = Iv(x).
\]

These together yield \( Iu^o \leq Iv \) as required.
For the second half of the lemma we apply Proposition 1.5(iv) to see that $I^* u^o \leq I^* u$. Now suppose that $v \in L^1_\lambda$ with $I^* v \leq I^* u$. Our object is to show that $I^* v \leq I^* u^o$. We do this in two cases depending on whether or not $u$ is integrable.

If $\int_R u \, d\lambda < \infty$ then for each positive integer $n$ set $u_n = \min(u, n)\chi_{(-\infty, n]}$ and choose $A_n \in \mathcal{A}$ such that $A_n u_n = u_n^o$. Fix $x \in \mathbb{R}$ and let $\chi = \chi_{[x, \infty)}$. Since $\chi \in L^1_\lambda$, Proposition 1.4 yields $I \chi \leq IA_n \chi$ for all $n > 0$ so Corollary 1.3 shows

$$I^* v(x) = \int_R \chi v \, d\lambda \leq \int_R (A_n \chi) v \, d\lambda$$

because $v$ is non-increasing. Now $A_n \chi$ is non-decreasing and $I^* v \leq I^* u$ so Corollary 1.3 shows that

$$\int_R (A_n \chi) v \, d\lambda \leq \int_R (A_n \chi) u \, d\lambda = \int_R (A_n \chi) (u - u_n) \, d\lambda + \int_R (A_n \chi) u_n \, d\lambda.$$

Notice that for any $n$, $A_n \chi \leq 1$. Since $\int_R u \, d\lambda < \infty$ and $u - u_n$ tends to zero pointwise, the Dominated Convergence Theorem shows that

$$\int_R (A_n \chi) (u - u_n) \, d\lambda \to 0 \text{ as } n \to \infty.$$

Proposition 1.5(ii) and the Monotone Convergence Theorem show that as $n \to \infty$,

$$\int_R (A_n \chi) u_n \, d\lambda = \int_R \chi (A_n u_n) \, d\lambda = \int_R \chi u_n^o \, d\lambda \to \int_R \chi u^o \, d\lambda = I^* u^o(x).$$

Putting these together yields $I^* v(x) \leq I^* u^o(x)$ and since $x$ was arbitrary this completes the proof of the case $\int_R u \, d\lambda < \infty$.

To handle the case $\int_R u \, d\lambda = \infty$ we first observe that by the Monotone Convergence Theorem it is enough to prove the result for $v$ satisfying $\int_R v \, d\lambda < \infty$. Next we need the following fact which depends on the $\sigma$-finiteness of $\lambda$: If $\int_R v \, d\lambda < \infty$ and $\int_R u \, d\lambda = \infty$ with $I^* v \leq I^* u$ then there exists a $w \in L^1_\lambda$ with $w \leq u$ such that $\int_R w \, d\lambda < \infty$ and $I^* v \leq I^* w$. The construction of such a $w$, an easy exercise in measure theory, is left to the reader. Having $w$ we use the previous case to get

$$I^* v \leq I^* w^o \leq I^* u^o.$$

Note that since $w \leq u$ it is a consequence of Proposition 1.5(ii) that $w^o \leq u^o$.

Now we present an analogue of (2.1) with $f$ and $g$ restricted to be non-increasing. The level function appears here in its role of least non-increasing majorant of $u$ with respect to the order $\leq$. 

Theorem 2.3. Suppose \( \lambda \) satisfies (1.2) and \( u \in L^+_\lambda \). If \( f \in L^1_\lambda \) then

\[
(2.5) \quad \sup_{g \in L^1_\lambda \atop Ig \leq If} \int_R gu \, d\lambda = \int_R fu^o \, d\lambda.
\]

Proof. By Proposition 1.5(ii) and the Monotone Convergence Theorem it is enough to prove the theorem assuming that \( u \) is bounded and vanishes on \([M, \infty)\) for some \( M \). If \( g \in L^1_\lambda \) and \( Ig \leq If \) then by Proposition 1.5(i) and Corollary 1.3 applied twice we have

\[
\int_R gu \, d\lambda \leq \int_R gu^o \, d\lambda \leq \int_R fu^o \, d\lambda
\]

since \( u^o \) is non-increasing. This proves the inequality \( \leq \) of (2.5).

For the reverse inequality apply Proposition 1.5(iii) to choose \( A_u \in A \) such that \( A_uu = u^o \). Then Proposition 1.4(ii) shows that \( A UIF \in L^1_\lambda \) and \( IAUF \leq If \). Thus

\[
\sup_{g \in L^1_\lambda \atop Ig \leq If} \int_R gu \, d\lambda \geq \int_R (AUF)u \, d\lambda = \int_R f(IAUF) \, d\lambda = \int_R fu^o \, d\lambda.
\]

This completes the proof.

Now that we have the analogue (2.5) of (2.1) it is natural to ask if there is an analogue of (2.2) with \( f \) and \( g \) restricted to be non-increasing. Surprisingly, the answer is no. The following example shows that no direct analogue is possible.

Example 2.4. Let \( \lambda \) be Lebesgue measure on \((0,3)\), that is, \( d\lambda(x) = \chi_{(0,3)}(x) \, dx \), and set \( u = 3\chi_{(0,1)} + \chi_{(2,3)} \). Then there is no function \( u^o \) which satisfies

\[
(2.6) \quad \inf_{g \in L^1_\lambda \atop Ig \geq If} \int_R gu \, d\lambda = \int_R fu^o \, d\lambda
\]

for all \( f \in L^1_\lambda \).

Proof. For each \( s \in (0,4) \) set

\[
f_s = 4\chi_{(0,1]} + s\chi_{(1,3)}.
\]

Observe that

\[
\inf_{g \in L^1_\lambda \atop Ig \geq If_s} \int_R gu \, d\lambda = \inf_{g \in L^1_\lambda \atop Ig \geq If_s} 3 \int_0^1 g + \int_2^3 g \geq \inf_{g \in L^1_\lambda \atop Ig \geq If_s} 3 \int_0^1 g \geq 3 \int_0^1 f_s = 12.
\]
Also, since $\int_0^1 g - \int_1^2 g \geq 0$ for $g \in L^1_\lambda$, we have

$$\inf_{\substack{g \in L^1_\lambda \\text{if} \ g \geq f_s}} \int_R g u \, d\lambda = \inf_{\substack{g \in L^1_\lambda \\text{if} \ g \geq f_s}} \int_0^1 g + \left( \int_0^1 g - \int_1^2 g \right) + \int_0^3 g \geq \int_0^1 f_s + \int_0^3 f_s = 8 + 2s$$

These two observations yield

\begin{equation}
\inf_{\substack{g \in L^1_\lambda \\text{if} \ g \geq f_s}} \int_R g u \, d\lambda \geq \max(12, 8 + 2s).
\end{equation}

In fact this inequality is equality, as we show in two cases. If $0 < s \leq 2$ then we set $g_s = 4\chi_{(0,1]} + 2s\chi_{(1,2]}$, note that $g_s \in L^1_\lambda$ and $Ig_s \geq If_s$, and conclude that

$$\inf_{\substack{g \in L^1_\lambda \\text{if} \ g \geq f_s}} \int_R g u \, d\lambda \leq \int_R g_s u \, d\lambda = 12.$$ 

If $2 \leq s \leq 4$ then we set $g_s = 4\chi_{(0,2]} + (2s - 4)\chi_{(2,3)}$, again note that $g_s \in L^1_\lambda$ and $Ig_s \geq If_s$, and conclude that

$$\inf_{\substack{g \in L^1_\lambda \\text{if} \ g \geq f_s}} \int_R g u \, d\lambda \leq \int_R g_s u \, d\lambda = 8 + 2s.$$ 

Equality in (2.7) turns (2.6), with $f$ replaced by $f_s$, into

$$\max(12, 8 + 2s) = \int_R f_s u \, d\lambda = 4 \int_0^1 u \, d\lambda + s \int_1^3 u.$$

It is clear that this cannot hold for all $s \in (0, 4)$ no matter what the function $u$ may be as the right hand side has constant slope while the left hand side does not.

A similar situation occurs when we consider restricting (2.1) and (2.2) to non-decreasing functions $f$ and $g$. The level function provides an analogue of (2.2) but there is no analogue of (2.1).

**Theorem 2.5.** Suppose $\lambda$ satisfies (1.2) and $u \in L^+_\lambda$. If $f \in L^1_\lambda$ then

\begin{equation}
\inf_{\substack{g \in L^1_\lambda \\text{if} \ g \geq f}} \int_R g u \, d\lambda = \int_R f u \, d\lambda.
\end{equation}

**Proof.** If $g \in L^1_\lambda$ and $Ig \geq If$ then by Proposition 1.5(iv) and Corollary 1.3 applied twice we have

$$\int_R g u \, d\lambda \geq \int_R g u \, d\lambda \geq \int_R f u \, d\lambda.$$
because \( u^o \) is non-increasing. This proves the inequality \( \geq \) of (2.8).

