PERTURBED WEIGHTED HARDY INEQUALITIES

Peter Weidemaier and Gord Sinnamon

Fraunhofer-Institut für Kurzzeitdynamik and University of Western Ontario

January 16, 1998

Abstract. A perturbation is introduced into the usual weighted Hardy inequalities yielding a new inequality that is used in looking at the trace problem on Lebesgue spaces with mixed norms.

1. Introduction

The inequality

$$\int_0^A x^{-\varepsilon\gamma} \int_0^{x^\gamma} y^\varepsilon f(y)\,dy\,dx \le (\gamma\varepsilon)^{-1} \int_0^{A^\gamma} f(y)\,dy, \quad f \ge 0, \tag{1.1}$$

is an important tool for proving sharp results (see [5]) about the regularity of the trace on $\partial\Omega \times (0,T)$ for functions in the space $L_p(0,T;W_p^2(\Omega)) \cap W_p^1(0,T;L_p(\Omega))$. The inequality belongs to a well-known class of weighted Hardy inequalities (see Proposition 2.2) which are employed in various contexts to compare norms of functions to norms of their averages. When considering the trace problem for the more general class of functions $L_q(0,T;W_p^2(\Omega)) \cap W_q^1(0,T;L_p(\Omega))$, the related inequality

$$\left(\int_0^T \left(\int_0^A x^{-1-\beta-\varepsilon\gamma} \int_0^{x^\beta} \int_0^{x^\gamma} y^\varepsilon \rho(y,t+a)\,dy\,da\,dx\right)^p dt\right)^{1/p}$$
$$\le p(\gamma\varepsilon)^{-1} \left(\int_0^{T+A^\beta} \left(\int_0^{A^\gamma} \rho(y,\tau)\,dy\right)^p d\tau\right)^{1/p}, \quad \rho \ge 0, \tag{1.2}$$

arises (see [6]). The aim of this paper, realized in Corollary 2.4, is to prove the inequality (1.2). In Theorem 2.3 we prove a somewhat more general result in order to place (1.2) properly in the setting of weighted Hardy inequalities.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 46E35, 46E30.

Key words and phrases. Hardy inequality, Sobolev space, trace.

The second author acknowledges the support of the Natural Sciences and Engineering Research Council of Canada.

The results of Theorem 2.3 we call *perturbed weighted Hardy inequalities.* We regard the average $x^{-\beta}\int_0^{x^\beta}(\dots)$ as a perturbation since without it (1.2) would follow immediately from (1.1).

2. Perturbed Weighted Hardy Inequalities

We begin with a lemma that is interesting in its own right. Although it is easy to see that the integral average $\frac{1}{x}\int_0^x f$ of an L_1 function is not necessarily in L_1, the lemma shows that if $f = f(x,t) \in L_p(L_1) = L_p(L_1(dx), dt)$ then the integral average $\frac{1}{x}\int_0^x f(x, t+a)\,da$ is again in $L_p(L_1)$ for $1 \le p < \infty$. A similar averaging operator was studied in [4] where its boundedness on mixed norm weighted spaces was shown to characterize Muckenhoupt's A_p weight condition.

The important feature of this lemma for the purpose of application to the regularity of the trace is to get the $L_p(L_1)$ norm rather than the $L_1(L_p)$ norm on the right hand side. This renders the task more difficult because, in view of Minkowski's integral inequality, the former norm is the smaller of the two.

Lemma 2.1. *Suppose that* $1 \le p < \infty$, $0 < \beta, \gamma < \infty$, *and* $0 < A, T \le \infty$. *Then*

$$\left(\int_0^T \left(\int_0^A x^{-\beta}\int_0^{x^\beta} f(x,t+a)\,da\,dx\right)^p dt\right)^{1/p} \le p\left(\int_0^{T+A^\beta}\left(\int_0^A f(x,\tau)\,dx\right)^p d\tau\right)^{1/p}, \quad f \ge 0. \tag{2.1}$$

Proof. We proceed by duality. Let $\phi \in L_{p'}(O,T)$ with $\|\phi\|_{L_{p'}(0,T)} \le 1$ and set

$$I_\phi = \left|\int_0^T \phi(t)\int_0^A x^{-\beta}\int_0^{x^\beta} f(x,t+a)\,da\,dx\,dt\right|.$$

Here p' is the conjugate index of p satisfying $1/p + 1/p' = 1$. Note that $1 < p' \le \infty$. To establish (2.1) it is enough to show that

$$I_\phi \le p\left(\int_0^{T+A^\beta}\left(\int_0^A f(x,\tau)\,dx\right)^p d\tau\right)^{1/p}. \tag{2.2}$$

Extend ϕ to be zero off $(0,T)$ so that $\phi \in L_{p'}(\mathbf{R})$ and note that $\|\phi\|_{L_{p'}(\mathbf{R})} \le 1$.

