Western Science

Department of Mathematics

AM9626 Course Outline

1. Course Information

Course Information
Course name: Numerical Algebraic Geometry
Course number: AM9626
Term: Fall 2024

List of Prerequisites
A course in linear algebra, a course in abstract algebra (or group theory), and basic programming skills. Background in algebraic geometry, complex analysis, numerical analysis, or topology will be helpful, but ultimately, is not required.

Auditing
Permission to audit the course, due to reasons which prohibit enrollment, will be considered. To be considered for auditing, please contact the professor directly.

2. Instructor Information

Instructors	Email	Office	Office Hours
Dr. Taylor Brysiewicz	tbrysiew@uwo.ca		

Office hours are available in a hybrid format (i.e. in-person and via zoom)

3. Course Syllabus, Schedule, Delivery Mode

This course is a broad introduction to the world of numerical algebraic geometry.

Numerical algebraic geometry is a computational paradigm for studying algebraic varieties, i.e., solutions to polynomial systems. Algorithms in numerical algebraic geometry work by manipulating numerical approximations of points on varieties. In this sense, numerical algebraic geometry is the geometry of computational algebraic geometry - contrary to algebraic approaches (e.g. Gröbner bases, regular chains, etc) which perform exact algebraic manipulations on polynomials.

Relaxing the need for exact computation offers enormous computational benefits - numerically, systems with millions of solutions can be efficiently and reliably solved on a personal laptop computer. Moreover, the algorithms for doing so are trivially parallelizable and can be certified, turning them into rigorous mathematical statements for which the computation is the proof.

Learning Outcomes

- Given a zero-dimensional polynomial system, construct an appropriate homotopy to find its solutions. This includes squaring-up the system, finding an appropriate start system and start solutions, and applying the gamma trick.
- Implement a basic multivariate predictor-corrector method, using Newton's method and Euler's method.
- Identify the possible end-game behaviours of multivariate homotopies and how they are handled.
- Compute several types of degree bounds associated to a zero-dimensional polynomial system and interpret/compare the results. This includes its Bezout bound, its BKK bound, and its multihomogeneous Bezout bound.
- Describe the differences of symbolic and numerical computation in regards to speed, accuracy, and proof.
- Implement a basic monodromy solve algorithm.
- Apply the trace test for verifying the successful computation of a witness superset.
- Describe the monodromy break-up algorithm for producing witness sets from a witness superset.
- Interpret the results of the certification procedures of alpha certification and the Krawczyk's method.
- Identify the basic constituents of a parameter homotopy. This include writing down the appropriate branched cover, defining the discriminant, and drawing conclusions about the connected components of the complement of the discriminant in regards to the number of real solutions.

Lecture Schedule and Delivery Mode

The hybrid lectures.
For online participants, the zoom link will be provided to you by the Fields Institute.

Class begins: September 10, 2024
Fall Reading Week: October 12 – October 20, 2024
Classes end: December 5, 2024

The tentative schedule is below:

Week	Section Title	Description
1	Polynomial systems and an introduction to numerical algebraic geometry	We will lead with examples of polynomial systems coming from applications and theory to illustrate the kinds of problems we will be talking about in the course. We will showcase algorithms in numerical algebraic geometry by applying them to these systems, leaving their explanation for the rest of the course.
2,3	Univariate Homotopy Continuation	We will describe homotopy continuation in the univariate setting. Topics: Newton's Method, Euler's Method, Discriminants, Bezout Start System, Roots of Unity, Homotopy, Davidenko Differential Equation, Varieties in C, the Cauchy Endgame.
4	Multivariate Homotopy Continuation	We will extend our homotopy continuation algorithm to multiple variables. Topics: Jacobians, Multivariate Newton and Euler, Bezout Start System, Squaring-up, and Endpoints on Positive-Dimensional Varieties.

5,6	Parameter Homotopies	We will realize our homotopy algorithms in a broader context of parameter homotopies. Topics: Discriminants Revisited, Incidence Varieties, Branched Covers, The Bezout Family, The Polyhedral Family, and The Parameter Continuation Theorem.
7,8	Monodromy	We will define monodromy groups, how to use them to reveal symmetry in families of polynomial systems, and how to use them in a new way to solve a polynomial system. Topics: Monodromy Loop, Monodromy Permutation, Transitive Groups, Imprimitive/Primitive groups, Monodromy Solving, Decomposable Branched Covers, Computing the Monodromy Groups, and Galois group
9,10,11	Witness Sets	We will define witness sets and witness supersets, describe their computation, and what one can learn about a variety from this datatype. Topics: Witness Set, Witness Superset, Trace Test, Monodromy Breakup, and Cascade Algorithm.
12	Certification	We will learn two ways of certifying approximate solutions to polynomial systems and what exactly it means have such a certificate. Topics: Alpha Theory, Krawczyk's Method, Banach Fixed Point Theorem

4. Course Materials

Course Website:

https://sites.google.com/view/taylorbrysiewicz/teaching/numericalalgebraicgeometry

Students are responsible for checking the course site on a regular basis for news and updates. This is the primary method by which information will be disseminated to all students in the class.
All course material will be posted to the course website.