For the reverse inequality we apply Theorem 6.1 to \( u \) to obtain intervals \( J_{\text{left}} \) and \( J_{\text{right}} \) and an operator \( A \in \mathcal{A} \) such that \( u^o = Au \) off \( J_{\text{left}} \cup J_{\text{right}} \). Note that all the intervals of \( A \) are contained in \( \mathbb{R} \setminus (J_{\text{left}} \cup J_{\text{right}}) \). Define \( g \) by \( g = Af \) on \( \mathbb{R} \setminus J_{\text{right}} \) and

\[
g = \lim_{x \to \infty} \frac{1}{\lambda((-\infty, x] \cap J_{\text{right}})} \int_{(-\infty, x] \cap J_{\text{right}}} f \, d\lambda
\]
on \( J_{\text{right}} \). Because \( f \) is non-decreasing the limit is non-decreasing and therefore exists. It is easy to check that \( g \) is non-decreasing as well. By Proposition 1.4(iii) we see that \( Ig \geq If \) on \( \mathbb{R} \setminus J_{\text{right}} \) and for \( x \in J_{\text{right}} \)

\[
Ig(x) = \int_{(-\infty, x] \setminus J_{\text{right}}} g \, d\lambda + \int_{(-\infty, x] \cap J_{\text{right}}} g \, d\lambda \\
\geq \int_{(-\infty, x] \setminus J_{\text{right}}} f \, d\lambda + \int_{(-\infty, x] \cap J_{\text{right}}} f \, d\lambda = If(x)
\]
as well. To complete the proof we show that

\[
\int_{\mathbb{R}} gu \, d\lambda = \int_{\mathbb{R}} fu^o \, d\lambda.
\]

Proposition 1.4(i) implies

\[
\int_{\mathbb{R} \setminus J_{\text{right}}} gu \, d\lambda = \int_{\mathbb{R} \setminus J_{\text{right}}} fu^o \, d\lambda
\]
so we need only show that

\[
\int_{J_{\text{right}}} gu \, d\lambda = \int_{J_{\text{right}}} fu^o \, d\lambda.
\]

That is, by Theorem 6.1(iii), that

\[
\left( \lim_{x \to -\infty} \frac{1}{\lambda((-\infty, x] \cap J_{\text{right}})} \int_{(-\infty, x] \cap J_{\text{right}}} f \, d\lambda \right) \int_{J_{\text{right}}} u \, d\lambda
\]

\[
= \int_{J_{\text{right}}} f \, d\lambda \left( \limsup_{x \to -\infty} \frac{1}{\lambda((-\infty, x] \cap J_{\text{right}})} \int_{(-\infty, x] \cap J_{\text{right}}} u \, d\lambda \right).
\]

It is not difficult to recognise both sides as

\[
\limsup_{x \to -\infty} \frac{1}{\lambda((-\infty, x] \cap J_{\text{right}})} \int_{(-\infty, x] \cap J_{\text{right}}} u \, d\lambda \int_{(-\infty, x] \cap J_{\text{right}}} f \, d\lambda
\]
to complete the proof.
Example 2.6. Let $\lambda$ be Lebesgue measure on $(0,3)$, that is, $d\lambda(x) = \chi_{(0,3)}(x) \, dx$, and set $u = 3\chi_{(1,2)} + \chi_{(2,3)}$. Then there is no function $u_0$ which satisfies

$$\tag{2.9} \sup_{g \in L^1_{\lambda} \atop 1 \leq \int f} \int_{\mathbb{R}} gu \, d\lambda = \int_{\mathbb{R}} fu_0 \, d\lambda$$

for all $f \in L^1_{\lambda}$.

Proof. For each $s \in (0,4)$ set $f_s = s\chi_{(0,2]} + 4\chi_{(2,3]}$. Then

$$\sup_{g \in L^1_{\lambda} \atop 1 \leq \int f} \int_{\mathbb{R}} gu \, d\lambda \leq \sup_{g \in L^1_{\lambda} \atop 1 \leq \int f} \int_{\mathbb{R}} f_s \, d\lambda = 6s + 4.$$

Also, since $\int_1^2 g \leq \int_2^3 g$ for $g \in L^1_{\lambda}$, we have

$$\sup_{g \in L^1_{\lambda} \atop 1 \leq \int f} \int_{\mathbb{R}} gu \, d\lambda \leq \sup_{g \in L^1_{\lambda} \atop 1 \leq \int f} \int_{\mathbb{R}} f_s \, d\lambda = 4s + 8.$$

These two observations yield

$$\tag{2.10} \sup_{g \in L^1_{\lambda} \atop 1 \leq \int f} \int_{\mathbb{R}} gu \, d\lambda \leq \min(6s + 4, 4s + 8).$$

We demonstrate that this inequality is equality. If $0 < s \leq 2$ then $g_s = 2s\chi_{(1,2]} + 4\chi_{(2,3]}$ is in $L^1_{\lambda}$ and it is easy to check that $I g_s \leq I f_s$. Therefore,

$$\sup_{g \in L^1_{\lambda} \atop 1 \leq \int f} \int_{\mathbb{R}} gu \, d\lambda \geq \int_{\mathbb{R}} g_s u \, d\lambda = 6s + 4.$$

If $2 \leq s \leq 4$ then we set $g_s = (s + 2)\chi_{(1,3]}$. Again $g_s \in L^1_{\lambda}$ and $I g_s \leq I f_s$. We have

$$\sup_{g \in L^1_{\lambda} \atop 1 \leq \int f} \int_{\mathbb{R}} gu \, d\lambda \geq \int_{\mathbb{R}} g_s u \, d\lambda = 4s + 8.$$
Equality in (2.10) turns (2.9), with $f$ replaced by $f_s$, into
\[
\min(6s + 4, 4s + 8) = \int_R f_s u_0 \, d\lambda = s \int_0^2 u_0 + 4 \int_2^3 u_0.
\]

It is clear that this cannot hold for all $s \in (0, 4)$ no matter what the function $u_0$ may be as the right hand side has constant slope while the left hand side does not.

The non-decreasing analogue of the level function is obtained by simply flipping the real line end for end. Thus if $\lambda(x, \infty) < \infty$ for $x \in \mathbb{R}$ and $u \in L^+_{\lambda}$ then we set $\lambda_1(x) = \lambda(-x)$, $u_1(x) = u(-x)$, and let $u_1^\circ$ be the level function of $u_1$ with respect to $\lambda_1$. The non-decreasing level function of $u$ with respect to $\lambda$ is then $u_1^\circ(-x)$. This construction simultaneously yields the least non-decreasing majorant of $u$ with respect to the order $\leq_1$ and the greatest non-decreasing minorant of $u$ with respect to the order $\leq_1$. Lemma 2.2 and Theorems 2.3 and 2.5 have obvious counterparts for the non-decreasing level function that we leave to the reader.

3. Transferring Monotonicity

In [13] the notion of transferring monotonicity from the kernel of an operator to the weight was introduced to study a special case of the weighted Hardy inequality. The results of the previous section allow us to better express the ideas behind that notion and place them in a more general setting. A result related to Theorem 3.5 may be found in [3, Proposition 2.12].

As before, the measure $\lambda$ is a $\sigma$-finite measure on $(-\infty, \infty)$ for which non-increasing functions are $\lambda$-measurable. Let $\mu$ be any measure on any set and let $X$ be a Banach Function Space of $\mu$-measurable functions. Define the linear operator $K$ by
\[
Kf(x) = \int_R k(x, t)f(t) \, d\lambda(t)
\]
where the kernel $k(x, t)$ is a non-negative $(\mu \times \lambda)$-measurable function.

**Theorem 3.1.** Suppose $k(x, t)$ is non-increasing in $t$ for each $x$. Then the least constant $C$, finite or infinite, for which
\[
\|Kf\|_X \leq C \int_R fu \, d\lambda, \quad f \in L^+_{\lambda},
\]
holds is unchanged when $u$ is replaced by $u_1^\circ$. That is,
\[
(3.1) \sup_{f \geq 0} \frac{\|Kf\|_X}{\int_R fu \, d\lambda} = \sup_{f \geq 0} \frac{\|Kf\|_X}{\int_R u_1^\circ \, d\lambda}.
\]

**Proof.** Since $u_1^\circ \leq u \lambda$-almost everywhere the inequality “$\leq$” in (3.1) is immediate. To establish the reverse inequality we apply (2.2) of Theorem 2.1 to get
\[
\sup_{f \geq 0} \frac{\|Kf\|_X}{\int_R fu_1^\circ \, d\lambda} = \sup_{f \geq 0} \frac{\|Kf\|_X}{\inf_{I \geq 1} \int_R gu \, d\lambda} = \sup_{f \geq 0} \frac{\|Kf\|_X}{\int_R gu \, d\lambda}.
\]
Now if $If \leq Ig$ then the monotonicity of $k$ and Corollary 1.3 shows that $Kf \leq Kg$ and since $X$ is a Banach Function Space we have $\|Kf\|_X \leq \|Kg\|_X$. Thus
\[
\sup_{f \geq 0} \frac{\|Kf\|_X}{\int_R fu_1 d\lambda} \leq \sup_{f \geq 0} \sup_{Ig \leq If} \frac{\|Kg\|_X}{\int_R gu d\lambda} \leq \sup_{g \geq 0} \frac{\|Kg\|_X}{\int_R gu d\lambda}.
\]
This completes the proof.

The substitution $x \to -x$ gives the corresponding result for non-decreasing kernels.

**Corollary 3.2.** Suppose $k(x, t)$ is non-decreasing in $t$ for each $x$. Then the least constant $C$, finite or infinite, for which
\[
\|Kf\|_X \leq C \int_R fu d\lambda, \quad f \in L^+_\lambda,
\]
holds is unchanged when $u$ is replaced by $u_1$. That is,
\[
\sup_{f \geq 0} \frac{\|Kf\|_X}{\int_R fu d\lambda} = \sup_{f \geq 0} \frac{\|Kf\|_X}{\int_R fu_1 d\lambda}.
\]

Next we look at the reversed inequality.

**Theorem 3.3.** Suppose $k(x, t)$ is non-increasing in $t$ for each $x$. Then the least constant $C$, finite or infinite, for which
\[
\int_R fu d\lambda \leq C\|Kf\|_X, \quad f \in L^+_\lambda,
\]
holds is unchanged when $u$ is replaced by $u^\uparrow$. That is,
\[
(3.2) \quad \sup_{f \geq 0} \frac{\int_R fu d\lambda}{\int_R \|Kf\|_X} = \sup_{f \geq 0} \frac{\int_R fu^\uparrow d\lambda}{\int_R \|Kf\|_X}.
\]

**Proof.** Since $u \leq u^\uparrow$ $\lambda$-almost everywhere the inequality “≤” in (3.2) is clear. For the reverse inequality we apply (2.1) of Theorem 2.1 to get
\[
\sup_{f \geq 0} \frac{\int_R fu^\uparrow d\lambda}{\|Kf\|_X} = \sup_{f \geq 0} \frac{\sup_{Ig \leq If} \int_R gu d\lambda}{\|Kf\|_X} = \sup_{f \geq 0} \frac{\int_R gu d\lambda}{\|Kf\|_X}.
\]
Now if $Ig \leq If$ then the monotonicity of $k$ and Corollary 1.3 shows that $Kg \leq Kf$ and since $X$ is a Banach Function Space we have $\|Kg\|_X \leq \|Kf\|_X$. Thus
\[
\sup_{f \geq 0} \frac{\int_R fu^\uparrow d\lambda}{\|Kf\|_X} \leq \sup_{f \geq 0} \sup_{Ig \leq If} \frac{\int_R gu d\lambda}{\|Kg\|_X} \leq \sup_{g \geq 0} \frac{\int_R gu d\lambda}{\|Kg\|_X}.
\]
This completes the proof.
Corollary 3.4. Suppose \( k(x, t) \) is non-decreasing in \( t \) for each \( x \). Then the least constant \( C \), finite or infinite, for which

\[
\int_{\mathbb{R}} fu \, d\lambda \leq C \| Kf \|_X, \quad f \in L^1_\lambda,
\]

holds is unchanged when \( u \) is replaced by \( u^\uparrow \). That is,

\[
\sup_{f \geq 0} \frac{\int_{\mathbb{R}} fu \, d\lambda}{\| Kf \|_X} = \sup_{f \geq 0} \frac{\int_{\mathbb{R}} fu^\uparrow \, d\lambda}{\| Kf \|_X}.
\]

We can also transfer monotonicity in weighted norm inequalities restricted to monotone functions.