Once we take the absolute value inside the integral defining I_ϕ we may apply Tonelli's Theorem to interchange the order of integration. We obtain

$$\begin{aligned} I_\phi &\le \int_0^T |\phi(t)|\int_0^A x^{-\beta}\int_0^{x^\beta} f(x,t+a)\,da\,dx\,dt \\ &= \int_0^A x^{-\beta}\int_0^{x^\beta}\int_0^T |\phi(t)| f(x,t+a)\,dt\,da\,dx \\ &= \int_0^A x^{-\beta}\int_0^{x^\beta}\int_a^{T+a} |\phi(\tau-a)| f(t,\tau)\,d\tau\,da\,dx \end{aligned}$$

where we have made the substitution $\tau = t + a$. Since $a \le x^\beta$ and $x \le A$ we see that $\tau \le T + a \le T + A^\beta$ so we may extend the range of the inner integral from $(a, T + a)$ to $(0, T + A^\beta)$. Following this estimate with another interchange yields

$$\begin{aligned} I_\phi &\le \int_0^A x^{-\beta} \int_0^{x^\beta} \int_0^{T+A^\beta} |\phi(\tau - a)| f(x,\tau)| \, d\tau \, da \, dx \\ &= \int_0^{T+A^\beta} \int_0^A x^{-\beta} \int_0^{x^\beta} |\phi(\tau - a)| \, da f(x,\tau) \, dx \, d\tau \\ &\le \int_0^{T+A^\beta} M\phi(\tau) \left(\int_0^A f(x,\tau) \, dx \right) d\tau. \end{aligned}$$

The last inequality follows from the definition of the one-sided Hardy-Littlewood Maximal Function, $M\phi(\tau)$. It is shown in [3] that for $1 < p' \le \infty$,

$$M\psi(\tau) = \sup_{h>0} \frac{1}{h} \int_0^h |\psi(\tau - a)| \, da$$

satisfies $\|M\psi\|_{L_{p'}(\mathbf{R})} \le p\|\psi\|_{L_{p'}(\mathbf{R})}$ for all $\psi \in L_{p'}(\mathbf{R})$. Taking $\psi = \phi$ we have

$$\|M\phi\|_{L_{p'}(0,T+A^\beta)} \le \|M\phi\|_{L_{p'}(\mathbf{R})} \le p\|\phi\|_{L_{p'}(\mathbf{R})} \le p.$$

Applying Hölder's inequality with indices p' and p to the last estimate of I_ϕ yields

$$\begin{aligned} I_\phi \le& \|M\phi\|_{L_{p'}(0,T+A^\beta)} \left(\int_0^{T+A^\beta} \left(\int_0^A f(x,\tau) \, dx \right)^p d\tau \right)^{1/p} \\ \le& p \left(\int_0^{T+A^\beta} \left(\int_0^A f(x,\tau) \, dx \right)^p d\tau \right)^{1/p}. \end{aligned}$$

This proves (2.2) and completes the proof.

The weighted Hardy inequalities that we intend to perturb are given in the following proposition. Note that (1.1) is just the special case $r = s = 1$.

Proposition 2.2. [1] *Suppose that* $1 \le s \le r < \infty$, $0 < \gamma, \varepsilon < \infty$, $0 < A \le \infty$, *and* $\mu = 1 - 1/s + 1/r$. *Then*

$$\begin{aligned} &\left(\int_0^A x^{-1-\varepsilon\gamma r} \left(\int_0^{x^\gamma} y^{-1+\varepsilon+1/s} f(y) \, dy \right)^r dx \right)^{1/r} \\ &\qquad\qquad \le \gamma^{-1/r} (\mu/\varepsilon)^\mu \left(\int_0^{A^\gamma} f(y)^s \, dy \right)^{1/s}, \quad f \ge 0. \end{aligned}$$