Optional Textbook:

The Numerical Solution of Systems of Polynomials Arising in Engineering and Science by Andrew Sommese and Charles Wampler

Other resources (articles and software) will be posted to course website.

Technical Requirements

A working version of the programming language **julia.**
If remote: A stable internet connection, computer with working microphone/webcam

5. Methods of Evaluation

The overall course grade will be calculated as listed below:
Assignments (8 (of 10)) 60%

Attendance	20%
Project	20% = 5%+5%+10% for proposal, draft 1, and final version respectively

There are 5 due dates for assignments and each assignment has two parts, a "theory" part and an "applications/coding" part. This totals 10 assignments, worth 7.5% each. The lowest two scores (15%) are dropped.

Assignment due dates are
October 1, October 22, November 5, November 19, and December 5

The project will be to write a short paper (no more than 4 pages) summarizing a computational analysis of some polynomial system via techniques in numerical algebraic geometry. The student is free to choose the polynomial system, subject to approval.

The project proposal is due October 1
The first project draft is due November 14
The final version is due December 5

6. Student Absences

If you are unable to meet a course requirement due to illness or other serious circumstances, please follow the procedures below.

Assessments worth less than 10% of the overall course grade:

Missed work will be handled via extensions whenever appropriate. The extension will be no more than two weeks after the original due date. Any missed homework can be turned in by the end of the term for half credit.

6. Accommodation and Accessibility

Religious Accommodation
When a course requirement conflicts with a religious holiday that requires an absence from the University or prohibits certain activities, students should request accommodation for their absence in writing at least two weeks prior to the holiday to the course instructor and/or the Academic Counselling office of their Faculty of Registration. Please consult University's list of recognized religious holidays (updated annually) at

https://multiculturalcalendar.com/ecal/index.php?s=c-univwo.

Accommodation Policies
Students with disabilities are encouraged to contact Accessible Education, which provides recommendations for accommodation based on medical documentation or psychological and cognitive testing. The policy on Academic Accommodation for Students with Disabilities can be found at:

https://www.uwo.ca/univsec/pdf/academic_policies/appeals/Academic Accommodation_disabilities.pdf.

7. Academic Policies

The website for Registrarial Services is http://www.registrar.uwo.ca.

In accordance with policy,

https://www.uwo.ca/univsec/pdf/policies_procedures/section1/mapp113.pdf,

the centrally administered e-mail account provided to students will be considered the individual's official university e-mail address. It is the responsibility of the account holder to ensure that e-mail received from the University at their official university address is attended to in a timely manner.

Scholastic offences are taken seriously and students are directed to read the appropriate policy, specifically, the definition of what constitutes a Scholastic Offence, at the following Web site:

http://www.uwo.ca/univsec/pdf/academic_policies/appeals/scholastic_discipline_undergrad.pdf.

8. Support Services

Please visit the Science & Basic Medical Sciences Academic Counselling webpage for information on adding/dropping courses, academic considerations for absences, appeals, exam conflicts, and many other academic related matters: https://www.uwo.ca/sci/counselling/.

Students who are in emotional/mental distress should refer to Mental Health@Western (https://uwo.ca/health/) for a complete list of options about how to obtain help.

Western is committed to reducing incidents of gender-based and sexual violence and providing compassionate support to anyone who has gone through these traumatic events. If you have experienced sexual or gender-based violence (either recently or in the past), you will find information about support services for survivors, including emergency contacts at

https://www.uwo.ca/health/student_support/survivor_support/get-help.html.

To connect with a case manager or set up an appointment, please contact support@uwo.ca.

Please contact the course instructor if you require lecture or printed material in an alternate format or if any other arrangements can make this course more accessible to you. You may also wish to contact Accessible Education at

http://academicsupport.uwo.ca/accessible_education/index.html

if you have any questions regarding accommodations.

Learning-skills counsellors at the Student Development Centre (https://learning.uwo.ca) are ready to help you improve your learning skills. They offer presentations on strategies for improving time management, multiple-choice exam preparation/writing, textbook reading, and more. Individual support is offered throughout the Fall/Winter terms in the drop-in Learning Help Centre, and year-round through individual counselling.

Western University is committed to a thriving campus as we deliver our courses in the mixed model of both virtual and face-to-face formats. We encourage you to check out the Digital Student Experience website to manage your academics and well-being: https://www.uwo.ca/se/digital/.

Additional student-run support services are offered by the USC, https://westernusc.ca/services/.