Theorem 3.5. Suppose \( k(x, t) \) is non-increasing in \( t \) for each \( x \). Then the least constant \( C \), finite or infinite, for which

\[
\int_{\mathbb{R}} fu \, d\lambda \leq C \| Kf \|_X, \quad f \in L^1_\lambda,
\]

holds is unchanged when \( u \) is replaced by \( u^o \). That is,

\[
(3.3) \quad \sup_{f \in L^1_\lambda} \frac{\int_{\mathbb{R}} fu \, d\lambda}{\| Kf \|_X} = \sup_{f \in L^1_\lambda} \frac{\int_{\mathbb{R}} fu^o \, d\lambda}{\| Kf \|_X}.
\]

Proof. Since \( Iu \leq Iu^o \), Corollary 1.3 yields the inequality "\( \leq \)" in (3.3). To establish the reverse inequality we apply (2.5) of Theorem 2.3 to get

\[
\sup_{f \in L^1_\lambda} \frac{\int_{\mathbb{R}} f u \, d\lambda}{\| Kf \|_X} = \sup_{f \in L^1_\lambda} \frac{\int_{\mathbb{R}} f u^o \, d\lambda}{\| Kf \|_X}.
\]

Now if \( Ig \leq If \) then the monotonicity of \( k \) and Corollary 1.3 shows that \( Kg \leq Kf \) and since \( X \) is a Banach Function Space we have \( \| Kg \|_X \leq \| Kf \|_X \). Thus

\[
\sup_{f \in L^1_\lambda} \frac{\int_{\mathbb{R}} f u \, d\lambda}{\| Kf \|_X} = \sup_{f \in L^1_\lambda} \frac{\int_{\mathbb{R}} f u^o \, d\lambda}{\| Kf \|_X}.
\]

This completes the proof.
Theorem 3.6. Suppose \( k(x,t) \) is non-increasing in \( t \) for each \( x \). Then the least constant \( C \), finite or infinite, for which

\[
\|Kf\|_X \leq C \int_R fu d\lambda, \quad f \in L^1_\Lambda,
\]

holds is unchanged when \( u \) is replaced by \( u^o \). That is,

\[
(3.4) \quad \sup_{f \in L^1_\Lambda} \frac{\|Kf\|_X}{\int_R fu d\lambda} = \sup_{f \in L^1_\Lambda} \frac{\|Kf\|_X}{\int_R f u^o d\lambda}.
\]

Proof. Since \( I^* u \geq I^* u^o \), Corollary 1.3 yields the inequality “\( \leq \)” in (3.4). To establish the reverse inequality we apply (2.7) of Theorem 2.5 to get

\[
\sup_{f \in L^1_\Lambda} \frac{\|Kf\|_X}{\int_R fu d\lambda} = \sup_{f \in L^1_\Lambda} \frac{\|Kf\|_X}{\inf_{g \in L^1_\Lambda} \inf_{I_g \geq I_f} \int_R gu d\lambda} = \sup_{f \in L^1_\Lambda} \sup_{g \in L^1_\Lambda} \frac{\|Kf\|_X}{\int_R gu d\lambda}.
\]

Now if \( I^* f \leq I^* g \) then the monotonicity of \( k \) and Corollary 1.3 shows that \( Kf \leq Kg \) and since \( X \) is a Banach Function Space we have \( \|Kf\|_X \leq \|Kg\|_X \). Thus

\[
\sup_{f \in L^1_\Lambda} \frac{\|Kf\|_X}{\int_R fu d\lambda} \leq \sup_{f \in L^1_\Lambda} \sup_{g \in L^1_\Lambda} \frac{\|Kg\|_X}{\int_R gu d\lambda} = \sup_{g \in L^1_\Lambda} \frac{\|Kg\|_X}{\int_R gu d\lambda}.
\]

This completes the proof.

The results for non-decreasing kernels corresponding to Theorems 3.5 and 3.6 involve the non-decreasing level function. (See the remark at the end of Section 2.) We leave their formulation to the reader.

As an application of these results we present a companion result to [11, Theorem 4.4] and then a special case related to [9, Theorem 1]. Let \( \Lambda = I1 \) so that \( \Lambda(x) = \lambda((−\infty, x]) \). Define the operator \( P \) by

\[
Pf = (If)/\Lambda + If(\infty)/\Lambda(\infty).
\]

Note that the second term in the definition of \( P \) is absent if \( \lambda \) is an infinite measure. When working in rearrangement invariant spaces it is natural to assume that the underlying measure is either non-atomic or purely atomic with all atoms having equal measure. This ensures, among other things, that the associate space of a rearrangement invariant space is again rearrangement invariant. Under this assumption [11, Theorem 4.4] shows that

\[
\sup_{g \in L^1_\Lambda} \int_R fg d\lambda \approx \|Pf\|_X
\]
provided $P : X \to X$ is bounded. Lemma 1.2 shows that we can take the supremum over $g = \lambda^*G \in \lambda^*L^+_\lambda$ rather than $g \in L^+_\lambda$ so we can evaluate

$$
\sup_{G \in L^+_\lambda} \int_R uG d\lambda \|I^*G\|_{X'}
$$

where $u = If$. Here $u$ is non-decreasing but by transferring monotonicity we can evaluate the above supremum for arbitrary $u \in L^+_\lambda$.

The boundedness of $P : X \to X$ is equivalent to the upper Boyd index of $X$ being less than 1. For more information about Boyd indices see [2, 7].

**Theorem 3.7.** Let $\lambda$ be a $\sigma$-finite measure on $R$ that is either non-atomic or purely atomic with all atoms having equal measure. Suppose that $X$ is a rearrangement invariant Banach function space of $\lambda$-measurable functions. If $P : X \to X$ is bounded then

$$
\sup_{G \in L^+_\lambda} \frac{\int_R uG d\lambda}{\|I^*G\|_{X'}} \approx \|u^\dagger/\Lambda\|_X + u^\dagger(\infty)\|1\|_X/\Lambda(\infty).
$$

Here 1 represents the constant function with value 1.

Note that if $\lambda(R) = \infty$ the second term on the right hand side is absent.

**Proof.** The kernel of $I^*$ is $\chi_{[x,\infty)}(t)$ which is non-decreasing in $t$ for each $x$. We may apply Corollary 3.4 to get

$$
\sup_{G \in L^+_\lambda} \int_R uG d\lambda = \sup_{G \in L^+_\lambda} \|I^*G\|_{X'}.
$$

Since $u^\dagger$ is non-decreasing it can be approximated from below by integrals. Thus

$$
\sup_{G \in L^+_\lambda} \frac{\int_R u^\dagger G d\lambda}{\|I^*G\|_{X'}} = \sup_{G \in L^+_\lambda} \sup_{f \in L^+_\lambda \text{ if } u^\dagger f \leq u^\dagger} \int_R (If)G d\lambda = \sup_{G \in L^+_\lambda} \sup_{f \in L^+_\lambda \text{ if } u^\dagger f \leq u^\dagger} \int_R f(I^*G) d\lambda.
$$

Using the fact that a non-increasing function can be approximated from below by integrals we have

$$
\sup_{G \in L^+_\lambda} \sup_{f \in L^+_\lambda \text{ if } u^\dagger f \leq u^\dagger} \int_R f(I^*G) d\lambda = \sup_{G \in L^+_\lambda} \sup_{g \in L^+_\lambda \text{ if } u^\dagger g \leq u^\dagger} \int_R f(I^*G) d\lambda = \sup_{G \in L^+_\lambda} \sup_{g \in L^+_\lambda \text{ if } u^\dagger g \leq u^\dagger} \int_R g d\lambda.
$$

The inner supremum above is equivalent to $\|Pf\|_X$ by [11, Theorem 4.4] so we have

$$
\sup_{G \in L^+_\lambda} \int_R u^\dagger G d\lambda \approx \sup_{f \in L^+_\lambda \text{ if } u^\dagger f \leq u^\dagger} \|Pf\|_X \approx \sup_{f \in L^+_\lambda \text{ if } u^\dagger f \leq u^\dagger} \|If/\Lambda\|_X + If(\infty)\|1\|_X/\Lambda(\infty)
$$

$$
= \|u^\dagger/\Lambda\|_X + u^\dagger(\infty)\|1\|_X/\Lambda(\infty).
$$

This completes the proof.
Corollary 3.8. Suppose that \(1 < p < \infty\), \(1/p + 1/p' = 1\), and \(v\) is a non-negative, Lebesgue measurable function defined on \((0, \infty)\) which is finite almost everywhere. Then

\[
(3.5) \quad \sup_{g \in L^+_\lambda} \left( \int_0^\infty \left( \int_0^\infty \left( \int_0^x g \right)^{p'} v(x) \, dx \right)^{1/p'} \right) \approx \left( \int_0^\infty \left( \int_0^x v^{-p} v(x) \, dx \right)^{1/p} + u^1(\infty) \left( \int_0^\infty v \right)^{-1/p'} \right).
\]

Proof. Since \(v\) is finite almost everywhere, the measure \(\lambda\) defined by

\[
d\lambda(x) = v(x) \chi_{(0, \infty)}(x) \, dx
\]

is \(\sigma\)-finite and non-atomic. With respect to this underlying measure the weighted Lebesgue space \(L^p_v\) having norm

\[
\|f\|_{L^p_v} = \left( \int_0^\infty |f|^p v \right)^{1/p}
\]

is rearrangement invariant and its associate space is \(L^{p'}_{v'}\). Moreover, since \(1 < p < \infty\), the upper Boyd index of \(L^p_v\) is \(1/p\) which is less than 1. Therefore the conclusion of Theorem 3.5 holds. We have

\[
\sup_{G \in L^+_\lambda} \int_0^\infty uGv \left( \int_0^\infty \left( \int_0^x Gv \right)^{p'} v(x) \, dx \right)^{1/p'} \approx \left( \int_0^\infty u^1(x)^p \left( \int_0^x v^{-p} v(x) \, dx \right)^{1/p} + u^1(\infty) \left( \int_0^\infty v \right)^{1/p} \left( \int_0^\infty v \right)^{-1} \right).
\]

With \(g = Gv\) this reduces to (3.5) to complete the proof.