Theorem 2.3. *Suppose that* $1 \le s \le r \le p < \infty$, $0 < \beta, \gamma, \varepsilon < \infty$, $0 < A < T \le \infty$ *and* $\mu = 1 - 1/s + 1/r$. *Then*

$$\left(\int_0^T \left(\int_0^A x^{-1-\beta-\varepsilon\gamma r} \int_0^{x^\beta} \left(\int_0^{x^\gamma} y^{-1+\varepsilon+1/s} \rho(y, t+a)\, dy\right)^r da\, dx\right)^{p/r} dt\right)^{1/p}$$
$$\le (p/r\gamma)^{1/r} (\mu/\varepsilon)^\mu \left(\int_0^{T+A^\beta} \left(\int_0^{A^\gamma} \rho(y,\tau)^s\, dy\right)^{p/s} d\tau\right)^{1/p}, \quad \rho \ge 0.$$

Proof. Apply Lemma 2.1 with p replaced by p/r and $f(x,\tau)$ replaced by

$$x^{-1-\varepsilon\gamma r} \left(\int_0^{x^\gamma} y^{-1+\varepsilon+1/s} \rho(y,\tau)\, dy\right)^r$$

to get the first inequality in

$$\left(\int_0^T \left(\int_0^A x^{-1-\beta-\varepsilon\gamma r} \int_0^{x^\beta} \left(\int_0^{x^\gamma} y^{-1+\varepsilon+1/s} \rho(y, t+a)\, dy\right)^r da\, dx\right)^{p/r} dt\right)^{r/p}$$
$$\le (p/r) \left(\int_0^{T+A^\beta} \left(\int_0^A x^{-1-\varepsilon\gamma r} \left(\int_0^{x^\gamma} y^{-1+\varepsilon+1/s} \rho(y,\tau)\, dy\right)^r dx\right)^{p/r} d\tau\right)^{r/p}$$
$$\le (p/r)[\gamma^{-1/r}(\mu/\varepsilon)^\mu]^r \left(\int_0^{T+A^\beta} \left(\int_0^{A^\gamma} \rho(y,\tau)^s\, dy\right)^{p/s} d\tau\right)^{r/p}.$$

The second inequality relies on Proposition 2.2. Raising both sides to the power $1/r$ completes the proof.

Remarks. 1. Hölder's inequality shows that $\left(x^{-\beta}\int_0^{x^\beta} \dots\right)^r \le x^{-\beta}\int_0^{x^\beta} (\dots)^r$ so Theorem 2.3 also applies if the perturbation is made inside the rth power rather than outside as stated.

2. The constant, $\gamma^{-1/r}(\mu/\varepsilon)^\mu$, given in Proposition 2.2 is not best possible when $1 < s < r$. The best possible constant,

$$\gamma^{-1/r}\varepsilon^{-\mu}(1/s')^{1/s'}(1/r)^{1/r} \left(\frac{r(1-\mu)\Gamma(1/(1-\mu))}{\Gamma(1/(r(1-\mu))\Gamma(1/(r'(1-\mu))}\right)^{1-\mu},$$

follows from the results of [2]. The obvious improvement in Theorem 2.3 may be made.

Corollary 2.4. *If* $p \ge 1$, $0 < \beta, \gamma, \varepsilon < \infty$, *and* $0 < A, T \le \infty$ *then inequality* (1.2) *holds.*

Proof. Take $r = s = 1$ in Theorem 2.3.

References

1. O. V. Besov, V. P. Il'in, and S. M. Nikols'kii, "Integral Representations of Functions and Imbedding Theorems, Volume 1" Wiley, New York, 1978.
2. G. A. Bliss, An integral inequality, *J. Lond. Math. Soc.* **5** (1930), 40–46.
3. G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities" Cambridge University Press, London, 1934.
4. G. Sinnamon, The A_p condition and a positive, linear operator, *Proceeding of the Royal Society of Edinburgh* **118A** (1991), 153–156.
5. P. Weidemaier, The trace theorem $W_p^{2,1}(\Omega_T) \ni f \mapsto \nabla_x f \in W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(\partial\Omega_T)$ revisited, *Comment. Math. Univ. Carolinae* **32** (1991), 307–314.
6. P. Weidemaier, Existence results in $L_p - L_q$ spaces for second order parabolic equations with inhomogeneous Dirichlet boundary conditions, "Proceedings of the Third European Conference on Elliptic and Parabolic Problems, Pont-à-Mousson, France, June 1997" Pitman Research Notes in Mathematics, Longman, to appear 1998.

Fraunhofer-Institut für Kurzzeitdynamik, Eckerstr. 4, D-79104 Freiburg, GERMANY and Department of Mathematics, University of Western Ontario, London, Ontario, N6A 5B7, CANADA

E-mail address: weide@emi.fhg.de and sinnamon@uwo.ca