4. Quasi-Concave Functions

We call a function \(h : (0, \infty) \to [0, \infty)\) quasi-concave and write \(h \in \Omega_{0,1}\), provided \(h(t)\) is non-decreasing and \(t^{-1}h(t)\) is non-increasing. The two quasi-concave envelopes of a Lebesgue measurable function \(u : (0, \infty) \to [0, \infty)\) are the least quasi-concave majorant of \(u\), given by

\[
\bar{u}(x) = x \operatorname{ess sup}_{t \geq x} t^{-1} \operatorname{ess sup}_{0 \leq s \leq t} u(s),
\]

and the greatest quasi-concave minorant of \(u\), given by

\[
u(x) = x \operatorname{ess inf}_{0 \leq t \leq x} t^{-1} \operatorname{ess inf}_{s \geq t} u(s).
\]

It is easy to check that \(\bar{u} \leq u \leq \nu\), that \(\bar{u}\) and \(\nu\) are both quasi-concave, and that they are envelopes in the following sense: If \(u \leq h\) and \(h\) is quasi-concave then \(\bar{u} \leq h\). Also, if \(h \leq u\) and \(h\) is quasi-concave then \(h \leq \nu\).

In this section we take the measure \(\lambda\) to be Lebesgue measure on \((0, \infty)\) and recall the definitions of \(u^1\), \(u^1_\downarrow\), \(u^1\), and \(u^1_\uparrow\) given in Section 2.
Theorem 4.1. Suppose that for each $x$, $k(x,t)$ is a quasi-concave function of $t$. Then the least constant $C$, finite or infinite, for which

$$\int_0^\infty f u \leq C \|Kf\|_X, \quad f \geq 0,$$

is unchanged when $u$ is replaced by $\bar{u}$. That is,

$$\sup_{f \geq 0} \int_0^\infty f u \|Kf\|_X = \sup_{f \geq 0} \int_0^\infty f \bar{u} \|Kf\|_X.$$

Proof. Since $k(x,t)$ is non-decreasing in $t$ we may apply Corollary 3.4 to get

$$\sup_{f \geq 0} \int_0^\infty f u \|Kf\|_X = \sup_{f \geq 0} \int_0^\infty f \bar{u} \|Kf\|_X.$$

Let $l(x,t) = t^{-1}k(x,t)$ and define the operator $L$ by

$$Lg(x) = \int_0^\infty l(x,t)g(t) dt.$$

Note that if we set $g(t) = tf(t)$ then $Kf = Lg$. Now $l(x,t)$ is non-increasing in $t$ so by Theorem 3.3 we have

$$\sup_{f \geq 0} \int_0^\infty f u \|Kf\|_X = \sup_{g \geq 0} \int_0^\infty g w \|Lg\|_X = \sup_{g \geq 0} \int_0^\infty g w \|Lg\|_X$$

where $w(t) = t^{-1}u^\dagger(t)$. The definition of $\bar{u}$ shows that $xw^\dagger(x) = \bar{u}(x)$. Therefore

$$\sup_{g \geq 0} \int_0^\infty g w \|Lg\|_X = \sup_{f \geq 0} \int_0^\infty f \bar{u} \|Kf\|_X$$

which completes the proof.

In just the same way the next theorem follows from Corollary 3.2 and Theorem 3.1. We omit the details.

Theorem 4.2. Suppose that for each $x$, $k(x,t)$ is a quasi-concave function of $t$. Then the least constant $C$, finite or infinite, for which

$$\|Kf\|_X \leq C \int_0^\infty f u, \quad f \geq 0,$$

is unchanged when $u$ is replaced by $u$. That is,

$$\sup_{f \geq 0} \int_0^\infty f u \|Kf\|_X = \sup_{f \geq 0} \int_0^\infty f u \|Kf\|_X.$$
With our choice of \( \lambda \) the operators \( I \) and \( I^* \) become

\[
I f(x) = \int_0^x f \quad \text{and} \quad I^* f(x) = \int_x^\infty f.
\]

The composition \( II^* \) maps the cone of non-negative functions to the cone of quasi-concave functions. It is well known, see for example [12, Lemma 2.3], that every quasi-concave function is equivalent to an increasing limit of functions in the image \( II^*(L^+_\lambda) \). Work of [5, 6, 12] has characterized weighted Lebesgue space imbeddings of the cone of quasi-concave functions. In the next theorem we apply a special case of [5, Theorem 5.1(ii)].

**Theorem 4.3.** Suppose \( 1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1, \) and \( f \in L^+_\lambda \). Then

\[
\sup_{f \geq 0} \frac{\int_0^\infty fg}{\left( \int_0^\infty (II^* f)^{p} v \right)^{1/p}} \approx \|g\|_{L^{p'}_{\sigma}}
\]

where

\[
\sigma(x) = \frac{1}{x} \left( \int_0^x t^{p} v(t) dt \right) \left( x^{p} \int_0^\infty v(t) dt \right) \left( \int_0^x t^{p} v(t) dt + x^{p} \int_x^\infty v(t) dt \right)^{-p'-1}.
\]

**Proof.** Interchanging the order of integration in the composition \( II^* \) yields

\[
II^* f(x) = \int_0^x \int_s^\infty f(t) dt \, ds = \int_0^\infty \min(x, t) f(t) \, dt.
\]

The kernel, \( \min(x, t) \), is a quasi-concave function of \( t \) for each \( x \) so by Theorem 4.1 we have

\[
\sup_{f \geq 0} \frac{\int_0^\infty fg}{\left( \int_0^\infty (II^* f)^{p} v \right)^{1/p}} = \sup_{f \geq 0} \frac{\int_0^\infty f\tilde{g}}{\left( \int_0^\infty (II^* f)^{p} v \right)^{1/p}}.
\]

The quasi-concave function \( \tilde{g} \) is equivalent to an increasing limit of functions in \( II^*(L^+_\lambda) \) so

\[
\sup_{f \geq 0} \frac{\int_0^\infty f\tilde{g}}{\left( \int_0^\infty (II^* f)^{p} v \right)^{1/p}} \approx \sup_{h \geq 0} \sup_{h \leq \tilde{g}} \frac{\int_0^\infty f(II^* h)}{\left( \int_0^\infty (II^* f)^{p} v \right)^{1/p}} = \sup_{h \geq 0} \sup_{h \leq \tilde{g}} \frac{\int_0^\infty (II^* f) h}{\left( \int_0^\infty (II^* f)^{p} v \right)^{1/p}}.
\]

It is easy to check that

\[
II^*(L^+_\lambda) \subset \{ tf^{**}(t) : f \in L^+_\lambda \} \subset \Omega_{0,1}
\]
where \( f^* \) is the non-increasing rearrangement of \( f \) and \( f^{**}(t) = \frac{1}{t} \int_0^t f^* \). So for any \( h \in L^+_\lambda \) we have

\[
\sup_{f \geq 0} \frac{\int_0^\infty (II^* f)h}{\left( \int_0^\infty (II^* f)^p v \right)^{1/p}} = \sup_{f \geq 0} \frac{\int_0^\infty (tf^{**}(t))h(t) dt}{\left( \int_0^\infty (tf^{**}(t))^p v(t) dt \right)^{1/p}}.
\]

By [5, Theorem 5.1(ii)], with \( q = 1, u \equiv 1, w(s) = sh(s) \) and \( v(s) \) replaced by \( s^p v(s) \), the last expression is equivalent to

\[
\left( \int_0^\infty \left( \int_0^\infty \min(1, x/t) th(t) dt \right)^{p'} \sigma(x) dx \right)^{1/p'}.
\]

This reduces to \( \|II^* h\|_{L^p'} \) so we have

\[
\sup_{f \geq 0} \frac{\int_0^\infty fg}{\left( \int_0^\infty (II^* f)^p v \right)^{1/p}} \approx \sup_{h \geq 0} \|II^* h\|_{L^p'} = \|\bar{g}\|_{L^p'}
\]
as required.

As a consequence we are able to give necessary and sufficient conditions for an inequality studied in [14].

**Corollary 4.4.** Suppose that \( 1 < p < \infty, 1/p + 1/p' = 1, \) and \( v, h \in L^+_\lambda \). Then

\[
\sup_{g \in L^+_\lambda} \frac{\int_0^\infty gh}{\left( \int_0^\infty (\frac{1}{x} \int_0^x g) v(x) dx \right)^{1/p}} \approx \|H\|_{L^p'}
\]

where \( H(x) \) is the least non-increasing majorant of \( \frac{1}{x} \int_0^x h \) and

\[
\tau(x) = \frac{1}{x} \left( \frac{1}{x^p} \int_0^x v(t) dt \right) \left( \int_x^\infty v(t) \frac{dt}{t^p} \right) \left( \frac{1}{x^p} \int_0^x v(t) dt + \int_x^\infty v(t) \frac{dt}{t^p} \right)^{-p' - 1}.
\]

**Proof.** The supremum over \( L^+_\lambda \) can be replaced by a supremum over \( L^*_\lambda \) so we have

\[
\sup_{g \in L^+_\lambda} \frac{\int_0^\infty gh}{\left( \int_0^\infty (\frac{1}{x} \int_0^x g) v(x) dx \right)^{1/p}} = \sup_{f \in L^+_\lambda} \frac{\int_0^\infty (I^* f)h}{\left( \int_0^\infty (I^* f)^p v(x) dx \right)^{1/p}} = \sup_{f \in L^+_\lambda} \frac{\int_0^\infty f Ih}{\left( \int_0^\infty (I^* f)^p x^{-p} v(x) dx \right)^{1/p}} \approx \|\bar{H}\|_{L^p_{x'}}.
\]
The equivalence follows from Theorem 4.3 with $g$ replaced by $Ih$ and $v(x)$ replaced by $x^{-p}v(x)$. Here $\sigma_1$ is given by

$$\sigma_1(x) = \frac{1}{x} \left( \int_0^x v(t) \, dt \right) \left( x^p \int_x^\infty v(t) \, \frac{dt}{t^p} \right) \left( \int_0^x v(t) \, dt + x^p \int_x^\infty v(t) \, \frac{dt}{t^p} \right)^{-p' - 1}.$$ 

Now $Ih$ is non-decreasing so

$$\bar{I}h(x) = x \text{ess sup} t^{-1} \text{ess sup} Ih(s) = x \text{ess sup} t^{-1} Ih(t) = xH(x).$$

Taking the factor of $x$ into the weight we have

$$\|\bar{I}h\|_{L_{\sigma_1}^p} = \|H\|_{L_{\tau}^p}$$

to complete the proof.

5. Proof of Theorem 2.1

Some preparation is required before we give our proof of Theorem 2.1 but the intermediate results are themselves worthy of note. In particular, Lemma 5.2 and Corollary 5.3 are useful tools since they make precise the notion of pushing mass mentioned in the sketch proof of (2.1). We begin by showing that even when the function $u$ is not well behaved, each of the envelopes of $u$ is constant except where it is close to $u$.

**Lemma 5.1.** Suppose $u \in L_\lambda^+$, $a < b$ and $y \in \mathbb{R}$. If

(5.1) \{x \geq y : a < u^1(x) \leq b\} \neq \emptyset

then

(5.2) \lambda\{x \geq y : a < u(x) \leq u^1(x) \leq b\} > 0.

Similarly, if

(5.3) \{x \leq y : a \leq u^1(x) < b\} \neq \emptyset

then

(5.4) \lambda\{x \leq y : a \leq u^1(x) \leq u(x) < b\} > 0.

**Proof.** To prove the first implication we suppose that (5.1) holds and (5.2) fails and derive a contradiction. Fix $x \geq y$ such that $a < u^1(x) \leq b$. If $t \geq x$ then $u^1(t) \leq u^1(x) \leq b$ so either $u^1(t) \leq a$ or $a < u^1(t) \leq b$. For $\lambda$-almost every $t \geq x$
satisfying \( u^i(t) \leq a \) we have \( u(t) \leq u^i(t) \leq a \). Since (5.2) fails and \( x \geq y \) we also have \( u(t) \leq a \) for \( \lambda \)-almost every \( t \geq x \) satisfying \( a < u^i(t) \leq b \). It follows that \( u(t) \leq a \) for \( \lambda \)-almost every \( t \geq x \). Therefore

\[
u^i(x) = \operatorname{ess sup}_{t \geq x} u(t) \leq a
\]

which contradicts the choice of \( x \).

To prove the second implication we suppose that (5.3) holds and (5.4) fails and derive a contradiction. Fix \( x \leq y \) such that \( a < u^i(x) < b \). If \( t \leq x \) then \( u^i(t) \geq u^i(x) \geq a \) so either \( u^i(t) \geq b \) or \( u^i(t) < a \). For \( \lambda \)-almost every \( t \leq x \) satisfying \( u^i(t) \geq b \) we have \( u(t) \geq u^i(t) \geq b \). Since (5.4) fails and \( x \leq y \) we also have \( u(t) \geq b \) for \( \lambda \)-almost every \( t \leq x \) satisfying \( a < u^i(t) < b \). If follows that \( u(t) \geq b \) for \( \lambda \)-almost every \( t \leq x \). Therefore

\[
u^i(x) = \operatorname{ess inf}_{t \leq x} u(t) \geq b
\]

which contradicts the choice of \( x \).

In the next lemma we show how the mass of a function \( f \) can be “pushed” to the right and onto a small subset. Recall that \( L^+_\lambda(S) \) is the collection of non-negative \( \lambda \)-measurable functions which vanish off \( S \).

**Lemma 5.2.** Suppose \( x \in \mathbb{R} \) and \( E \) is a \( \lambda \)-measurable subset of \((-\infty, x] \) satisfying \( \lambda(E \cap (y, \infty)) > 0 \) for all \( y < x \). If \( f \in L^+_\lambda((-\infty, x]) \) then there exists a function \( g \in L^+_\lambda(E) \) such that \( Ig \leq If \) and \( I^*g \geq I^*f \).

**Proof.** We look at the simple case first. If \( x \in E \) and \( x \) is an atom for \( \lambda \) then the \( \sigma \)-finiteness of \( \lambda \) ensures that \( 0 < \lambda\{x\} < \infty \). In this case we can push all the mass of \( f \) onto a single point in \( E \). Set

\[
g = \left( \int_{\mathbb{R}} f \, d\lambda \right) \frac{\chi_{\{x\}}}{\lambda\{x\}}.
\]

Since \( f \) is zero on \( [x, \infty) \) we have

\[
Ig(y) = \begin{cases} 
0 & y < x \\
\int_{\mathbb{R}} f \, d\lambda & y \geq x
\end{cases} \leq If(y)
\]

and

\[
I^*g(y) = \begin{cases} 
\int_{\mathbb{R}} f \, d\lambda & y \leq x \\
0 & y > x
\end{cases} \geq I^*f(y).
\]

In the remaining case, either \( x \notin E \) or \( x \) is not an atom for \( \lambda \). Then we have \( \lambda(E \cap (y, x)) > 0 \) for all \( y < x \). Hence we can choose an increasing sequence \( y_1 < y_2 < \ldots \) converging to \( x \) such that

\[
\lambda(E \cap (y_n, y_{n+1}]) > 0.
\]
The \( \sigma \)-finiteness of \( \lambda \) allows us to choose subsets \( E_n \subset E \cap (y_n, y_{n+1}] \) of finite, positive \( \lambda \)-measure. Now let \( y_0 = -\infty \) and set

\[
g = \sum_{n=1}^{\infty} \left( \int_{(y_{n-1}, y_n]} f \, d\lambda \right) \frac{\chi_{E_n}}{\lambda(E_n)}.
\]

Here the mass of \( f \) has been cascaded to the right with each interval’s mass being pushed onto a small subset of the adjacent interval.

To see that \( I f \leq I g \) and \( I^* g \leq I^* f \) we argue as follows: If \( y \geq x \) then, since \( f \) and \( g \) are zero on \([x, \infty)\),

\[
I g(y) = \int_{\mathbb{R}} g \, d\lambda = \sum_{n=1}^{\infty} \int_{(y_{n-1}, y_n]} f \, d\lambda = \int_{\mathbb{R}} f \, d\lambda = I f(y).
\]

Also \( I^* g(y) = 0 = I^* f(y) \) whenever \( y \geq x \). If \( y < x \) then we choose \( N \) so that \( y \in (y_{N-1}, y_N] \). None of the sets \( E_N, E_{N+1}, \ldots \) intersect \(( -\infty, y]\) so

\[
I g(y) \leq \sum_{n=1}^{N-1} \int_{(y_{n-1}, y_n]} f \, d\lambda = I f(y_{N-1}) \leq I f(y)
\]

and

\[
I^* g(y) \geq \sum_{n=N}^{\infty} \int_{(y_{n-1}, y_n]} f \, d\lambda = \int_{(y_{N-1}, x]} f \, d\lambda = \int_{(y_{N-1}, \infty)} f \, d\lambda \geq I^* f(y).
\]

This completes the proof.

Of course, mass can also be pushed to the left.

**Corollary 5.3.** Suppose \( x \in \mathbb{R} \) and \( E \) is a \( \lambda \)-measurable subset of \([x, \infty)\) satisfying \( \lambda(E \cap (-\infty, y]) > 0 \) for all \( y > x \). If \( f \in L^+_{\lambda}([x, \infty)) \) then there exists a function \( g \in L^+_{\lambda}(E) \) such that \( I g \geq I f \) and \( I^* g \leq I^* f \).

**Lemma 5.4.** Suppose that \( f, u \in L^+_{\lambda} \) and \( \varepsilon > 0 \). Then there exists a function \( g \in L^+_{\lambda} \) such that \( I g \leq I f \) and

\[
\int_{\mathbb{R}} g u \, d\lambda \geq \int_{\mathbb{R}} f u^1 \, d\lambda - \varepsilon.
\]

**Proof.** If \( \int_{\mathbb{R}} f u^1 \, d\lambda = 0 \) then \( g = f \) satisfies the conclusion of the lemma. Otherwise, choose \( \alpha > 1 \) so close to 1 that

\[
\alpha^{-1} \int_{\mathbb{R}} f u^1 \, d\lambda \geq \int_{\mathbb{R}} f u^1 \, d\lambda - \varepsilon/2.
\]
Define
\[ J_{\infty} = \{ x \in \mathbb{R} : u^1(x) = \infty \}, \]
\[ J_n = \{ x \in \mathbb{R} : \alpha^{n-1} < u^1(x) \leq \alpha^n \}, \quad n \in \mathbb{Z}, \]
\[ J_{-\infty} = \{ x \in \mathbb{R} : u^1(x) = 0 \}. \]

Since \( u^1 \) is non-negative and non-increasing we see that each \( J_n, n \in \mathbb{Z} \cup \{ \pm \infty \} \), is an interval (possibly a singleton or an empty set) and that \( \mathbb{R} \) is the disjoint union of the \( J_n \)'s. We construct the desired function \( g \) by defining \( f_n = f \chi_{J_n} \) and constructing functions \( g_n \) to satisfy
\[ Ig_n \leq If_n, \quad n \in \mathbb{Z} \cup \{ \pm \infty \}, \]
(5.5) \[
\int_{\mathbb{R}} g_n u \, d\lambda \geq \alpha^{-1} \int_{\mathbb{R}} f_n u^1 \, d\lambda, \quad n \in \mathbb{Z} \cup \{ -\infty \}, \quad \text{and} \]
\[
\int_{\mathbb{R}} g_\infty u \, d\lambda \geq \alpha^{-1} \int_{\mathbb{R}} f_\infty u^1 - \varepsilon/2. \]

Since \( f = \sum_{n \in \mathbb{Z} \cup \{ \pm \infty \}} f_n \), it is easy to see that the function \( g = \sum_{n \in \mathbb{Z} \cup \{ \pm \infty \}} g_n \) will satisfy the conclusion of the lemma.

If \( J_n = \emptyset \) for some \( n \) then \( g_n = 0 \) satisfies (5.5). If \( n = -\infty \) then \( g_\infty = 0 \) satisfies (5.5).

If \( n \in \mathbb{Z} \) and \( J_n \neq \emptyset \) we let \( x_n = \sup J_n \) be the right endpoint of \( J_n \) and set
\[ E_n = \{ x \in \mathbb{R} : \alpha^{n-1} < u(x) \leq u^1(x) \leq \alpha^n \} \subset (-\infty, x_n]. \] 

If \( y > x_n \) then \( J_n \cap [y, \infty) \neq \emptyset \) so Lemma 5.1 yields
\[ \lambda(E_n \cap [y, \infty)) > 0. \]

Now \( f_n \in L_\lambda^+((-\infty, x_n]) \) so by Lemma 5.2 there exists a \( g_n \in L_\lambda^+(E_n) \) such that \( Ig_n \leq If_n \) and \( I^* g_n \geq I^* f_n \). Since \( g_n \) is zero off \( E_n \) and \( f_n \) is zero off \( J_n \), we have
\[
\int_{\mathbb{R}} g_n u \, d\lambda \geq \alpha^{n-1} \int_{\mathbb{R}} g_n \, d\lambda = \alpha^{n-1} I^* g_n (-\infty) \]
\[
\geq \alpha^{n-1} I^* f_n (-\infty) = \alpha^{n-1} \int_{\mathbb{R}} f_n \, d\lambda \geq \alpha^{-1} \int_{\mathbb{R}} f_n u^1 \, d\lambda. \]

The remaining case is \( n = \infty \) and \( J_{\infty} \neq \emptyset \). If \( \int_{\mathbb{R}} f_\infty u^1 \, d\lambda = 0 \) then we can take \( g_\infty = 0 \). Otherwise, \( f_\infty \) does not vanish so \( \int_{\mathbb{R}} f_\infty \, d\lambda > 0 \). Set \( x_\infty = \sup J_{\infty} \). If \( x_\infty \in J_{\infty} \) then \( u^1(x_\infty) = \infty \) so for \( k = 1, 2, \ldots \) using the definition of \( u^1 \) we can choose subsets
\[ U_k \subset \{ x \geq x_\infty : u(x) > 2^k \} \]
of finite positive $\lambda$-measure and set

$$g_\infty = \left( \int_{(-\infty,x_{\infty}\!]} f_\infty \, d\lambda \right) \sum_{k=1}^{\infty} 2^{-k} \frac{\chi_{U_k}}{\lambda(U_k)}.$$

Otherwise, $u^\downarrow(x_{\infty} - 1/k) = \infty$ for each positive integer $k$, so we can choose subsets

$$U_k \subset \{ x \geq x_{-\infty} - 1/k : u(x) > 2^k \}$$

of finite positive $\lambda$-measure and set

$$g_\infty = \sum_{k=1}^{\infty} \left( \int_{(-\infty,x_{\infty}-1/k]} f_\infty \, d\lambda \right) 2^{-k} \frac{\chi_{U_k}}{\lambda(U_k)}.$$

Either way the integral of $g_\infty u$ is infinite because

$$2^{-k} \int \frac{\chi_{U_k}}{\lambda(U_k)} u \, d\lambda \geq 1.$$

If $y \geq x_{\infty}$ then

$$Ig_\infty(y) \leq \int_{(-\infty,x_{\infty}\!]} f_\infty \, d\lambda \leq If_\infty(y).$$

If $y < x_{\infty}$ then either none of the $U_k$’s intersects $(-\infty, y]$ or only those $U_k$ with $k$ satisfying $x_{\infty} - 1/k \leq y$ intersect $(-\infty, y]$. Therefore either $Ig_\infty(y) = 0$ or

$$Ig_\infty(y) \leq \sum_{1/k \geq x_{\infty} - y} \left( \int_{(-\infty,x_{\infty}-1/k]} f_\infty \, d\lambda \right) 2^{-k} \leq If_\infty(y).$$

This completes the proof.

The corresponding result for the lower envelope is just different enough that a separate proof is required.

**Lemma 5.5.** Suppose that $f, u \in L^+_\lambda$ and $\varepsilon > 0$. Then there exists a function $g \in L^+_\lambda$ such that $Ig \geq If$ and

$$\int_R gu \, d\lambda \leq \int_R fu^\downarrow \, d\lambda + \varepsilon.$$

**Proof.** If $\int_R fu^\downarrow \, d\lambda = \infty$ then $g = f$ satisfies the conclusion of the lemma. Otherwise, choose $\alpha > 1$ so close to 1 that

$$\alpha \int_R fu^\downarrow \, d\lambda < \int_R fu^\downarrow \, d\lambda + \varepsilon/2.$$
Define
\[ J_\infty = \{ x \in \mathbb{R} : u_\downarrow(x) = \infty \}, \]
\[ J_n = \{ x \in \mathbb{R} : \alpha^n \leq u_\downarrow(x) < \alpha^{n+1}, \quad n \in \mathbb{Z} \}, \]
\[ J_{-\infty} = \{ x \in \mathbb{R} : u_\downarrow(x) = 0 \}. \]

Since \( u_\downarrow \) is non-negative and non-increasing we see that each \( J_n, n \in \mathbb{Z} \cup \{\pm \infty\} \), is an interval (possibly a singleton or an empty set) and that \( \mathbb{R} \) is the disjoint union of the \( J_n \)'s. We construct the desired function \( g \) by defining \( f_n = f\chi_{J_n} \) and constructing functions \( g_n \) to satisfy
\[ \int_{\mathbb{R}} g_n u \, d\lambda \leq \int_{\mathbb{R}} f_n u_\downarrow \, d\lambda, \quad n \in \mathbb{Z} \cup \{ \infty \}, \]
and
\[ \int_{\mathbb{R}} g_{-\infty} u \, d\lambda \leq \varepsilon/2. \]

Since \( f = \sum_{n \in \mathbb{Z} \cup \{\pm \infty\}} f_n \), it is easy to see that the function \( g = \sum_{n \in \mathbb{Z} \cup \{\pm \infty\}} g_n \) will satisfy the conclusion of the lemma.

If \( J_n = \emptyset \) for some \( n \) then \( g_n = 0 \) satisfies (5.6). If \( n = \infty \) then
\[ \int_{\mathbb{R}} f_\infty u_\downarrow \, d\lambda \leq \int_{\mathbb{R}} f_\infty \, d\lambda < \infty \]
so \( f_\infty \) necessarily vanishes \( \lambda \)-almost everywhere. Thus \( g_\infty = 0 \) satisfies (5.6).

If \( n \in \mathbb{Z} \) and \( J_n \neq \emptyset \) we set \( x_n = \inf J_n \) and
\[ E_n = \{ x \in \mathbb{R} : \alpha^n \leq u_\downarrow(x) \leq u(x) < \alpha^{n+1} \} \subset [x_n, \infty). \]
If \( y < x_n \) then \( J_n \cap (-\infty, y] \neq \emptyset \) so Lemma 5.1 yields
\[ \lambda(E_n \cap (-\infty, y]) > 0. \]

Now \( f_n \in L^+_\lambda([x_n, \infty)) \) so by Corollary 5.3 there exists a \( g \in L^+_\lambda(E_n) \) such that \( Ig_n \geq If_n \) and \( I^*g_n \leq I^*f_n \). Since \( g_n \) is zero off \( E_n \) and \( f_n \) is zero off \( J_n \), we have
\[ \int_{\mathbb{R}} g_n u \, d\lambda \leq \alpha^{n+1} \int_{\mathbb{R}} g_n \, d\lambda = \alpha^{n+1} I^*g_n(-\infty) \]
\[ \leq \alpha^{n+1} I^*f_n(-\infty) = \alpha^{n+1} \int_{\mathbb{R}} f_n \, d\lambda \leq \alpha \int_{\mathbb{R}} f_n u_\downarrow \, d\lambda. \]

The remaining case is \( n = -\infty \) and \( J_{-\infty} \neq \emptyset \). Set \( x_{-\infty} = \inf J_{-\infty} \). If \( x_{-\infty} \in J_{-\infty} \) then \( u_\downarrow(x_{-\infty}) = 0 \) so for \( k = 1, 2, \ldots \) we can choose subsets
\[ U_k \subset \{ x \leq x_{-\infty} : u(x) < 2^{-k} \} \]
of finite positive $\lambda$-measure. If $x_{-\infty} \notin J_{-\infty}$ then $u_1(x_{-\infty} + 1/k) = 0$ for each positive integer $k$, so we can choose subsets

$$U_k \subset \{x \leq x_{-\infty} + 1/k : u(x) < 2^{-k}\}$$

of finite positive $\lambda$-measure. Set

$$g_{-\infty} = \frac{\varepsilon}{2} \sum_{k=1}^{\infty} \frac{\chi_{U_k}}{\lambda(U_k)}.$$

The choice of $U_k$ ensures that the integral of $g_{-\infty} u$ is small. We have

$$\int_{\mathbb{R}} g_{-\infty} u \, d\lambda = \frac{\varepsilon}{2} \sum_{k=1}^{\infty} \frac{\int_{U_k} u \, d\lambda}{\lambda(U_k)} \leq \frac{\varepsilon}{2} \sum_{k=1}^{\infty} 2^{-k} = \frac{\varepsilon}{2}.$$

If $x_{-\infty} \in J_{-\infty}$ then $I f$ is zero on $(-\infty, x_{-\infty})$ and, since all the $U_k$’s are contained in $(-\infty, x_{-\infty})$, $I g$ is infinite on $[x_{-\infty}, \infty)$. Hence $I g \geq I f$. If $x_{-\infty} \notin J_{-\infty}$ then $I f$ is zero on $(-\infty, x_{-\infty})$ and, since infinitely many of the $U_k$’s are contained in $(x_{-\infty}, x)$ for any $x > x_{-\infty}$, $I g$ is infinite on $(x_{-\infty}, \infty)$. Again $I g \geq I f$. This completes the proof.

Proof of Theorem 2.1. We prove only (2.1) and (2.2) since (2.3) and (2.4) follow from them by making the substitution $x \to -x$ throughout.

If $I g \leq I f$ then, since $u(x) \leq u^+(x) \lambda$-almost everywhere, we may use Corollary 1.3 to get

$$\int_{\mathbb{R}} g u \, d\lambda \leq \int_{\mathbb{R}} g u^+ \, d\lambda \leq \int_{\mathbb{R}} f u^+ \, d\lambda.$$

If $I g \geq I f$ then, since $u(x) \geq u_1(x) \lambda$-almost everywhere, we may use Corollary 1.3 to get

$$\int_{\mathbb{R}} g u \, d\lambda \geq \int_{\mathbb{R}} g u_1 \, d\lambda \geq \int_{\mathbb{R}} f u_1 \, d\lambda.$$

Thus

$$\sup_{I g \leq I f} \int_{\mathbb{R}} g u \, d\lambda \leq \int_{\mathbb{R}} f u^+ \, d\lambda \quad \text{and} \quad \inf_{I g \geq I f} \int_{\mathbb{R}} g u \, d\lambda \geq \int_{\mathbb{R}} f u_1 \, d\lambda.$$

For the reverse inequalities we use Lemmas 5.4 and 5.5. For each $\varepsilon > 0$ we apply Lemma 5.4 to get a function $g_\varepsilon$ satisfying $I g_\varepsilon \leq I f$ and

$$\int_{\mathbb{R}} f u^+ \, d\lambda \leq \int_{\mathbb{R}} g_\varepsilon u \, d\lambda + \varepsilon.$$

Now

$$\sup_{I g \leq I f} \int_{\mathbb{R}} g u \, d\lambda \geq \sup_{\varepsilon > 0} \int_{\mathbb{R}} g_\varepsilon u \, d\lambda \geq \int_{\mathbb{R}} f u^+ \, d\lambda.$$
We apply Lemma 5.5 to functions $f$ and $u$ to get a function $g_\varepsilon$ satisfying $Ig_\varepsilon \geq If$ and
\[
\int_R fu_\downarrow d\lambda \geq \int_R g_\varepsilon u d\lambda - \varepsilon.
\]
Now
\[
\inf_{Ig \geq If} \int_R gu d\lambda \leq \inf_{\varepsilon > 0} \int_R g_\varepsilon u d\lambda \leq \int_R fu_\downarrow d\lambda.
\]
This completes the proof.

6. The Structure of General Level Functions

In Proposition 1.5(iii) the structure of the level function of a bounded function, supported on $(-\infty, M]$ for some $M$ is given in terms of an averaging operator $A \in \mathcal{A}$. Although one would expect that taking increasing limits of such level functions would destroy this simple structure we show here that it does not. The structure of the level function of an arbitrary function remains attractively simple.

\textbf{Theorem 6.1.} Suppose $u$ is non-negative and $\lambda$-measurable. Then there exists an $A \in \mathcal{A}$ and (possible empty) intervals $J_{\text{left}}$ and $J_{\text{right}}$ such that

i) if $J_{\text{left}} \neq \emptyset$ then $\inf J_{\text{left}} = -\infty$ and $u^o \equiv \infty$ on $J_{\text{left}}$;

ii) $u^o = Au$ $\lambda$-almost everywhere off $J_{\text{left}} \cup J_{\text{right}}$;

iii) if $J_{\text{right}} \neq \emptyset$ then $\sup J_{\text{right}} = \infty$, $\lambda(J_{\text{right}}) = \infty$, and on $J_{\text{right}}$

\[
\int_{(-\infty, x] \cap J_{\text{right}}} u_n d\lambda \leq \int_{(-\infty, x] \cap J_{\text{right}}} \limsup_{x \to \infty} u_{n, o} d\lambda.
\]

\textit{Proof.} Set $u_n = \min(u, n)\chi_{(-\infty, n]}$ so that $u_n \uparrow u$ and, by Proposition 1.5(ii), $u_{n, o} \uparrow u^o$ pointwise on $\mathbb{R}$. Each $u_n$ is bounded and supported on $(-\infty, n]$ so there is an operator $A_n \in \mathcal{A}$ such that $u_{n, o} = A_n u_n$. We call the intervals of $A_n$ the \textit{level intervals} of $u_n$ and note that $u_{n, o}$ is constant on each of its level intervals. Also note that level intervals have positive $\lambda$-measure by definition. If an interval $J$ has the property that every level interval of $u_n$ that intersects $J$ is contained in $J$ we say that the interval \textit{respects} $u_n$. If $J$ respects $u_n$ then $A_n \chi_J = \chi_J$ so

\[
\int_J u_n d\lambda = \int_R A_n \chi_J u_n d\lambda = \int_R \chi_J A_n u_n d\lambda = \int_J u_{n, o} d\lambda
\]

and for any $x \in J$ the interval $(-\infty, x] \setminus J$ also respects $u_{n, o}$ so, by Proposition 1.5(i),

\[
\int_{(-\infty, x] \setminus J} u_n d\lambda = \int_{(-\infty, x]} u_n d\lambda - \int_{(-\infty, x] \setminus J} u_n d\lambda \leq \int_{(-\infty, x]} u_{n, o} d\lambda - \int_{(-\infty, x] \setminus J} u_{n, o} d\lambda = \int_{(-\infty, x] \cap J} u_{n, o} d\lambda
\]
These two together show that if $J$ respects $u_n$ and $x \in \mathbb{R}$ then
\[
\int_J u_n \, d\lambda = \int_J u_n^o \, d\lambda \quad \text{and} \quad \int_{(-\infty,x] \cap J} u_n \, d\lambda \leq \int_{(-\infty,x] \cap J} u_n^o \, d\lambda.
\]

Notice that for any interval $J$ the interval $(u_n^o)^{-1}(u_n^o(J))$ respects $u_n$. In particular, if $s < t$ then with $J = (u_n^o)^{-1}(u_n^o(s,\infty))$ we have
\[
\int_{[s,t]} u_n \, d\lambda \leq \int_{(-\infty,t] \cap J} u_n \, d\lambda \leq \int_{(-\infty,t] \cap J} u_n^o \, d\lambda \leq u^o(s)\lambda(-\infty,t].
\]
Letting $n \to \infty$ we conclude that
\[
(6.2) \quad \int_{[s,t]} u \, d\lambda \leq u^o(s)\lambda(-\infty,t] \quad \text{and} \quad \int_{(s,t]} u \, d\lambda \leq u^o(s+)\lambda(-\infty,t].
\]

Set $J_{\text{left}} = (u^o)^{-1}(-\infty)$. Since $u^o$ is non-increasing, $J_{\text{left}} = \emptyset$ or else $\inf J_{\text{left}} = -\infty$ and we have proved part (i). For $0 \leq y < \infty$ set
\[
J_y = (u^o)^{-1}(y).
\]
The $J_y$ are (possibly empty or singleton) intervals that partition $\mathbb{R} \setminus J_{\text{left}}$. Since $\lambda(-\infty,x] < \infty$ for all $x \in \mathbb{R}$ at most one of the intervals $J_y$ has infinite $\lambda$-measure. If no $J_y$ has infinite $\lambda$-measure then let $J_{\text{right}} = \emptyset$. Otherwise, let $J_{\text{right}}$ be the unique interval $J_y$ having infinite $\lambda$-measure and note that $\sup J_{\text{right}} = \infty$. To define the averaging operator $A$ we specify its intervals. See the definition of $A$ preceding Proposition 1.4. We take the intervals of $A$ to be all the intervals $J_y$ having interior and satisfying $0 < \lambda(J_y) < \infty$.

To show that $u^o = Au$ off $J_{\text{left}} \cup J_{\text{right}}$ we first show that $u^o(x) = u(x)$ for $\lambda$-almost every $x$ in some $J_y$ with no interior or with zero $\lambda$-measure. The inequalities (6.2) show that $u$ is locally $\lambda$-integrable on $\mathbb{R} \setminus J_{\text{left}}$ so the set of points in the $\lambda$-Lebesgue sets \cite[p156]{4} of all the functions $u, u_1, u_2, \ldots$ has full $\lambda$-measure. We consider only these points. Also, since the $J_y$ are disjoint, at most countably many have interior. The union of those $J_y$ with interior and zero $\lambda$-measure also has zero $\lambda$-measure so we may disregard points in such intervals.

The remaining points $x$ each lie in some interval $J_y$ with no interior, Thus $J_y = \{x\}$. If $u_n(x) = u_n^o(x)$ for infinitely many $n$ then we have $u(x) = u^o(x)$ as required. If not, then for some $N$, $x$ is in a level interval $L_n$ of $u_n$ for all $n \geq N$. We have $0 < \lambda(L_n) < \infty$ and $u_n^o$ is constant on $L_n$, taking the value
\[
\frac{1}{\lambda(L_n)} \int_{L_n} u \, d\lambda.
\]
Since $J_y = \{x\}$, for any $a, d$ with $a < x < d$ we have $u^o(a) > u^o(x) > u^o(d)$ and hence $u_n^o(a) > u_n^o(x) > u_n^o(d)$ for sufficiently large $n$. It follows that $x \in L_n \subset (a,d)$.
for sufficiently large $n$. We have shown that the diameter of $L_n$ converges to zero with $n$. Since $x$ is in the $\lambda$-Lebesgue set of $u$ and $u_m$ for all $m$ we have

$$
\lim_{n \to \infty} \frac{1}{\lambda(L_n)} \int_{L_n} u \, d\lambda = u(x) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{\lambda(L_n)} \int_{L_n} (u - u_m) \, d\lambda = u(x) - u_m(x)
$$

for all $m$. Now

$$
|u(x) - u^\circ(x)| = \lim_{n \to \infty} \left| \left( \frac{1}{\lambda(L_n)} \int_{L_n} u \, d\lambda \right) - u^\circ_n(x) \right|
$$

$$
= \lim_{n \to \infty} \frac{1}{\lambda(L_n)} \int_{L_n} (u - u_n) \, d\lambda
$$

$$
\leq \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{\lambda(L_n)} \int_{L_n} (u - u_m) \, d\lambda
$$

$$
= \lim_{m \to \infty} u(x) - u_m(x) = 0.
$$

Therefore $u(x) = u^\circ(x)$ for $\lambda$-almost every $x \in \mathbb{R} \setminus (J_{\text{left}} \cup J_{\text{right}})$ off the intervals of $A$.

To complete the proof of part (ii) we must show that

$$
y = \frac{1}{\lambda(J_y)} \int_{J_y} u \, d\lambda
$$

for all $y$ such that $J_y$ has interior and $0 < \lambda(J_y) < \infty$. Since

$$
\lim_{n \to \infty} \int_{J_y} u^\circ_n \, d\lambda = \int_{J_y} u^\circ \, d\lambda = y\lambda(J_y) < \infty \quad \text{and} \quad \lim_{n \to \infty} \int_{J_y} u_n \, d\lambda = \int_{J_y} u \, d\lambda
$$

it is enough to show that

$$
(6.3) \quad \lim_{n \to \infty} \left| \int_{J_y} (u^\circ_n - u_n) \, d\lambda \right| = 0.
$$

To do this we let $s = \inf J_y$ and proceed in the case that $s \notin J_y$. The case $s \in J_y$ is similar. Since $s \notin J_y$ we have $u^\circ(s+) = y$ and (6.2) implies that for any $t > s$,

$$
\int_{(s,t]} u \, d\lambda \leq y\lambda(-\infty, t] < \infty.
$$

Therefore, for any $\varepsilon > 0$ we may choose $a$, $b$, $c$, and $d$ satisfying

$$
-\infty \leq a < b \leq c < d \leq \infty,
$$

$$
[b, c] \subset J_y \subset (a, d),
$$

$$
y\lambda((a, b) \cup (c, d)) < \varepsilon/4, \quad \text{and}
$$

$$
\int_{(s,b)\cup(c,d)} u \, d\lambda < \varepsilon/2.
$$
(Note that if \( \sup J_y = \infty \) then \( \lambda(J_y) < \infty \) implies \( \lambda(R) < \infty \) so this is possible even if \( d \) is forced to be \( \infty \).) Since \( s \notin J_y \) and \( b \in J_y \), we have either \( a = -\infty \) or else \( u^\alpha(a) > u^\alpha(b) \) and hence \( u_n^\alpha(a) > u_n^\alpha(b) \) for sufficiently large \( n \). Also, \( c \in J_y \) and \( d \notin J_y \) so either \( d = \infty \) or else \( u^\alpha(c) > u^\alpha(d) \) and hence \( u_n^\alpha(c) > u_n^\alpha(d) \) for sufficiently large \( n \). Therefore, if we set

\[
J_y(n) = (u_n^\alpha)^{-1}(u_n^\alpha[b, c]),
\]
we have \([b, c] \subset J_y(n) \subset (a, d)\) and we can estimate the symmetric difference of \( J_y \) and \( J_y(n) \) by

\[
J_y \triangle J_y(n) \equiv (J_y \setminus J_y(n)) \cup (J_y(n) \setminus J_y) \subset (a, b) \cup (c, d)
\]
for sufficiently large \( n \). The choice of \( J_y(n) \) ensures that \( J_y(n) \) respects \( u_n \) so we have

\[
\int_{J_y(n)} (u_n^\alpha - u_n) \, d\lambda = 0 \quad \text{and} \quad \int_{(-\infty, s] \cap J_y(n)} (u_n^\alpha - u_n) \, d\lambda \geq 0
\]
and therefore

\[
\lim_{n \to \infty} \left| \int_{J_y} (u_n^\alpha - u_n) \, d\lambda \right| \\
\leq \lim_{n \to \infty} \int_{J_y \triangle J_y(n)} (u_n^\alpha + u_n) \, d\lambda \\
\leq \int_{(-\infty, s] \cap J_y(n)} u_n \, d\lambda + \int_{(s, b) \cup (c, d)} u_n \, d\lambda \\
\leq \lim_{n \to \infty} \int_{J_y \triangle J_y(n)} u_n \, d\lambda + \int_{(-\infty, s] \cap J_y(n)} u_n^\alpha \, d\lambda + \int_{(s, b) \cup (c, d)} u \, d\lambda \\
\leq \lim_{n \to \infty} 2y\lambda((a, b) \cup (c, d)) + \int_{(s, b) \cup (c, d)} u \, d\lambda < \varepsilon.
\]

Here we have used the fact that \( u_n^\alpha \leq u^\alpha = y \) on \( J_y \) and \( u_n^\alpha \leq u_n^\alpha(b) \leq u^\alpha(b) = y \) on \( J_y(n) \). Since \( \varepsilon \) was arbitrary, this proves (6.3) and completes part (ii).

If \( J_{\text{right}} = \emptyset \) there is nothing more to prove. Otherwise \( J_{\text{right}} = J_y \) for the unique \( y \) satisfying \( \lambda(J_y) = \infty \). We have already argued that \( \sup J_{\text{right}} = \infty \). To prove (6.1) we set \( s = \sup J_y \) as before. The two cases \( s \in J_y \) and \( s \notin J_y \) are similar again so this time we look at the case \( s \in J_y \). We have \( u^\alpha(s) = y \) and (6.2) implies that for any \( t > s \),

\[
\int_{[s, t]} u \, d\lambda \leq y\lambda(-\infty, t] < \infty.
\]
Therefore, for any \( \varepsilon > 0 \) and any \( x > s \) we may choose \( a, b, \) and \( c \) satisfying
\[
-\infty \leq a < b < x \leq c < \infty,
\]
\([b, c] \subset J_y \subset (a, \infty),\)
\[y \lambda((a, b)) < \varepsilon/4, \text{ and}\]
\[\int_{[s, b)} u \, d\lambda < \varepsilon/2.\]

Since \( a \notin J_y \) and \( b \in J_y \) we have either \( a = -\infty \) or else \( u^o(a) > u^o(b) \) and hence we may choose \( N \) so that \( u^o_n(a) > u^o_n(b) \) for \( n > N \). Therefore, if we set
\[J_y(n) = (u_n^o)^{-1}(u_n^o[b, c]),\]
we have \([b, c] \subset J_y(n) \subset (a, \infty)\) for \( n > N \) and, provided \( t \in J_y(n) \), we can estimate the symmetric difference of \((-\infty, t] \cap J_y \) and \((-\infty, t] \cap J_y(n) \) by
\[\left[(-\infty, t] \cap J_y \right] \Delta \left[(-\infty, t] \cap J_y(n) \right] \subset (a, b)\]
for \( n > N \). Again \( J_y(n) \) respects \( u_n \) so for all \( t \in \mathbb{R} \) we have
\[
\int_{(-\infty, t] \cap J_y(n)} (u_n - u_n^o) \, d\lambda \leq 0 \quad \text{and thus} \quad \int_{(-\infty, s] \cap J_y(n)} (u_n - u_n^o) \, d\lambda \leq 0.
\]

For \( n > N \) and any \( t \in J_y(n) \),
\[
\begin{align*}
&\left| \int_{(-\infty, t] \cap J_y} (u_n - u_n^o) \, d\lambda - \int_{(-\infty, t] \cap J_y(n)} (u_n - u_n^o) \, d\lambda \right| \\
&\leq \int_{[(-\infty, t] \cap J_y] \Delta [(-\infty, t] \cap J_y(n)]} (u_n + u_n^o) \, d\lambda \\
&\leq \int_{[(-\infty, t] \cap J_y] \Delta [(-\infty, t] \cap J_y(n)]} u_n^o \, d\lambda + \int_{(-\infty, s] \cap J_y(n)} u_n \, d\lambda + \int_{[s, b)} u \, d\lambda \\
&\leq \int_{[(-\infty, t] \cap J_y] \Delta [(-\infty, t] \cap J_y(n)]} u_n^o \, d\lambda + \int_{(-\infty, s] \cap J_y(n)} u_n^o \, d\lambda + \int_{[s, b)} u \, d\lambda \\
&\leq 2y \lambda(a, b) + \int_{[s, b)} u \, d\lambda < \varepsilon.
\end{align*}
\]

Here we have used the fact that \( u_n^o \leq u^o \) on \( J_y \) and \( u_n^o \leq u_n^o(b) \leq u^o(b) = y \) on \( J_y(n) \).

Since \( x \in [b, c] \subset J_y(n) \) for \( n > N \), \( \varepsilon > 0 \) was arbitrary, and
\[
\lim_{n \to \infty} \int_{(-\infty, x] \cap J_y(n)} (u_n - u_n^o) \, d\lambda \leq 0
\]
it follows that
\[
\int_{(-\infty,x]\cap J_y} (u - u^o) \, d\lambda = \lim_{n \to \infty} \int_{(-\infty,x]\cap J_y} (u_n - u^o_n) \, d\lambda \leq 0.
\]

This proves the inequality “≥” in (6.1). For the other inequality we fix \( \varepsilon > 0 \) and increase \( N \) so that for \( n > N \), \( u^o_n(c) > u^o(c) - \varepsilon \). Set \( t_n = \sup J_y(n) \). If \( t_n \in J_y(n) \) then
\[
\left| \int_{(-\infty,t_n]\cap J_y} (u_n - u^o_n) \, d\lambda - \int_{(-\infty,t_n]\cap J_y(n)} (u_n - u^o_n) \, d\lambda \right| < \varepsilon
\]
but
\[
\int_{(-\infty,t_n]\cap J_y(n)} (u_n - u^o_n) \, d\lambda = \int_{J_y(n)} (u_n - u^o_n) \, d\lambda = 0
\]
so
\[
\int_{(-\infty,t_n]\cap J_y} u \, d\lambda \geq \int_{(-\infty,t_n]\cap J_y} u_n \, d\lambda \\
\geq \int_{(-\infty,t_n]\cap J_y} u^o_n \, d\lambda - \varepsilon \\
\geq u^o_n(c) \lambda((-\infty,t_n]\cap J_y) - \varepsilon \\
\geq (u^o(c) - \varepsilon) \lambda((-\infty,t_n]\cap J_y) - \varepsilon.
\]

Therefore
\[
\frac{1}{(-\infty,t_n]\cap J_y} \int_{(-\infty,t_n]\cap J_y} u \, d\lambda \geq u^o(c) - \varepsilon - \varepsilon / \lambda((-\infty,t_n]\cap J_y).
\]

If \( t_n \notin J_y(n) \) then a similar argument shows that
\[
\frac{1}{(-\infty,t_n]\cap J_y} \int_{(-\infty,t_n]\cap J_y} u \, d\lambda \geq u^o(c) - \varepsilon - \varepsilon / \lambda((-\infty,t_n]\cap J_y).
\]

For \( n > N, x \leq t_n \) so
\[
\sup_{t \geq x} \frac{1}{\lambda((-\infty,t]\cap J_y)} \int_{(-\infty,t]\cap J_y} u \, d\lambda \geq u^o(c) - \varepsilon - \varepsilon / \lim sup_{n \to \infty} \lambda((-\infty,t]\cap J_y).
\]

Since \( \varepsilon \) was arbitrary this proves the inequality “≤” in (6.1) to complete the proof of part (iii) and the theorem.
References


Department of Mathematics, University of Western Ontario, London, Ontario, N6A 5B7, CANADA

E-mail address: sinnamon@uwo.ca